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ACTIVE FLUTTER SUPPRESSION OF A 3D-WING: PRELIMINARY DESIGN AND ASSESSMENT

PART II - ASSESSMENT OF SYSTEM PERFORMANCES

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This paper refers to active flutter suppression; the configuration of interest is a 3D-wing dynamically controlled through a conventional trailing-edge flap.

In Part I of the paper the problem has been stated and the construction of the mathematical model as well as the design of the controller have been addressed.

The text that follows attempts an evaluation of the system performances from a structural perspective, with special accent on the sub-critic regime.

Keywords: flutter suppression, gust response, wing loads

1. The performance of the flutter suppressor in the sub-critic domain

▪ **For a thorough evaluation** of the flutter suppressor it is interesting to evaluate its performances in the sub-critic domain - that is for a test speed *below* the system's nominal flutter speed, $\bar{U}_{test} < \bar{U}_F$ - under some "natural" loading conditions for the airplane, such as the one emerging from a *gust encounter* (in this formulation one easily recognizes an attempt to treat the general problem of designing an active *gust load alleviation* system).

In this scope, all relationships necessary for the evaluation of the structural dynamic response under the mentioned condition must be first established.

Note. In the derivations that follow reference will be made to some equations appearing in Part I; this text will be further designated as reference [1].

▪ **Let's consider again** the previously analyzed straight cantilever wing under the influence of a *deterministic* gust $w_G(t)$ as in Fig. 1.

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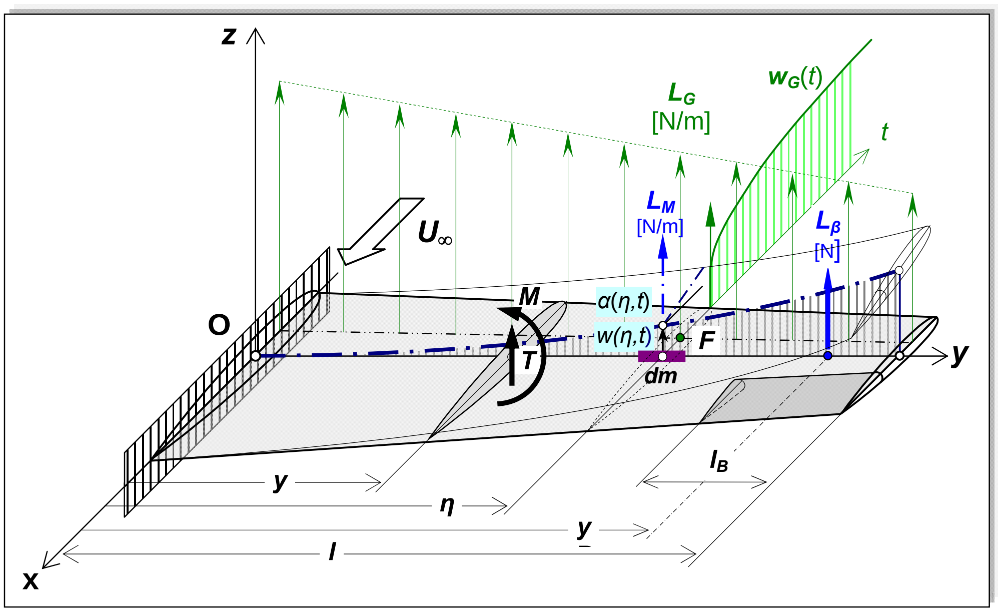

Fig. 1. Straight wing encountering a deterministic gust - Sign conventions

▪ **The gust must first** be introduced into the equations. We assume the gust as *uniform* over the wing span; with the *stationary* aero-model and the *strip theory*, the running gust force - acting in the aerodynamic center F - is given by

$$\uparrow L_G\left(y,t\right)[\mathrm{N/m}]=\frac{\rho}{2}U^2 c(y)\cdot C_L^{\alpha}\,\frac{w_G(t)}{U} \tag{1}$$

The corresponding *generalized* forces to be entered into the *Lagrange* equations - see (3) in [1] - are next calculated from the running lift and moment associated with the gust, through definitions analogous to (11) in [1]; using the procedures given there, one gets the following nondimensionalized expressions in the *Laplace* variable $\overline{s}$ for the *uniform wing* (compare with (13) in [1]), in which $H_{\dots}$ are again some influence factors

$$\mathbf{Q}^G(\overline{U})=\begin{Bmatrix} Qw_i \\ Q\alpha_j \end{Bmatrix}^{ST-G}=\begin{matrix}[\pi\rho b^3\omega_R^2\cdot l]\times \\ [\pi\rho b^4\omega_R^2\cdot l]\times\end{matrix}\begin{bmatrix} C_L^{\alpha}/\pi\cdot\overline{U}^2\cdot Hw_i \\ C_L^{\alpha}/\pi\cdot(\frac{1}{2}+a)\cdot\overline{U}^2\cdot H\alpha_j \end{bmatrix}\cdot\frac{w_G(\overline{s})}{U} \tag{2}$$

$$Hw_i=\int_0^1 Fw_i(\eta)\cdot d\eta\;;\;H\alpha_j=\int_0^1 F\alpha_j(\eta)\cdot d\eta$$

With these, the state-space system - see (21) to (24) in [1] - augmented with the gust entries reads

$$\overline{s}\cdot\mathbf{X}=\mathbf{A}(\overline{U})\cdot\mathbf{X}+\mathbf{B}(\overline{U})\cdot\beta(\overline{s})+\mathbf{G}(\overline{U})\cdot\frac{w_G(\overline{s})}{U}\;;\;\mathbf{G}(\overline{U})=\begin{bmatrix}\mathbf{0}\dots \\ \mathbf{M}^{-1}\mathbf{Q}^G(\overline{U})\end{bmatrix} \tag{3}$$

▪ **With the previously determined** controller, this last system can easily be "specialized" for numerical simulation; thus, following (34) to (38) in [1], we write again the regulated system in "*physical*" time *t*, that is

$$\dot{\mathbf{x}} = \left[\ \tilde{\mathbf{A}} - \tilde{\mathbf{B}} \cdot \tilde{\mathbf{K}}_{\text{opt}}\ \right] \cdot \mathbf{x} + \tilde{\mathbf{G}} \cdot \frac{w_G(t)}{U}\ ;\ \tilde{\mathbf{G}} = \omega_R \cdot \left[\mathbf{T}_{\boldsymbol{\omega}_\mathbf{R}}{}^{-1} \cdot \mathbf{G}\left(\bar{U}\right) \right] \tag{4}$$

▪ **The step-gust-response**

As a first application, let's build the response of the system to a standard *unbound step gust* under the following conditions:

$$\bar{U}_{test} = 4\ ;\ \frac{w_G}{U} = 0.1 \times 1(t) \tag{5}$$

Notes. The value of the test speed has been intentionally selected in the initially *stable* domain of the nominal system. As for the gust intensity, one can easily verify that, for the case under consideration, this would correspond to a magnitude in the order of

$$\bar{U}_{test} = \frac{U_{test}}{\omega_R} = 4 \rightarrow U_{test} = 4 \cdot \omega_R = 4 \cdot 33[\text{m/s}] = 132[\text{m/s}] \Rightarrow w_G = 13.2 \times 1(t)[\text{m/s}]$$

which is fully *realistic* in view of the current regulations in aircraft design…

The response of the two system variables is depicted in Fig. 2 for the *nominal* and the *controlled* system respectively.

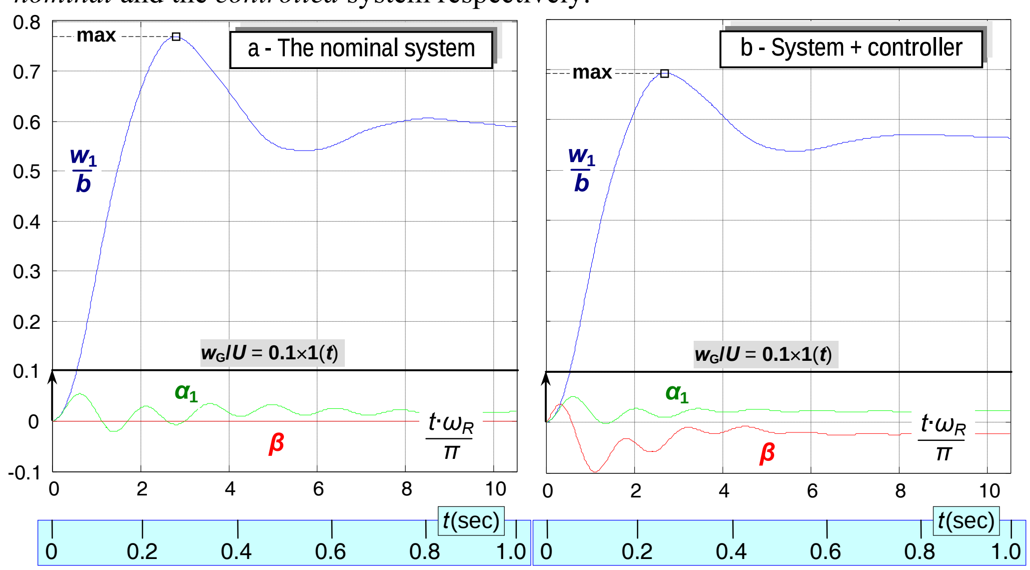

Fig. 2. Wing ("1×1" model) in sub-critical domain: $\bar{U}_{test} = 4$ - Step-gust response

The results are - apparently! - concluding: the control leads to a *reduction* of the maximum response values - in terms of the amplitude*s* of the generalized displacements - in the order of (8-10) %; the actual values are listed below

	The nominal system	System + controller	Difference...
$(w_1/b)_{max}$	0.7684	0.6925	(≈) – 10 %
$(\alpha_1)_{max}$	0.0546	0.0503	(≈) – 8 %

***We conclude*:** The indicated technique can be retained as a *possible* way for the design of a more efficient structure.

2. The direct calculation of wing loading and stresses

A more rational approach in assessing the efficiency of the controller from a structural point of view is to directly take into account the wing *loading and stresses* as they develop *in time*. For this task, explicit mathematical relationships are needed; the section that follows gives in short the appropriate methodology (we refer again to the wing in Fig. 1).

- **The loads acting** on the wing in a dynamic process are

– The "active" running lift induced by the *gust* (strip theory, etc…)

$$\uparrow L_G\left(y,t\right)[\text{N/m}] = \frac{\rho}{2}U^2 c(y)\cdot C_L^{\alpha}\,\frac{w_G(t)}{U} \tag{6}$$

– The "active" (concentrated!) lift associated with the *flap deflection*

$$\uparrow L_\beta\,[\text{N}] = \frac{\rho}{2}U^2(c_B l_B)\cdot C_L^{\beta}\beta(t) \tag{7}$$

– The "reactive" running inertia force associated with the *vibration*; with $m(y)$[kg/m] and $S_\alpha(y)$[kg×m/m] the running *mass* and *static moment*, one writes successively

$$\uparrow w(x,y,t) = w(y,t) - x\cdot\alpha(y,t) \Rightarrow \downarrow F_{\ddot{w}+\ddot{\alpha}}\left(y,t\right)[\text{N/m}] = m\cdot\ddot{w}(y,t) - S_\alpha\cdot\ddot{\alpha}(y,t) \tag{8}$$

– The aerodynamic forces induced by the *motion* of the wing itself. These must be computed in accord with the aero-model adopted, in particular the mentioned *quasistationary Fung* one (Part I, Chapter 2, eqs. (15)), that is

$$\uparrow L_M\left[\text{N/m}\right] = L^{QS-F} = \frac{\rho}{2}U^2 c\cdot C_L = \ldots = \rho b U^2 C_L^{\alpha}\cdot\left[-\frac{\dot{w}}{U} + \alpha + \left(\tfrac{1}{2} - a\right)\frac{b}{U}\dot{\alpha}\right] \tag{9}$$

- **The resulting shear and bending moment** (positive signs as in Fig. 1) can now be computed following the general definitions from the beam theory:

$$\begin{cases} T\left(y,t\right)[\text{N}] = \int_y^l \left[L_G + L_\beta + L_M - F_{\ddot{w}+\ddot{\alpha}}\right]_{"\eta"}\cdot d\eta = \ldots \\ M\left(y,t\right)[\text{N}\cdot\text{m}] = \int_y^l \left[L_G + L_\beta + L_M - F_{\ddot{w}+\ddot{\alpha}}\right]_{"\eta"}\cdot(y-\eta)\cdot d\eta = \ldots \end{cases} \tag{10}$$

In the expressions above, (w,α) are the *local* displacement and rotation as functions of time. With the method of *assumed modes*, these are given by the series (1) in Part I; for clarity we repeat them here together with the "1×1" case

$$\begin{cases} \text{- bending: } w(y,t) = \sum_{i=1}^{NW} Fw_i(y)\cdot w_i(t) \\ \text{- torsion: } \alpha(y,t) = \sum_{j=1}^{NA} F\alpha_j(y)\cdot \alpha_j(t) \end{cases} \rightarrow \begin{cases} w(y,t) = Fw_1(y)\cdot w_1(t) \\ \alpha(y,t) = F\alpha_1(y)\cdot \alpha_1(t) \end{cases} \quad (11)$$

▪ **Compact formulae can be established** through very simple operations. For the sake of clarity we illustrate the process for the shear force induced by the gust. As done previously, we concentrate on the "uniform" wing: we write then

$$\uparrow T^G(y,t)[\mathrm{N}] = \int_y^l \frac{\rho}{2}U^2 c\cdot C_L^\alpha \frac{w_G(t)}{U}\cdot d\eta = \frac{\rho}{2}U^2(2b)\cdot C_L^\alpha \frac{w_G(t)}{U}\int_y^l d\eta = \cdots$$

$$\cdots = \underbrace{\left[\pi\rho b^3\omega_R^2\cdot l\right]}_{"f"}\left(\frac{U}{b\omega_R}\right)^2 \frac{C_L^\alpha}{\pi}\underbrace{\left[1-\frac{y}{l}\right]}_{"T_G"}\cdot\frac{w_G(t)}{U}$$

and, with the nondimensional *Laplace* transform and known notations, compact

$$\uparrow T^G(\overline{y},\overline{s})[\mathrm{N}] = \underbrace{\left[\pi\rho b^3\omega_R^2\cdot l\right]}_{"f"}\overline{U}^2\frac{C_L^\alpha}{\pi}\underbrace{\left[1-\frac{y}{l}\right]}_{"T_G"}\cdot\frac{w_G(\overline{s})}{U} \;;\; T_G(\overline{y}) = 1-\overline{y} \quad (12)$$

in which a nondimensional function $T_G(\overline{y})$ appears...

All other formulae are being set-up in the same way. The resulting cumulative nondimensionalized *shear* and *bending moment* - via *Fung* quasistationary aerodynamics - are given in the following table

Table 3

Cumulative wing shear force and bending moment

$$\frac{\uparrow T(\overline{y},\overline{s})[\mathrm{N}]}{\pi\rho b^3\omega_R^2\cdot l} = -\mu\sum_{i=1}^{NW}\left[T_{w_i}(\overline{y})\cdot\frac{\overline{s}^2 w_i(\overline{s})}{b}\right] + \mu x_\alpha\sum_{j=1}^{NA}\left[T_{\alpha_j}(\overline{y})\cdot\overline{s}^2\alpha_j(\overline{s})\right]$$

$$+\overline{U}^2\frac{C_L^\alpha}{\pi}\sum_{j=1}^{NA}\left[T_{\alpha_j}(\overline{y})\cdot\alpha_j(\overline{s})\right] - \overline{U}\frac{C_L^\alpha}{\pi}\sum_{i=1}^{NW}\left[T_{w_i}(\overline{y})\cdot\frac{\overline{s}\,w_i(\overline{s})}{b}\right] + \left(\tfrac{1}{2}-a\right)\overline{U}\frac{C_L^\alpha}{\pi}\sum_{j=1}^{NA}\left[T_{\alpha_j}(\overline{y}\right.$$

$$+\overline{U}^2\frac{l_B}{l}\frac{C_L^\beta}{\pi}T_\beta(\overline{y})\cdot\beta(\overline{s})\Bigg|_{\overline{y}\le\overline{y}_B!...} + \overline{U}^2\frac{C_L^\alpha}{\pi}T_G(\overline{y})\cdot\frac{w_G(\overline{s})}{U}$$

$$\frac{M(\bar{y},\bar{s})[\mathrm{N\cdot m}]}{\pi\rho b^3\omega_R^2\cdot l\cdot l} = -\mu\sum_{i=1}^{NW}\left[M_{w_i}(\bar{y})\cdot\frac{\bar{s}^2 w_i(\bar{s})}{b}\right] + \mu x_\alpha\sum_{j=1}^{NA}\Big[M_{\alpha_j}(\bar{y})\cdot\bar{s}^2\alpha_j(\bar{s}$$

$$+\bar{U}^2\frac{C_L^\alpha}{\pi}\sum_{j=1}^{NA}\left[M_{\alpha_j}(\bar{y})\cdot\alpha_j(\bar{s})\right] - \bar{U}\frac{C_L^\alpha}{\pi}\sum_{i=1}^{NW}\left[M_{w_i}(\bar{y})\cdot\frac{\bar{s}w_i(\bar{s})}{b}\right] + \left(\tfrac{1}{2}-a\right)\bar{U}\frac{C_L^\alpha}{\pi}\sum_{j=1}^{NA}\Big[M_\alpha$$

$$+\bar{U}^2\frac{l_B}{l}\frac{C_L^\beta}{\pi}M_\beta(\bar{y})\cdot\beta(\bar{s})\Big|_{\bar{y}\le\bar{y}_B!\ldots} + \bar{U}^2\frac{C_L^\alpha}{\pi}M_G(\bar{y})\cdot\frac{w_G(\bar{s})}{U}$$

- **Notes**

1⁰. In the expressions above, a set of nondimensional *influence functions* appear with the following definitions (we recall that $Fw_i(y)$ and $F\alpha_j(y)$ come from the assumed mode representation in (11)):

$T_{w_i}(\bar{y}) = \int_{\bar{y}}^{1} Fw_i(\bar{\eta})\,d\bar{\eta}$	$T_{\alpha_j}(\bar{y}) = \int_{\bar{y}}^{1} F\alpha_j(\bar{\eta})\,d\bar{\eta}$	$T_\beta(\bar{y}) = 1\big\|_{\bar{y}\le\bar{y}_B!\ldots}$	$T_G(\bar{y}) = 1-\bar{y}$
$M_{w_i}(\bar{y}) = \int_{\bar{y}}^{1} Fw_i(\bar{\eta})(\bar{\eta}-\bar{y})\,d\bar{\eta}$	$M_{\alpha_j}(\bar{y}) = \int_{\bar{y}}^{1} F\alpha_j(\bar{\eta})(\bar{\eta}-\bar{y})\,d\bar{\eta}$	$M_\beta(\bar{y}) = \bar{y}_B-\bar{y}$ $(\bar{y}\le\bar{y}_B!\ldots)$	$M_G \overset{\mathrm{def!}}{=} \frac{1}{2}(1-\bar{y})^2$

2⁰. For illustration, in Fig. 3 the two related functions T_{w_i}, M_{w_i} have been represented; as shown, these are determined by the free vibration modes in bending $Fw_i(y)$ - see Annex - which are "oscillatory" functions. The graphs show that only the *first* (and *second*, eventually…) bending modes contribute significantly to the wing stresses (similar conclusions are valid for T_{α_j} and M_{α_j}). This fact confirms the utility of the *reduced* "1×1" model for preliminary studies.

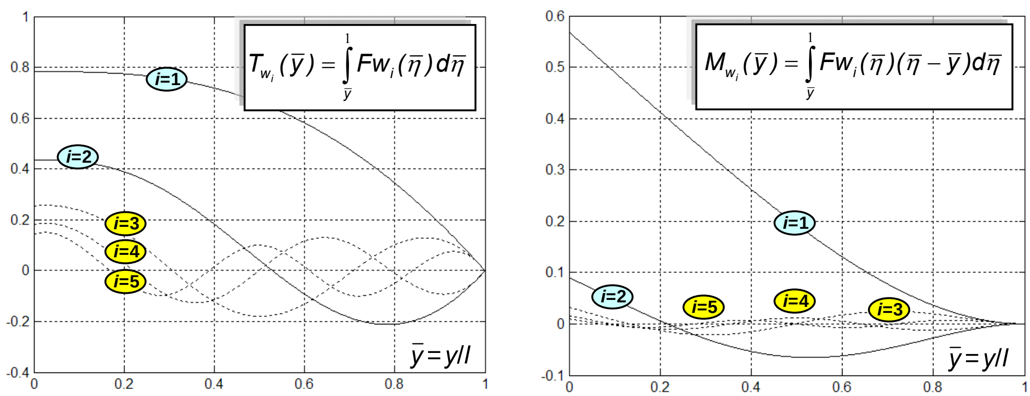

Fig. 3. Wing shear and bending - The influence functions T_{w_i} and M_{w_i}

▪ **Numerical simulation (System "1×1")**

As a continuation to the previous chapter, we will evaluate the effect of the flutter controller on the wing loading and stresses in *sub-critic* regime under the same loading conditions, namely the *step-gust*.

The starting point for the dynamic analysis is system (4) above, that is

$$\beta(t) = -\tilde{\mathbf{K}}_{\text{opt}} \cdot \mathbf{x}(t) \;\Rightarrow\; \dot{\mathbf{x}}(t) = \left[\, \tilde{\mathbf{A}} - \tilde{\mathbf{B}} \cdot \tilde{\mathbf{K}}_{\text{opt}} \,\right] \cdot \mathbf{x}(t) + \tilde{\mathbf{G}} \cdot \frac{w_G(t)}{U} \qquad (13)$$

In this one, for the basic "1×1" case, the state vector reads simply

$$\mathbf{x}(t) = \left\lfloor \frac{w_1(t)}{b} \quad \alpha_1(t) \quad \frac{\dot{w}_1(t)}{b} \quad \dot{\alpha}_1(t) \right\rfloor^T \qquad (14)$$

The corresponding formulae for shear and bending are set-up from the general expressions in Table 3, particularized for $NW = NA = 1$; further, for direct use in the dynamic process, these must be expressed in *physical* time -see Table 4- in which the *contributions* of the various variables are explicitly identified.

Notes

1^0. As expected, in the case of the *controlled* system, the wing loading "feels" the influence of the controller *twice*: through the wing dynamics ($w(t), \alpha(t)$) itself and, directly, through the flap movement $\beta(t)$, both appearing in fact as response to the external gust perturbation.

2^0. The complete functions $T(\bar{y},t)$ and $M(\bar{y},t)$ are useful for the detailed wing design. Out of these, the value of the *root shear*, that is $T(0,t)$, is especially valuable since, in case of a *free airplane*, it determines the accelerations experienced by the fuselage itself (the technical application in this sense is the so-called "ride comfort control"…).

A) As a first numerical simulation, the dynamic response of the controlled system in the *sub-critic* domain has been established together with the corresponding wing root shear $T(0,t)$ (Figure 4).

– The dynamic response is presented in Fig. 4-a, where, for clarity, all system variables have been individually displayed for comparison (note that the *relatively high* values of the time derivatives in comparison with the corresponding functions explain through the multiplicative effect of ω_R !...).

– In an analogous manner, Figure 4-b shows the contributions to the shear force of various functions of time that build up its expression. In context, it must be observed that, since the accelerations *do not* appear in the state vector (14), they must be constructed numerically from these last…

B) In a second **attempt**, the direct effect of the controller on the wing loading has been investigated. Figure 5 presents the root shear force $T(0,t)$ as it develops in time - as response to the same step gust - for the *nominal* and the *controlled* system, both in *sub-critic* regime.

Table 4

Wing shear force and bending moment for the "1×1" system

$\frac{\uparrow T(\bar{y},t)[\mathrm{N}]}{\pi\rho b^3\omega_R^2\cdot l} = \underbrace{-\mu\frac{1}{\omega_R^2}T_{w_1}(\bar{y})\cdot\frac{\ddot{w}_1(t)}{b}}_{T^{\ddot{w}}} \underbrace{-\bar{U}\frac{C_L^\alpha}{\pi}\frac{1}{\omega_R}T_{w_1}(\bar{y})\cdot\frac{\dot{w}_1(t)}{b}}_{T^{\dot{w}}} + \underbrace{(0)\cdot\frac{w_1(t)}{b}}_{T^w} + \underbrace{\mu x_\alpha\frac{1}{\omega_R^2}T_{\alpha_1}(\bar{y})\cdot\ddot{\alpha}_1(t)}_{T^{\ddot{\alpha}}} + \underbrace{\left(\tfrac{1}{2}-a\right)\bar{U}\frac{C_L^\alpha}{\pi}\frac{1}{\omega_R}T_{\alpha_1}(\bar{y})\cdot\dot{\alpha}_1(t)}_{T^{\dot{\alpha}}} + \underbrace{\bar{U}^2\frac{C_L^\alpha}{\pi}T_{\alpha_1}(\bar{y})\cdot\alpha_1(t)}_{T^{\alpha}} + \underbrace{\bar{U}^2\frac{l_B}{l}\frac{C_L^\beta}{\pi}T_\beta(\bar{y})\cdot\beta(t)\Big\vert_{\bar{y}\le\bar{y}_B!\ldots}}_{T^\beta} + \underbrace{\bar{U}^2\frac{C_L^\alpha}{\pi}T_G(\bar{y})\cdot\frac{w_G(t)}{U}}_{T^G}$
$\frac{\uparrow M(\bar{y},t)[\mathrm{N.m}]}{\pi\rho b^3\omega_R^2\cdot l} = \underbrace{-\mu\frac{1}{\omega_R^2}M_{w_1}(\bar{y})\cdot\frac{\ddot{w}_1(t)}{b}}_{M^{\ddot{w}}} \underbrace{-\bar{U}\frac{C_L^\alpha}{\pi}\frac{1}{\omega_R}M_{w_1}(\bar{y})\cdot\frac{\dot{w}_1(t)}{b}}_{M^{\dot{w}}} + \underbrace{(0)\cdot\frac{w_1(t)}{b}}_{M^w} + \underbrace{\mu x_\alpha\frac{1}{\omega_R^2}M_{\alpha_1}(\bar{y})\cdot\ddot{\alpha}_1(t)}_{M^{\ddot{\alpha}}} + \underbrace{\left(\tfrac{1}{2}-a\right)\bar{U}\frac{C_L^\alpha}{\pi}\frac{1}{\omega_R}M_{\alpha_1}(\bar{y})\cdot\dot{\alpha}_1(t)}_{M^{\dot{\alpha}}} + \underbrace{\bar{U}^2\frac{C_L^\alpha}{\pi}M_{\alpha_1}(\bar{y})\cdot\alpha_1(t)}_{M^{\alpha}} + \underbrace{\bar{U}^2\frac{l_B}{l}\frac{C_L^\beta}{\pi}M_\beta(\bar{y})\cdot\beta(t)\Big\vert_{\bar{y}\le\bar{y}_B!\ldots}}_{M^\beta} + \underbrace{\bar{U}^2\frac{C_L^\alpha}{\pi}M_G(\bar{y})\cdot\frac{w_G(t)}{U}}_{M^G}$

The graphs show that a conventional flutter suppressor leads to a *moderate* reduction - in the order of less than 5 % - in the maximum value of the shear load (similar results are obtained for the root bending moment $M(0,t)$ - not exposed).

Whether these effects *do* or *do not* have any practical significance remains to be judged; the present investigation is anyway valuable, at least because it introduces the subject and even exhausts it in the given circumstances.

Note. This section introduces a very important subject in the field of active structures, namely the *gust-load-alleviation-problem.* In a realistic design, this problem has to be approached *directly* and on the basis of a more sophisticated model that starts with the dynamics of the *whole* aircraft as a free body in space.

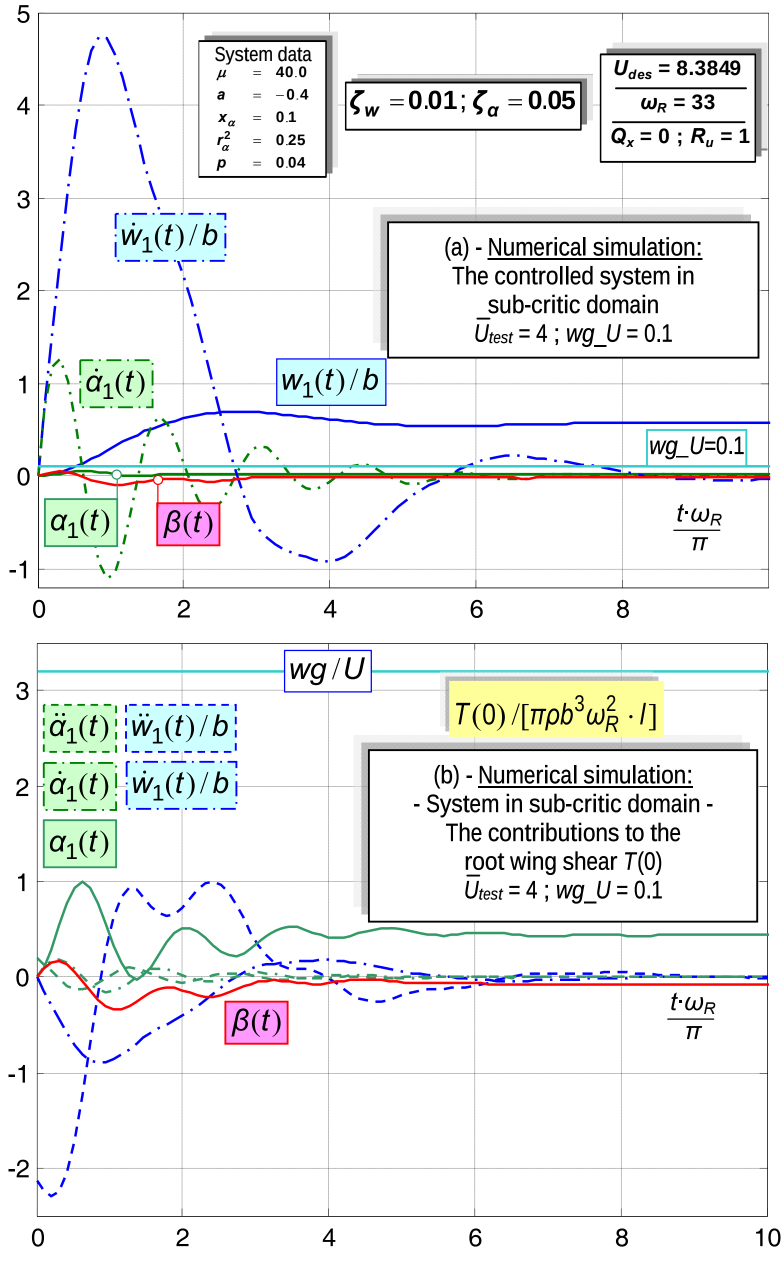

Fig. 4. System "1×1" - Step gust response in sub-critic regime

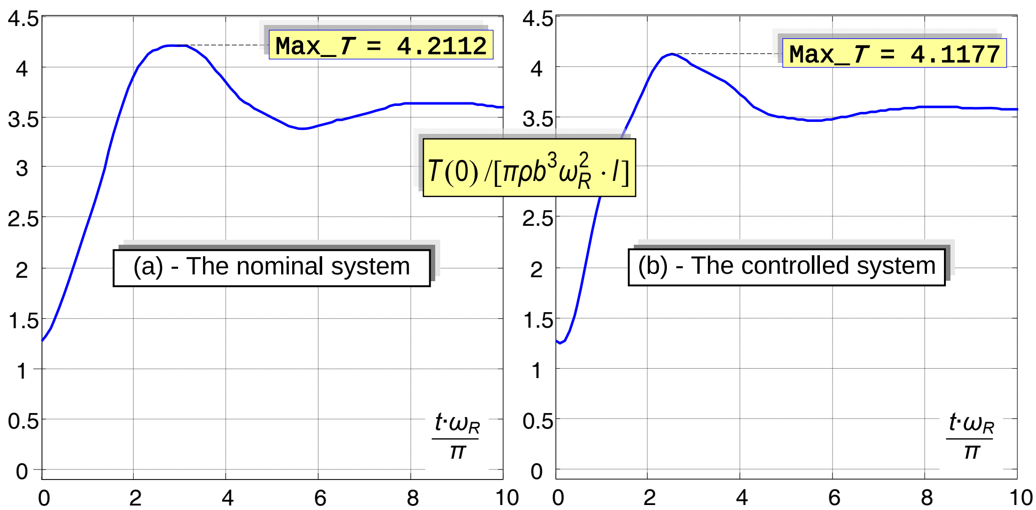

Fig. 5. System "1×1" - Step gust response… - The root shear force $T(0,t)$

3. Conclusions and perspectives

The present article approaches a very actual and important aspect in the field of airplane design, namely the concept of *active structures*; out of this, the article concentrates on the active *flutter suppression* problem. Two are the major objectives of the proposed text:

▪ **To establish and numerically** demonstrate a systematic methodology for the design of a flutter suppressor in case of a 3D-wing. The usefulness of a simplified model for preliminary design has been stated. For the design of the controller, the standard LQR algorithm has been used. As already said, the efficiency of this procedure has been validated experimentally by an extensive research program [2] which otherwise, as mentioned in Part I, established a national priority in the field of active structures.

▪ **To assess the efficiency** of the flutter suppressor in sub-critic domain from a pure structural point of view. In this sense, the dynamic response of the controlled system under the action of a gust has been investigated. Explicit formulae for the wing internal loading and have been derived.

The numerical simulation performed on a test case demonstrates at least a moderate efficiency of the flutter suppressor as a gust-load-alleviator.

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▪ **Several continuations** can be envisaged starting from this study from a *pure theoretical* point of view (for instance the use of a more refined aerodynamic model - including numerical methods [3]) or from a *practical perspective* (for example the extension of the problem to the complete aircraft free in space).

4. Annex: The representation functions $Fw_i(y)$, $F\alpha_j(y)$

The functions used in this text with the method of *assumed modes* are the *free uncoupled vibration modes* of a uniform cantilever beam. These can be found in any textbook on aeroelasticity and will be reproduced here for clarity (Fig. 6).

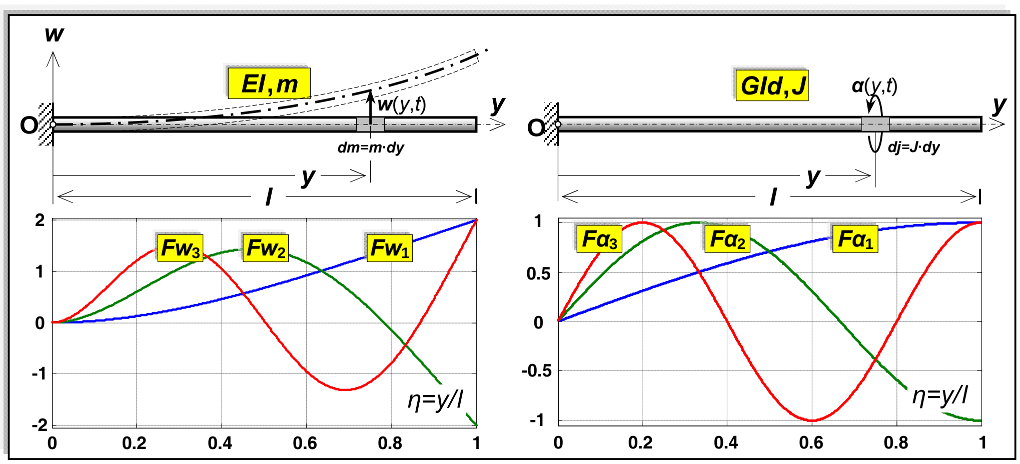

Fig. 6. *Bending* & *Torsion* vibration modes of the uniform beam

- **Bending.** Differential equation: $(EI \cdot w'')'' + m\ddot{w} = 0$, etc…

– The modes are [4] (the first *three* displayed):

$\omega_1 = (0.597)^2 \dfrac{\pi^2}{l^2}\sqrt{\dfrac{EI}{m}}$	$\omega_2 = (1.49)^2 \dfrac{\pi^2}{l^2}\sqrt{\dfrac{EI}{m}}$	…	$\omega_i = \left(i - \frac{1}{2}\right)^2 \dfrac{\pi^2}{l^2}\sqrt{\dfrac{EI}{m}}$ (for *i* sufficiently high)

$(\eta = y/l)$

$$Fw_i = \left[\left(\frac{\sin \pi N_i - \sinh \pi N_i}{\cosh \pi N_i + \cos \pi N_i}\right)(\sinh \pi N_i\eta - \sin \pi N_i\eta) + \cosh \pi N_i\eta - \cos \pi N_i\eta\right]$$

N_1 = 0.596864…	N_2 = 1.494175…	N_3 = 2.500246…	N_4 = 3.499989…	N_5 = 4.500000…	…

- **Torsion.** Differential equation: $(GId \cdot \alpha')' - J\ddot{\alpha} = 0$, etc…

– The modes are (the first *three* displayed):

$$\omega_j = (2j-1)\frac{\pi}{2l}\sqrt{\frac{GId}{J}} \; ; \; F\alpha_j = \sin(2j-1)\frac{\pi}{2}\frac{y}{l} \quad (n = 1, 2, 3, \cdots)$$

R E F E R E N C E S

- See also Part I -

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