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ITERATIVE SCHEMES FOR SOLVING NEW SYSTEM OF GENERAL EQUATIONS

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In this paper, we consider a new system of general equations, which can be used to study the odd-order and nonsymmetric boundary value problems. It is shown that Lax-Milgram Lemma and Reisz-Fréchet representation theorem can be obtained as special cases. We use the auxiliary principle technique to prove the existence of a solution to the general equations. This technique is also used to suggest some new iterative methods. The convergence analysis of the proposed methods is analyzed under some mild conditions. Ideas and techniques of this paper may stimulate further research.

1. Introduction

It is well known that a linear continuous functional can be represented by the inner product in a Hilbert space, the origin of which can be traced back to Riesz [24] and Fréchet [4]. This result is known as the Riesz-Fréchet representation theorem. By choosing the Hilbert Space and inner product appropriately, this theorem furnishes one of the major existence theory tools for even-order boundary value differential equations. The proof of the Riesz-Fréchet theorem also shows another fact which is omitted in the final statement; but is equivalent to it. This fact, the existence of a minimum to a certain quadratic form on a closed convex set, is very useful in variational problems

It is clear that the inner product is a bilinear function. Then the question arises whether such a representation result holds for an arbitrary bifunction. The answer to this is affirmative. Lax-Milgram [9] proved that a linear continuous function can be represented by an arbitrary bifunction under suitable conditions. This representation is known as the Lax-Milgram Lemma, which is a natural generalization of the Riesz-Fréchet theorem for continuous bilinear forms, plays a significant role in the development of various branches of mathematical and engineering sciences. From the day of discovery of the Riesz-Fréchet theorem, many important contributions have been made in this direction, see [1, 2, 4, 6, 8, 9, 10, 11, 12, 17, 18, 20, 22, 23, 24, 25, 26, 27, 28, 30, 31]. In every case, a new approach and method is applied to generalize some of these results and the ideas they used.

We would like to point out that the Lax-Milgram lemma is equivalent to the optimization problem, if the involved operator is positive and symmetric. It is known that only the even order and self-adjoint boundary value problems can be studied by the classical Lax-Milgram lemma. In fact, the Lax-Milgram lemma is the weak formulation of the boundary value problems. It have been observed that the involved operator may not be positive and symmetric. To tackle such problems, the operator may be made positive and symmetric with respect to an arbitrary map. For more details, see Fillopov [3], Noor et al.[19],Tonti [29] and the references therein.

In this paper, we introduce and study a new system of general equations with respect to an arbitrary operator. This system of general equations can be viewed as a weak formulation of the non-positive and nonsymmetric odd-order boundary value problems. It is shown that

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the Lax-Milgram lemma can be obtained as special cases. We use the auxiliary principle technique, which is mainly due to Lions et al.[10] and Glowinski et al. [7], to discuss the existence of a solution for the system of general equations. The auxiliary principle technique is used to suggest some iterative methods for solving the system of general equations. The convergence analysis of these methods is investigated under suitable pseudomonotonicity, which is a weaker condition.
In Section 2, we introduce new system of general equations and discuss its applications. It is shown that the third order boundary value problems can be studied in the framework of generalized equations. In section 3 and section 4, we use the auxiliary principle technique to discuss the existence of a solution as well as to suggest some iterative methods for solving the general equations. The convergence analysis of the proposed methods is considered under some mild conditions. Several new iterative methods for solving the generalized equations are obtained as novel applications of the results. The ideas and techniques of this paper may the starting point for further research.

2. **Formulations and basic facts**

Let H be a Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.
For given operators $L, g : H \longrightarrow H$ and a continuous linear functional f, we consider the problem of finding $u \in H$ such that

$$\langle Lu, g(v)\rangle = \langle f, g(v)\rangle, \quad \forall v \in H, \tag{1}$$

which is called the system of the general equations. This is also known as the general Lax-Milgram Lemma. The alternative formulation (1) can be viewed as the weak formulation of the odd-order boundary value problems.
We remark that the problem (1) is equivalent to finding $u \in H$ such that

$$\langle Lu, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{2}$$

This equivalent formulation is used to discuss the unique existence to a solution of the odd-order and nonsymmetric boundary value problems.
Clearly, the problem (1) reduces to the following problems of finding $u \in H$ such that

$$\langle Lu, v - g(u)\rangle = \langle f, v - g(u)\rangle, \quad \forall v \in H, \tag{3}$$

which is a special case of general variational inequality [13].
Also, the problem(1) can be written as: Find $u \in H$ such that

$$\langle Lu, g(v) - u)\rangle = \langle f, g(v) - u\rangle, \quad \forall v \in H. \tag{4}$$

We note that the problem (4) can be deuced from the general variational inequality [16].
Note that problems (2), (3) and (4) are distinctly different from each other.

We now discuss some special cases of general equations (1).
(I). If $\langle Lu, g(v)-u)\rangle = a(u, g(v))$, where $a(.,.) : H\times H \to H$ is a bifunction, then problem (1) reduces to finding $u \in H$ such that

$$a(u, g(v)) = \langle f, g(v)\rangle, \quad \forall v \in H, \tag{5}$$

which is called the general Lax-Milgram Lemma[9, 17, 18]. We would like to point that, if the bifunction $a(,.,)$ is not positive and symmetric, then it can be made positive and symmetric with respect an arbitrary functions. Consequently odd-order and nonsymmetric boundary value can be studied in the general framework of the problem (5). This is the novelty of the general Lax-Milgram lemma.

(II). If $g = I$, the identity operator, then the general Lax-Nilgram lemma (5) collapses to finding $u \in H$, such that

$$a(u, v) = \langle f, v\rangle, \quad \forall v \in H, \tag{6}$$

which is the classical Lax-Milgram Lemma.
(III). If $L = I$, then problem (1) reduces to finding $u \in H$ such that

$$\langle u, g(v)\rangle = \langle f, g(v)\rangle, \quad \forall v \in H, \tag{7}$$

which is called the general Riesz-Fréchet theorem and appears to be a new one.
(IV). If $L = I$ and $g = I$, and then problem (1) reduces to finding $u \in H$ such that

$$\langle u, v\rangle = \langle f, v\rangle, \quad \forall v \in H, \tag{8}$$

which is the celebrated Riesz-Fréchet representation theorem [11, 18].
We would like to emphasize that the problem (8) is equivalent to finding the minimum $u \in H$ of the functional

$$I[v] = \langle v, v\rangle - 2\langle f, v\rangle, \quad \forall v \in H, \tag{9}$$

which is called the energy (virtual work) functional and can be viewed as novel extension of the variational principles. The function $I[v]$ defined by (9 is a strongly convex function and this equivalent formulation can be used to discuss the existence and uniqueness of the representation theorems. This a fascinating feature of the Riesz-Fréchet representation theorem, which can be exploited to use the techniques and ideas of the quadratic programming optimization.

Remark 2.1. *For suitable and appropriate choice of the operators L, g, one can obtain various classes of new and old classes of generalized equations. This shows that the system of generalized equations is a unified one.*

We now recall some well known concepts and basic results [3, 29], which play significant part in deriving the main results.

Definition 2.1. [3, 29] *An operator $L : H \longrightarrow H$ with respect to an arbitrary operator $g : H \to H$ is said to be :*
(a). *g-symmetric , if and only if,*

$$\langle Lu, g(v)\rangle = \langle g(u), Lv\rangle, \quad \forall u, v \in H.$$

(b). *g-positive, if and only if,*

$$\langle Lu, g(u)\rangle \geq 0, \quad \forall u \in H.$$

(c). *g-coercive (g-elliptic), if there exists a constant $\alpha > 0$ such that*

$$\langle Lu, g(u)\rangle \geq \alpha\|g(u)\|^2, \quad \forall u \in H.$$

Note that g-coercivity implies g-positivity, but the converse is not true. It is also worth mentioning that there are operators which are not g-symmetric but g-positive. On the other hand, there are g-positive , but not g-symmetric operators. Furthermore, it is well-know [3, 29]] that, if, for a linear operator L, there exists an inverse operator L^{-1} on $R(L)$, the range of L, with $\overline{R(L)} = H$, then one can find an infinite set of auxiliary operators g such that the operator L is both g-symmetric and g-positive.
If the operator L is linear, g-positive, g-symmetric and the operator g is linear,
then the problem (1) is equivalent to finding a minimum of the function $I[v]$ on H, where

$$I[v] = \langle Lv, g(v)\rangle - 2\langle f, g(v)\rangle, \quad \forall v \in H, \tag{10}$$

which is a nonlinear programming problem and can be solved using the known techniques of the nonlinear optimization.

We now consider the problem of finding the minimum of the functional $I[v]$, defined by (10) and this is the main motivation of our next result.

Theorem 2.1. *Let the operator $L : H \longrightarrow H$ be linear, g-symmetric and let L be g-positive. If the operator $g : H \longrightarrow H$ is linear, then the function $u \in H$ minimizes the functional $I[v]$, defined by (10), if and only if,*

$$\langle Lu, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{11}$$

Proof. Let $u \in H$ satisfy (11). Then, using the g-positivity of (L, we have

$$\langle Lv, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle. \tag{12}$$

$\forall u, v \in H, \quad \epsilon \geq 0$, let $v_\epsilon = u + \epsilon(v - u) \in H$. Taking $v = v_\epsilon$ in (12) and using the fact that g is linear, we have

$$\langle Lv_\epsilon, g(v_\epsilon) - g(u)\rangle \geq \langle f, g(v_\epsilon) - g(u)\rangle. \tag{13}$$

We now define the function

$$h(\epsilon) = \epsilon\langle Lu, g(v) - g(u)\rangle + \frac{\epsilon^2}{2}\langle L(v-u), g(v) - g(u)\rangle - \epsilon\langle f, g(v) - g(u)\rangle, \tag{14}$$

such that

$$h'(\epsilon) = \langle Lu, g(v) - g(u)\rangle + \epsilon\langle L(v-u), g(v) - g(u)\rangle - \langle f, g(v) - g(u)\rangle \geq 0, \quad \text{by (13)} .$$

Using the g symmetry of L, we see that $h(\epsilon)$ is an increasing function on $[0, 1]$ and so $h(0) \leq h(1)$ gives us

$$\langle Lu, g(u)\rangle - 2\langle f, g(u)\rangle \leq \langle Lv, g(v)\rangle - 2\langle f, g(v)\rangle,$$

that is,

$$I[u] \leq I[v], \quad \forall v \in H,$$

which shows that $u \in H$ minimizes the functional $I[v]$, defined by (10).
Conversely, assume that $u \in H$ is the minimum of $I[v]$, then

$$I[u] \leq I[v], \quad \forall v \in H. \tag{15}$$

Taking $v = v_\epsilon \equiv u + \epsilon(v - u) \in H, \forall u, v \in H$ in (15), we have

$$I[u] \leq I[v_\epsilon].$$

Using (10), g-positivity and the linearity of L, we obtain

$$\langle Lu, g(v) - g(u)\rangle + \frac{\epsilon}{2}\langle L(g(v) - g(u)), g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle,$$

from which, as $\epsilon \longrightarrow 0$, we have

$$\langle Lu, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{16}$$

Replacing $g(v) - g(u)$ by $(g(u) - g(v))$ in inequality (16), we have

$$\langle Lu, g(v) - g(u)\rangle \leq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{17}$$

From (16) and (17, it follows that $u \in H$ satisfies

$$\langle Lu, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{18}$$

the required result (11). □

We now show that the third order boundary value problems can be studied via problem (1).

Example 2.1. *Consider the third order boundary value problem of finding u such that*

$$-\frac{D^3u}{dx^3} = f(x), \quad \forall x \in [a,b], \tag{19}$$

with boundary conditions

$$u(a) = 0, \quad u'(a) = 0, \quad u'(b) = 0, \tag{20}$$

where $f(x)$ is a continuous function. This problem can be studied in the general framework of the problem (1). *To do so, let*

$$H = \{u \in H_0^2[a,b] : u(a) = 0, \quad u'(a) = 0, \quad u'(b) = 0\}$$

be a Hilbert space, see [3, 7, 29]. *One can easily show that the energy functional associated with* (1) *is:*

$$\begin{aligned} I[v] &= -\int_a^b \frac{d^3v}{dx^3} v dx - 2\int_a^b f\frac{dv}{dx} dx, \quad \forall \frac{dv}{dx} \in H_0^2[a,b] \\ &= \int_a^b (\frac{d^2v}{dx^2})^2 - 2\int_a^b f\frac{dv}{dx} v dx \\ &= \langle Lv, g(v)\rangle - 2\langle f, g(v)\rangle, \end{aligned}$$

where

$$\langle Lu, g(v)\rangle = -\int_a^b \frac{d^3u}{dx^3}\frac{dv}{dx} dx = \int_a^b (\frac{d^2u}{dx^2})(\frac{d^2v}{dx^2}) dx, \tag{21}$$

and

$$\langle f, g(v)\rangle = \int_a^b f\frac{dv}{dx} dx,$$

where $g = \frac{d}{dx}$ is linear operator. It is clear that the operator L defined by (21) *is linear, g-symmetric, g-positive. Thus the minimum of the functional $I[v]$ defined on the Hilbert space H can be characterized by equation* (1). *This shows that the third order absolute boundary value problems can be studied in the framework of* (1).

Definition 2.2. *An operator $L : H \longrightarrow H$ is said to be;*
(i). Strongly monotone, if there exists a constant $\alpha > 0$, such that

$$\langle Lu - Lv, u - v\rangle \geq \alpha\|u - v\|^2, \quad \forall u, v \in H.$$

(ii). Lipschitz continuous, if there exists a constant $\beta > 0$, such that

$$\|Lu - Lv\| \leq \beta\|u - v\|, \quad \forall u, v \in H.$$

(iii) monotone, if

$$\langle Lu - Lv, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

(iv) firmly strongly monotone, if

$$\langle Lu - Lv, u - v\rangle \geq \|u - v\|^2, \quad \forall u, v \in H.$$

We remark that, if the operator L is both strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively, then from (i) and (ii), it follows that $\alpha \leq \beta$.

3. Main results

In this section, we use the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [10]and Glowinski et al [7], as developed by Noor [15] and Noor et al. [18, 20, 23]. The main idea of this technique is to consider an auxiliary problem related to the original problem. This way, one defines a mapping connecting the solutions of both problems. To prove the existence of solution of the original problem, it is enough to show that this connecting mapping is a contraction mapping and consequently has a unique solution of the original problem. Another novel feature of this approach is that this technique enables us to suggest some iterative methods for solving the generalized absolute value equations.

Theorem 3.1. *Let the operator L be a strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. Let the operator g be a firmly strongly monotone and Lipschitz continuous with constant β_1. If there exists a constant $\rho > 0$ such that*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \beta^2\nu(2-\nu)}}{\beta^2}, \quad \alpha > \beta\sqrt{\nu(2-\nu)}, \quad \nu < 1, \tag{22}$$

where

$$\nu = \sqrt{\beta_1^2 - 1}, \tag{23}$$

then the problem (1) *has a solution.*

Proof. We use the auxiliary principle technique to prove the existence of a solution of (2). For a given $u \in H$, consider the problem of finding $w \in H$ such that,

$$\langle \rho Lu, g(v) - g(w)\rangle + \langle g(w) - g(u), g(v) - g(w)\rangle = \langle \rho f, g(v) - g(w)\rangle, \quad \forall v \in H, \tag{24}$$

which is called the auxiliary problem, where $\rho > 0$ is a constant. It is clear that (24) defines a mapping w connecting the both problems (1) and (24)). To prove the existence of a solution of (1), it is enough to show that the mapping w defined by (24) is a contraction mapping. Let $w_1 \neq w_2 \in H$ (corresponding to $u_1 \neq u_2$) satisfy the auxiliary problem (24). Then

$$\begin{aligned} \langle \rho Lu_1, g(v) - g(w_1)\rangle &+ \langle g(w_1) - g(u_1), g(v) - g(w_1)\rangle \\ &= \langle \rho f, g(v) - g(w_1)\rangle, \quad \forall v \in H, \qquad (25) \\ \langle \rho Lu_2, g(v) - g(w_2)\rangle &+ \langle g(w_2) - g(u_2), g(v) - g(w_2)\rangle \\ &= \langle \rho f, g(v) - g(w_2)\rangle, \quad \forall v \in H. \qquad (26) \end{aligned}$$

Taking $v = w_2$ in (25) and $v = w_1$ in (26) and adding the resultant, we have

$$\begin{aligned} \|g(w_1) - g(w_2)\|^2 &= \langle g(w_1) - g(w_2), g(w_1) - g(w_2)\rangle \\ &= \langle g(u_1) - g(u_2) - \rho(Lu_1 - Lu_2). \end{aligned} \tag{27}$$

From (27), we have

$$\|g(w_1) - g(w_2)\|^2 \le \|g(u_1) - g(u_2) - \rho(Lu_1 - Lu_2)\|,$$

from which, it follows that

$$\begin{aligned} \|w_1 - w_2| &\le \|g(w_1) - g(w_2)\| \\ &\le \|g(u_1) - g(u_2) - \rho(Lu_1 - Lu_2)\| \\ &\le \|u_1 - u_2 - g(u_1) - g(u_2)\| + \|u_1 - u_2 - \rho(Lu_1 - Lu_2)\|. \end{aligned} \tag{28}$$

Using the strongly monotonicity and Lipschitz continuity of the operator L with constants $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned}\|u_1 - u_2 - \rho(Lu_1 - Lu_2)\|^2 &= \langle u_1 - u_2 - \rho(Lu_1 - Lu_2), u_1 - u_2 - \rho(Lu_1 - Lu_2)\rangle \\ &= \langle u_1 - u_2, u_1 - u_2\rangle - 2\rho\langle Lu_1 - Lu_2, u_1 - u_2\rangle \\ &\quad +\rho^2\langle Lu_1 - Lu_2, Lu_1 - Lu_2\rangle \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2. \end{aligned} \quad (29)$$

Similarly, using the strongly firmly monotonicity and Lipschitz continuity of the operator g with constant β_1, we have

$$\|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 \leq \{\sqrt{\beta_1^2 - 1}\}\|u_1 - u_2\|^2. \quad (30)$$

Combining (28),(29) and (30), we have

$$\begin{aligned}\|w_1 - w_2\| &\leq \big(\sqrt{\beta_1^2 - 1} + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)}\big)\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned} \quad (31)$$

where

$$\begin{aligned}\theta &= \big(\sqrt{\beta_1^2 - 1} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\big) \\ &= \nu + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}, \end{aligned}$$

and

$$\nu = \sqrt{\beta_1^2 - 1}.$$

From (22), it follows that $\theta < 1$, so the mapping w is a contraction mapping and consequently, it has a fixed point $w(u) = u \in H$ satisfying the problem (2). □

Remark 3.1. *We point out that the solution of the auxiliary problem* (24) *is equivalent to finding the minimum of the functional* $I[w]$, *where*

$$I[w] = \frac{1}{2}\langle g(w) - g(u), g(w) - g(u)\rangle - \rho\langle Lu - f, g(w) - g(u)\rangle,$$

which is a differentiable convex functional associated with the inequality (24), if the operator g is differentiable. This alternative formulation can be used to suggest iterative methods for solving general equations. This auxiliary functional can be used to find a kind of gap function, whose stationary points solves the problem (2), see [5].

It is clear that, if $w = u$, then w is a solution of (1). This observation shows that the auxiliary principle technique can be used to suggest the following iterative method for solving the general equations (2).

Algorithm 3.1. *For a given initial value u_0, compute the approximate solution x_{n+1} by the iterative scheme*

$$\langle Lu_n + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle = \langle f, g(v) - g(u_{n+1})\rangle, \forall v \in H.$$

From Algorithm 3.1, one can easily obtain the Picard type iterative method for solving the general equation (7) and appears to be a new one.

Algorithm 3.2. *For a given initial value u_0, compute the approximate solution x_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) - \rho(Lu_n - f), \quad n = 0, 1, 2, 3...$$

We again use the auxiliary principle technique to suggest an implicit method for solving the problem (2).
For a given $u \in H$, consider the problem of finding $w \in H$ such that,

$$\langle \rho Lw, g(v) - g(w)\rangle + \langle g(w) - g(u), g(v) - g(w)\rangle = \rho\langle f, g(v) - g(w)\rangle, \quad \forall v \in H, \tag{32}$$

which is called the auxiliary problem. We note that the auxiliary problems (24) and (32) are quite different.
Clearly $w = u \in H$ is a solution of (2). This observation allows us to suggest the following iterative method for solving the problem (2).

Algorithm 3.3. *For a given initial value u_0, compute the approximate solution x_{n+1} by the iterative scheme*

$$\langle \rho Lu_{n+1} + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle = \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H, \tag{33}$$

which is an implicit method.
From this implicit method, we can obtain the following iterative method for solving (5)

Algorithm 3.4. *For a given initial value u_0, compute the approximate solution x_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) - \rho Lu_{n+1} - f), \quad n = 0, 1, 2, 3...$$

This is a new implicit method for solving the equations (7).
To implement the implicit method (3.3), one uses the explicit method as a predictor and implicit method as a predictor. Consequently, we obtain the two-step method for solving the problem (2).

Algorithm 3.5. *For a given initial value u_0, compute the approximate solution x_{n+1} by the iterative schemes*

$$\begin{aligned} \langle \rho Lu_n + g(y_n) - g(u_n), g(v) - g(u_{n+1})\rangle &= \langle \rho f, g(v) - g(y_n)\rangle, \forall v \in H, \\ \langle \rho Ly_n + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle &= \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H, \end{aligned}$$

which is known as two-step iterative method for solving problem (2).
Based on the above arguments, we can suggest a new two-step(predictor-corrector) method for solving the equations (5).

Algorithm 3.6. *For a given initial value u_0, compute the approximate solution x_{n+1} by the iterative schemes*

$$\begin{aligned} g(y_n) &= g(u_n) - \rho Lu_n - f \\ g(u_{n+1}) &= g(u_n) - \rho Ly_n - f, \quad n = 0, 1, 2... \end{aligned}$$

For the convergence analysis of the iterative methods, we need the following concept.

Definition 3.1. *The operator L is said to be pseudo g-monotone, if*

$$\begin{aligned} &\langle Lu, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \\ &\Rightarrow \\ &\langle Lv, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \end{aligned}$$

We now consider the convergence analysis of Algorithm 3.3 and this is the main motivation of our next result.

Theorem 3.2. *Let $u \in H$ be a solution of problem (2) and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. If L is a g-monotone operator, then*

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(u_n) - g(u)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2. \tag{34}$$

Proof. Let $u \in H : g(u) \in H$ be a solution of (2). Then

$$\langle Lu, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H,$$

which implies that

$$\langle Lv, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{35}$$

since the operator L is a pseudo g-monotone.
Taking $v = u_{n+1}$ in (35) and $v = u$ in (33), we have

$$\langle Lu_{n+1}, g(u_{n+1}) - g(u)\rangle \geq \langle f, g(u_{n+1}) - g(u)\rangle, \quad \forall v \in H, \tag{36}$$

and

$$\langle \rho Lu_{n+1} + g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle = \langle \rho f, g(u) - g(u_{n+1})\rangle, \forall v \in H. \tag{37}$$

From (37), we have

$$\begin{aligned}\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \quad &\geq \quad \rho\langle (Lu_{n+1}, g(u_{n+1}) - g(u)\rangle \\ &\quad -\rho\langle f, g(u_{n+1}) - g(u)\rangle \geq 0,\end{aligned} \tag{38}$$

where we have used (36).
Using the relation $2\langle a, b\rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \quad \forall a, b \in H$, the Cauchy inequality and from (38), we have

$$\|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_n) - g(u_{n+1})\|^2,$$

which is the required (34). □

Theorem 3.3. *Let $\bar{u} \in H$ be a solution of (2) and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. Let L be a pseudo g-monotone operator and g^{-1} exist. If g is linear, then*

$$\lim_{n\to\infty} u_{n+1} = \bar{u}. \tag{39}$$

Proof. Let $\bar{u} \in H$ be a solution of (2). From ((34), it follows that the sequence $\{\|g(\bar{u}) - g(u_n)\|\}$ is noncreasing and consequently the sequence $\{g(u_n)\}$ is bounded. Also, from (34), we have

$$\sum_{n=0}^{\infty} \|g(u_{n+1}) - g(u_n)\|^2 \leq \|g(u_0) - g(\bar{u})\|^2,$$

which implies that

$$\lim_{n\to\infty} \|g(u_{n+1}) - g(u_n)\| = 0 \Longrightarrow \quad \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0, \tag{40}$$

since g is linear and g^{-1} exits.
Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequences $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converges to $\bar{u} \in H$. Replacing u_n by u_{n_j} in (33), taking the limit as $n_j \to \infty$ and using (40), we have

$$\langle L\hat{u}, g(v) - g(\hat{u})\rangle = \langle f, g(v) - g(\hat{u})\rangle, \quad \forall v \in H,$$

which shows that $\hat{u} \in H$ satisfies (1) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

From the above inequality, it follows that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$. □

We now use the auxiliary principle technique involving the Bregman function to suggest and analyze the proximal method for solving general equations (2). For the sake of completeness and to convey the main ideas of the Bregman distance functions, we recall the basic concepts and applications.
The Bregman distance function is defined as

$$B(u,w) = E(g(u)) - E(g(w)) - \langle E'(g(w)), g(u) - g(w)\rangle \geq \nu\|g(u) - g(w)\|^2, \tag{41}$$

using the strongly general convexity with modulus ν.
The function $B(u,w)$ is called the general Bregman distance function associated with general convex functions.
For $g = I$, we obtain the original Bregman distance function

$$B(u,w) = E(u) - E(w)) - \langle E'(w), u - w)\rangle \geq \nu\|u - w\|^2.$$

For the applications of Bregman distance functions, see [15, 20, 32] and the references
For a given $u \in H$, find a solution $w \in H$ satisfying

$$\langle \rho Lu + E'(g(w)) - E'(g(u)), g(v) - g(w)\rangle = \langle \rho f, g(v) - g(w)\rangle, \forall v \in H, \tag{42}$$

where $E'(u)$ is the differential of a strongly general convex function E.
Note that, if $w = u$, then w is a solution of (2). Thus, we can suggest the following iterative method for solving (2).

Algorithm 3.7. *For a given $u_0 \in H$, calculate the approximate solution by the iterative scheme*

$$\langle \rho Lu_n + E'(g(u_{n+1}) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle = \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H, \tag{43}$$

which is known as the proximal point method.
We again use the auxiliary principle technique involving the Bregman function to suggest and analyze the proximal implicit method for solving equations (2).
For a given $u \in H$, find $w \in H$ satisfying the auxiliary equation

$$\langle \rho Lw + E'(g(w)) - E'(g(u)), g(v) - g(w)\rangle = \langle \rho f, g(v) - g(w)\rangle, \forall v \in H, \tag{44}$$

where $E'(u)$ is the differential of a strongly general convex function E.
It is clear that, if $w = u$, then w is a solution of (2). Thus, we can suggest the following iterative method for solving (2).

Algorithm 3.8. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho Lu_{n+1} \quad + \quad & E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ & = \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H,\end{aligned}$$

which is known as the proximal implicit point method.

Remark 3.2. *One can consider the convergence analysis of Algorithm 3.7 and Algorithm 3.8 using the technique of Noor* [15] *and Noor et al.* [20]. *We would like to emphasize that for appropriate choice of the operators L, g one can suggest and analyze several new iterative methods for solving general equations and related problems. The implementation and comparison with other techniques need further efforts.*

Conclusion

In this paper, we have considered a new system of general equations, which includes Lax-Milgram Lemma and the Riesz-Fréchet representation theorems as special cases. It is shown that the third order boundary value problems can be studied in the framework of general equations. We have used the auxiliary principle technique to study the existence of the unique solution of the system of the general equations. Some new iterative methods are suggested for solving the equations using the auxiliary principle technique. The convergence analysis of these iterative methods is investigated using the pseudo monotonicity, which is weaker condition than monotonicity. This is a new approach for solving the general equations. We would like to emphasize that these ideas and techniques may motivate a number of novel applications and extensions of the general equations and their variant forms in these areas.

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REFERENCES

[1] L. Bers, F. John and M. Schechter, Partial differential equations, Academic Press, New York, 1966.

[2] W. Fechner, Functional inequalities motivated by the Lax-Milgram Lemma, J. Math. Anal. Appl. 402(2013), 411-414.

[3] V. M. Filippov, Variational Principles for Nonpotential Operators, Amer. Math. soc. Providence, Rode Island, USA, 1989.

[4] M. Fréchet; Sur les ensembles de fonctions et les operations lineaires, C. R. Acad. Sci., Paris, (1907), 1414-1416.

[5] M. Fukushima, Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, Math. Program. 53 (1992), 99-110.

[6] M.R. Galen, A version of Lax-Milgram theorem for locally convex spaces, J. Convex Anal. 16 (2009), 993-1002

[7] R. Glowinski, J. L. Lions and R. Tremolieres, Numerical Analysis of Variational Inequalities. Amsterdam: North-Holland; 1981.

[8] H. Kozono, Lax-Milgram theorem in Banach spaces and its generalization to the elliptic system of boundary value problems, Manuscripta Math. 141 (2013), 637-662.

[9] P. D. Lax and A. N. Milgram, Parabolic equations, Annals, Math. Study No. 33, Princeton, N.J., (1954), 167-190.

[10] J. L. Lions and G. Stampacchia, Variational inequalities, Commun. Pure Appl. Math. 20(1967), 493-512.

[11] M. A. Noor, Riesz-Fréchet Theorem and Monotonicity, MSc Thesis, Quenn's University, Kingston, Ontario, Canada, 1971.

[12] M. A. Noor, On Variational Inequalities, PhD Thesis, Brunel University, London, U.K. (1975).

[13] M. A. Noor, Qausi variational inequalities, Appl. Math. Letters, 1(4)(1988), 367-370.

[14] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Ana. Appl. 251(2000), 217-229.

[15] M. A. Noor, Some developments in general variational inequalities, 251(2004), 199-277.

[16] M. A. Noor, Differentiabel nonconvex functions and general variational inequalities, Appl. Math. Comput. 199(2008), 623-630.

[17] K. I. Noor and M. A. Noor, A generalization of the Lax-Milgram lemma, Canad. Math. Bull. 23 (2)(1980), 179-184.

[18] M. A. Noor and K. I. Noor, From representaion theorems to variational inequalities, in: Computational Mathematics and Variational Analysis (Edits: N. J. Daras, T. M. Rassias), Springer Optimization and Its Applications, 159(2020), 261-277. https:doi.org/10.1007/978-3-030-44625-3-15261

[19] M. A. Noor, K. I. Noor and B. B. Mohsen, Some new classes of general quasi variational inequalities, AIMS Math. 6(6)(2021),6406-6421.

[20] M. A. Noor, K. I. Noor and M. Th. Rassias, New trends in general avriational inequalities, Acat. Appl. Math. 170(1)(2020), 981-1046.

[21] M. A. Noor, K. I. Noor and Th. M. Rassias, Some aspects of variational inequalities, J. Comput. Appl. Math. 47(1933), 285-312.

[22] N. Nyamoradi and M.R. Hamidi, An extension of the Lax-Milgram theorem and its application to fractional differential equations, Electron. J. Differential Equations 2015 (2015), Article ID 95.

[23] Th. M. Rassias, M. A. Noor and K. I. Noor, Auxiliary principle technique for the general Lax-Milgram Lemma, J. Nonl. Funt. Analysis, 2018 (2018), Article ID 34,1-8

[24] F. Riesz; Sur une espece de geometrie analytique des systemes de fonctions sommables, C. R. Acad. Sci., Paris, (1907), 1409-1411.

[25] F. Riesz, Zur theorie des Hilbertschen raumes; Acta Acient. 7, (1934-1935), 34-38.

[26] D.R. Sahu, A. Pitea and M. Verma, A new iteration technique for nonlinear ope-rators as concerns convex programming and feasibility problems, Numer. Algorithms, 83(2)(2020), 421-449.

[27] B.S. Thakur, D. Thakur and M. Postolache, A new iteration scheme for approximating fixed points of nonexpansive mappings. Filomat, 30(10)(2016), 2711-2720.

[28] B.S. Thakur and M. Postolache, Existence and approximation of solutions for generalized extended nonlinear variational inequalities. J. Inequal. Appl. 2013(2013): 590.

[29] E. Tonti, E. Tonti, Variational formulation for every nonlinear problem, Intern. J. Eng. Sci. 22(11-12)(1984), 1343- 1371.

[30] Y. Yao, H. Li and M. Postolache, Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions. Optimization DOI: 10.1080/02331934.2020.1857757.

[31] L.J. Zhu, Y. Yao and M. Postolache, Projection methods with line search technique for pseudomonotone equilibrium problems and fixed point problems. U. Politeh. Buch. Ser. A 83(1)(2021), 3-14.

[32] D. L. Zhu and P. Marcotte, Cocoercivity and its role in the convergence of iterative schemes for solving variational inequalities, SIAM J. Optim. 6(1996), 714-726