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EXISTENCE AND ITERATION OF MONOTONE POSITIVE POLUTIONS FOR MULTI-POINT BVPS OF DIFFERENTIAL EQUATIONS

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By applying monotone iterative methods, we obtain not only the existence of monotone positive solutions for a kind of multi-point boundary value problems, but also establish iterative schemes for approximating the solutions. A boundary value problem that our results can readily apply, whereas the known results in the current literature do not cover, is presented at the end of the paper.

Keywords: Multi-point boundary-value problem; p-Laplacian; half-line; positive solutions; existence; uniqueness.

MSC2000: primary 49J40, 49K20; secondary 58E17, 65K10, 53C65.

1. Introduction

Recently an increasing interest has been observed in investigating the existence of positive solutions of boundary-value problems. This interest comes from situations involving nonlinear elliptic problems in annular regions. Erbe and Tang [19] noted that, if the boundary-value problem

$$-\Delta u = F(|x|, u) \quad \text{ in } R < |x| < \hat{R}$$

with

$$\begin{cases} u = 0 \quad \text{for } |x| = R, \quad u = 0 \quad \text{for } |x| = \hat{R}; \quad \text{or} \\ u = 0 \quad \text{for } |x| = R, \quad \frac{\partial u}{\partial |x|} = 0 \quad \text{for } |x| = \hat{R}; \quad \text{or} \\ \frac{\partial u}{\partial |x|} = 0 \quad \text{for } |x| = R, \quad u = 0 \quad \text{for } |x| = \hat{R} \end{cases}$$

is radially symmetric, then it can be transformed into the so called two-point Sturm-Liouville problem

$$\begin{cases} x''(t) = -f(t, x(t)), \quad 0 \le t \le 1, \\ \alpha x(0) - \beta x'(0) = 0, \\ \gamma x(1) + \delta x'(1) = 0. \end{cases}$$

where α, β, γ, δ are positive constants. Paper [19] may be the first one concerned with the existence of positive solutions of a boundary value problem. In the latest ten years, multi-point boundary-value problems (BVPs for short) for second order differential equations with $p-$Laplacian have gained much attention, see papers [1-18].

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Ma in [11,12] studied the following BVP

$$\begin{cases} [p(t)x'(t)]' - q(t)x(t) + f(t, x(t)) = 0, \quad t \in (0,1), \\ \alpha x(0) - \beta p(0)x'(0) = \sum_{i=1}^{m} a_i x(\xi_i), \\ \gamma x(1) + \delta p(1)x'(1) = \sum_{i=1}^{m} b_i x(\xi_i), \end{cases} \tag{1}$$

where $0 < \xi_1 < \cdots < \xi_m < 1$, $\alpha, \beta, \gamma, \delta \geq 0, a_i, b_i \geq 0$ with $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$. By using Green′s functions (which complicate the studies of BVP(1)) and Guo-Krasnoselskii fixed point theorem [7-10], the existence and multiplicity of positive solutions for BVP(1) were given.

In paper [20], the existence of positive solutions for the m-point boundary-value problem

$$\begin{cases} y''(t) = -f(t, y(t), y'(t)), \quad 0 \leq t \leq 1, \\ \alpha y(0) - \beta y'(0) = 0, \\ y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) \end{cases}$$

and

$$\begin{cases} y''(t) = -f(t, y(t), y'(t)), \quad 0 \leq t \leq 1, \\ \alpha y(0) + \beta y'(0) = 0, \\ y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) \end{cases}$$

was considered, where $\alpha > 0$, $\beta > 0$, the function f is continuous, and

$$f(t, y, y') \geq 0, \quad \text{for all } t \in [0,1],\ y \geq 0\ \ y' \in R.$$

The presence of the third variable y' in the function $f(t, y, y')$ causes some considerable difficulties, especially, in the case where an approach relies on a fixed point theorem on cones and the growth rate of $f(t, y, y')$ is sublinear or superlinear. The approach used in [20] is based on an analysis of the corresponding vector field on the (y, y') face-plane and on Kneser′s property for the solution′s funnel.

Recently, many authors studied the existence of multiple positive solutions of the following BVP consisting of the second order differential equation

$$[\phi(x'(t))]' + q(t)f(t, x(t), x'(t)) = 0, \quad t \in (0,1) \tag{2}$$

and one of the following boundary conditions

$$x(0) = \sum_{i=1}^{m} a_i x(\xi_i), \quad x(1) = \sum_{i=1}^{m} b_i x(\xi_i)$$

$$x(0) = \sum_{i=1}^{m} a_i x(\xi_i), \quad x'(1) = \sum_{i=1}^{m} b_i x'(\xi_i),$$

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and

$$x'(0) = \sum_{i=1}^{m} a_i x'(\xi_i), \quad x'(1) = \sum_{i=1}^{m} b_i x'(\xi_i),$$

where $0 < \xi_1 < \cdots < \xi_m < 1$, $a_i \geq 0, b_i \geq 0$ for all $i = 1, \cdots, m$, f is defined on $[0,1] \times [0, +\infty) \times R$, ϕ is called $p-$Laplacian, see papers [4-6] and [13-18,21].

To the author′s knowledge, there has been no paper concerning with the computational methods of the following boundary value problems

$$\begin{cases} [p(t)\phi(x'(t))]' + q(t)f(t,x(t),x'(t)) = 0, \quad t \in (0,1), \\ \alpha x(0) - \beta x'(0) = \sum_{i=1}^{m} a_i x(\xi_i), \\ x'(1) = \sum_{i=1}^{m} b_i x'(\xi_i) \end{cases} \tag{3}$$

where $0 < \xi_1 < \cdots < \xi_m < 1$, $\alpha, \beta \geq 0$, $a_i \geq 0, b_i \geq 0$ for all $i = 1, \cdots, m$, f is defined on $[0,1] \times [0,+\infty) \times [0,+\infty)$, $p \in C^1[0,1]$ and $q \in C^0[0,1]$, ϕ is called $p-$Laplacian, $\phi(x) = |x|^{r-2}x$ with $r > 1$, its inverse function is denoted by $\phi^{-1}(x)$ with $\phi^{-1}(x) = |x|^{s-2}x$ with $1/r + 1/s = 1$.

The purpose of this paper is to investigate the iteration and existence of positive solutions for BVP(3). By applying monotone iterative techniques, we will construct some successive iterative schemes for approximating the solutions in this paper.

The sequel of this paper is organized as follows: the main result is presented in Section 2, and some examples are given in Section 3.

2. **Main Results**

In this section, we first present some background definitions in Banach spaces.

Definition 2.1. Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ and $x + y \in P$ for all $x, y \in P$ and $a \geq 0$ and $x \in X$ and $-x \in X$ imply $x = 0$.

Definition 2.2. Let X be a real Banach space and P a cone in X. A map $\psi : P \to [0,+\infty)$ is a nonnegative continuous concave functional map provided ψ is nonnegative, continuous and satisfies $\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$, for all $x, y \in P$ and $t \in [0,1]$.

Definition 2.3. Let X be a real Banach space. An operator $T; X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Choose $X = C^1[0,1]$. We call $x \ll y$ for $x, y \in X$ if $x(t) \leq y(t)$ and $x'(t) \leq y'(t)$ for all $t \in [0,1]$. We define the norm

$$||x|| = \max\{\max_{t\in[0,1]} |x(t)|, \max_{t\in[0,1]} |x'(t)|\} \text{ for } x \in X.$$

It is easy to see that X is a semi-ordered real Banach space. Define the cone in X by

$$P = \{x \in X : x(t) \geq 0 \text{ and is concave and increasing on } [0,1]\}.$$

For a positive number H, denote the subset P_H by

$$P_H = \{x \in P : ||x|| < H\} \text{ and } \overline{P}_H = \{x \in P : ||x|| \leq H\}.$$

Suppose that

(A1) $\alpha, \beta \geq 0$, $a_i \geq 0$, $b_i \geq 0$ for all $i = 1, \cdots, m$ with $\sum_{i=1}^{m} a_i < \alpha$, and $\sum_{i=1}^{m} b_i \phi^{-1}\left(\frac{1}{p(\xi_i)} p(1)\right) < 1$;

(A2) $p \in C^1([0,1],(0,+\infty))$;

(A3) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, $q : [0,1] \to [0,+\infty)$ are continuous with $q(t)f(t,0,0) \not\equiv 0$ on each sub-interval of [0,1], $f(t,x_1,y_1) \leq f(t,x_2,y_2)$ for all $t \in [0,1], x_1 \leq x_2$ and $y_1 \leq y_2$.

Lemma 2.1. Suppose that $(A1),(A2)$ and $(A3)$ hold and x satisfies $x \in C^1[0,1]$ with $[p(t)\phi(x'(t))]' \leq 0$ on $[0,1]$. Then x is concave.

Proof. Suppose $x(t_0) = \max_{t\in[0,1]} x(t)$. If $t_0 < 1$, for $t \in (t_0,1)$, since $x'(t_0) \leq 0$, we have $p(t_0)\phi(x'(t_0)) \leq 0$. Then $p(t)\phi(x'(t)) \leq 0$ for all $t \in (t_0,1]$. Let

$$\tau(t) = \frac{\int_{t_0}^t \phi^{-1}\left(\frac{1}{p(s)}\right)ds}{\int_{t_0}^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds}.$$

Then $\tau \in C([t_0,1],[0,1])$ and is strictly increasing on $[t_0,1]$ since it is easy to see that

$$\frac{d\tau}{dt} = \frac{\phi^{-1}\left(\frac{1}{p(t)}\right)}{\int_{t_0}^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds} > 0,$$

$\tau(t_0) = 0$ and $\tau(1) = 1$. Thus

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\frac{\phi^{-1}\left(\frac{1}{p(t)}\right)}{\int_{t_0}^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds}$$

implies that

$$\frac{dx}{d\tau} = \int_{t_0}^1 \phi^{-1}\left(\frac{1}{p(s)}\right)ds\phi^{-1}\left(p(t)\phi(x'(t)\right).$$

Since

$$[p(t)\phi(x'(t))]' = -q(t)f(t,x(t),x'(t)) \leq 0,$$

we get that $\phi^{-1}\left(p(t)\phi(x'(t)\right)$ is decreasing as t increases.

Since t is increasing as τ increases, we get that $\frac{dx}{d\tau}$ is decreasing as τ increases. Then x is concave on $[0,1]$. If $t_0 = 1$, similarly to above discussion, we get that x is concave. The proof is completed.

Throughout the paper, δ, a_1, b_1 and d_1 are defined by

$$\begin{aligned}
\delta &= \phi\left(\frac{1}{\sum_{i=1}^m b_i\phi^{-1}\left(\frac{1}{p(\xi_i)}p(1)\right)}\right) - 1,\\
a_1 &= \frac{\sum_{i=1}^m a_i \int_0^{\xi_i}\phi^{-1}\left(\frac{1+\delta}{\delta p(s)}\int_0^1 q(u)du\right)ds}{\alpha - \sum_{i=1}^m a_i},\\
b_1 &= \frac{\beta\phi^{-1}\left(\frac{1+\delta}{\delta p(0)}\int_0^1 q(u)du\right)}{\alpha - \sum_{i=1}^m a_i},\\
d_1 &= \phi^{-1}\left(\frac{1+\delta}{\delta \min_{t\in[0,1]} p(t)}\int_0^1 q(u)du\right).
\end{aligned}$$

Lemma 2.2. Suppose that $(A1)-(A3)$ hold. If $x \in X$ is a solution of BVP(3), then

$$x(t) = B_x + \int_0^t \phi^{-1}\left(\frac{p(1)\phi(A_x)}{p(s)} + \frac{\int_s^1 q(u)f(u,x(u),x'(u))du}{p(s)}\right)ds \tag{4}$$

where

$$A_x \in \left[0, \phi^{-1}\left(\frac{\int_0^1 q(u)f(u,x(u),x'(u))du}{\delta p(1)}\right)\right] \tag{5}$$

satisfies

$$A_x = \sum_{i=1}^m b_i \phi^{-1}\left(\frac{p(1)}{p(\xi_i)}\phi(A_x) + \frac{\int_{\xi_i}^1 q(u)f(u,x(u),x'(u))du}{p(\xi_i)}\right) \tag{6}$$

and B_x satisfies

$$\begin{aligned} B_x = {} & \tfrac{1}{\alpha - \sum_{i=1}^m a_i} \\ & \left[\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\tfrac{p(1)}{p(s)}\phi(A_x) + \tfrac{\int_s^1 q(u)f(u,x(u),x'(u))du}{p(s)}\right) ds \right. \\ & \left. +\beta\phi^{-1}\left(\tfrac{p(1)}{p(0)}\phi(A_x) + \tfrac{\int_0^1 q(u)f(u,x(u),x'(u))du}{p(0)}\right)\right]. \end{aligned} \tag{7}$$

Proof. Since x is solution of (3), we get

$$p(t)\phi(x'(t)) = p(1)\phi(x'(1)) + \int_t^1 q(u)f(u,x(u),x'(u))du, \quad t \in [0,1].$$

Then

$$x(t) = x(0) + \int_0^t \phi^{-1}\left(\frac{p(1)}{p(s)}\phi(y'(1)) + \frac{\int_s^1 q(u)f(u,x(u),x'(u))du}{p(s)}\right) ds.$$

The BCs in (3) imply that

$$\begin{aligned} & \alpha x(0) - \beta\phi^{-1}\left(\frac{p(1)}{p(0)}\phi(x'(1)) + \frac{\int_0^1 q(u)f(u,x(u),x'(u))du}{p(0)}\right) \\ = {} & x(0)\sum_{i=1}^m a_i \\ & + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{p(1)}{p(s)}\phi(y'(1)) + \frac{\int_s^1 q(u)f(u,x(u),x'(u))du}{p(s)}\right) ds \end{aligned}$$

and

$$x'(1) = \sum_{i=1}^m b_i \phi^{-1}\left(\frac{p(1)}{p(\xi_i)}\phi(x'(1)) + \frac{\int_{\xi_i}^1 q(u)f(u,x(u),x'(u))du}{p(\xi_i)}\right). \tag{8}$$

It follows that

$$\begin{aligned} x(0) = {} & \frac{1}{\alpha - \sum_{i=1}^m a_i} \times \\ & \left[\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{p(1)}{p(s)}\phi(y'(1)) + \frac{\int_s^1 q(u)f(u,x(u),x'(u))du}{p(s)}\right) ds \right. \\ & \left. +\beta\phi^{-1}\left(\frac{p(1)}{p(0)}\phi(x'(1)) + \frac{\int_0^1 q(u)f(u,x(u),x'(u))du}{p(0)}\right)\right]. \end{aligned}$$

Let

$$G(c) = c - \sum_{i=1}^{m} b_i \phi^{-1}\left(\frac{p(1)}{p(\xi_i)}\phi(c) + \frac{\int_{\xi_i}^{1} q(u)f(u,x(u),x'(u))du}{p(\xi_i)}\right).$$

If

$$\int_{\xi_i}^{1} q(u)f(u,x(u),x'(u))du = 0 \text{ for each } i = 1,\cdots,m,$$

we get

$$G(c) = \left(1 - \sum_{i=1}^{m} b_i \phi^{-1}\left(\frac{1}{p(\xi_i)}p(1)\right)\right)c.$$

Then $G(c) = 0$ implies that $c = 0$. If $b_i = 0$ for all $i = 1,\cdots,m$, then $G(c) = 0$ implies that $c = 0$. If there exists $i \in \{1,\cdots,m\}$ such that

$$\int_{\xi_i}^{1} q(u)f(u,x(u),x'(u))du \neq 0$$

and there exists $j \in \{1,\cdots,m\}$ such that $b_j \neq 0$, it is easy to see that $G(0) \neq 0$ and

$$\frac{G(c)}{c} = 1 - \sum_{i=1}^{m} b_i \phi^{-1}\left(\frac{p(1)}{p(\xi_i)} + \frac{1}{\phi(c)}\frac{1}{p(\xi_i)}\int_{\xi_i}^{1} q(u)f(u,x(u),x'(u))du\right)$$

and $G(c)/c$ is continuous, increasing on $(0,+\infty)$ and on $(-\infty,0)$, respectively. One sees from $(A1)$ that

$$\lim_{c\to-\infty}\frac{G(c)}{c} = 1 - \sum_{i=1}^{m} b_i \phi^{-1}\left(\frac{1}{p(\xi_i)}p(1)\right) > 0, \quad \lim_{c\to 0^-}\frac{G(c)}{c} = +\infty.$$

Hence $G(c) < 0$ for all $c \in (-\infty,0)$. On the other hand, it follows from

$$\lim_{t\to 0^+}\frac{G(c)}{c} = -\infty$$

and

$$\begin{aligned}
&\frac{G\left(\phi^{-1}\left(\frac{\int_0^1 q(u)f(u,x(u),x'(u))du}{\delta p(1)}\right)\right)}{\phi^{-1}\left(\frac{\int_0^1 q(u)f(u,x(u),x'(u))du}{\delta p(1)}\right)} \\
= \;& 1 - \sum_{i=1}^{m} b_i \phi^{-1}\left(\frac{1}{p(\xi_i)}p(1)\right. \\
&\left. + \frac{\delta p(1)}{\int_0^1 q(u)f(u,x(u),x'(u))du}\frac{1}{p(\xi_i)}\int_{\xi_i}^{1} q(u)f(u,x(u),x'(u))du\right) \\
\geq \;& 1 - \sum_{i=1}^{m} b_i \phi^{-1}\left(\frac{1+\delta}{p(\xi_i)}p(1)\right) \\
= \;& 0
\end{aligned}$$

that there exists unique constant

$$c_0 \in \left[0, \phi^{-1}\left(\frac{\int_0^1 q(u)f(u,x(u),x'(u))du}{\delta p(1)}\right)\right] \tag{9}$$

such that $G(c_0) = 0$. Together with (8), we get that $c_0 = x'(1) = A_x$.

Hence we get that there exist constants A_x satisfying (5) and (6), and B_x satisfying (7) such that $x(t)$ satisfies (4). The proof is completed.

Lemma 2.3. Suppose that $(A1), (A2)$ and $(A3)$ hold. If $x \in X$ is a solution of BVP(3), then $x(t) > 0$ for all $t \in (0, 1)$.

Proof. Suppose x satisfies (3). It follows from the assumptions that $p(t)\phi(x')$ is decreasing on $[0, 1]$.

It follows from Lemma 2.1 that x is concave on [0,1]. Then x' is decreasing on [0,1]. It follows from Lemma 2.2 that $x'(1) \geq 0$. It follows that x is increasing on [0,1]. Then

$$x(0) - \alpha x'(0) = \sum_{i=1}^{m} a_i x(\xi_i) \geq \sum_{i=1}^{m} a_i x(0).$$

It follows that

$$\left(\alpha - \sum_{i=1}^{m} a_i\right) x(0) - \beta x'(0) \geq 0.$$

We get that $x(0) \geq 0$ since $(A1)$ and $x'(0) \geq 0$. Hence $x(t) > 0$ for $t \in (0, 1]$. The proof is complete.

Define the nonlinear operator $T : P \to X$ by

$$\begin{aligned}(Tx)(t) &= B_x \\ &\quad + \int_0^t \phi^{-1}\left(\frac{p(1)}{p(s)}\phi(A_x) + \frac{\int_s^1 q(u)f(u,x(u),x'(u))du}{p(s)}\right) ds\end{aligned}$$

for $x \in P$, where A_x satisfies (6), and B_x satisfies (7).

Lemma 2.4. Suppose that $(A1) - (A3)$ hold. It is easy to show that

(i) Tx satisfies

$$\begin{cases} [p(t)\phi((Tx)'(t))]' + q(t)f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ \alpha(Tx)(0) - \beta(Tx)'(0) = \sum_{i=1}^m a_i (Tx)(\xi_i), \\ (Tx)'(1) = \sum_{i=1}^m b_i (Tx)'(\xi_i); \end{cases} \tag{10}$$

(ii) $Ty \in P$ for each $y \in P$;

(iii) x is a solution of BVP(3) if and only if x is a solution of the operator equation $x = Tx$ in P;

(iv) $T : P \to P$ is completely continuous;

(v) $Tx_1 \ll Tx_2$ for all $x_1, x_2 \in P$ with $x_1 \ll x_2$.

Proof. The proofs of (i), (ii) and (iii) are simple.

To prove (iv), it suffices to prove that T is continuous on P and T is relative compact. It is similar to that of the proof of Lemma 2.9 in [18] or Lemmas in [16] and are omitted.

To prove (v), it is easy to see that A_x is increasing in x and consequently, using $(A3)$ and considering B_x as a function of A_x and f we get the monotonicity of B_x. Suppose $x_1 \ll x_2$, we get that $x_1(t) \leq x_2(t)$ and $x_1'(t) \leq x_2'(t)$ for all $t \in [0,1]$. Then one finds that

$$(Tx_1)(t) \leq (Tx_2)(t), \ \ (Tx_1)'(t) \leq (Tx_2)'(t), \ \ t \in [0,1].$$

It follows that $Tx_1 \ll Tx_2$. The proof is completed.

Theorem 2.1. Suppose that $(A1) - (A3)$ hold. Furthermore, suppose that there exists a constant $A > 0$ such that f satisfies

$$\max_{t\in[0,1]} f(t,2A,2A) \leq \phi(M), \quad t \in [0,1], \tag{11}$$

where

$$M = \min\left\{\frac{A}{a_1+b_1}, \ \frac{A}{d_1}\right\}.$$

Then BVP(3) has at least one positive solution $x \in P$ with

$$x = \lim_{n\to\infty} u_n \text{ or } x = \lim_{n\to\infty} v_n,$$

where

$$u_0(t) = 0, \quad v_0(t) = A + At$$

and

$$u_n(t) = (Tu_{n-1})(t), \quad v_n(t) = (Tv_{n-1})(t).$$

Proof: We first prove that $T : \overline{P}_{2A} \to \overline{P}_{2A}$. For $x \in \overline{P}_{2A}$, one has that

$$0 \leq x(t) \leq 2A, \ \ 0 \leq x'(t) \leq 2A, \ \ t \in [0,1].$$

Then (11) implies that

$$0 \leq f(t,x(t),x'(t)) \leq f(t,2A,2A) \leq \max_{t\in[0,1]} f(t,2A,2A) \leq \phi(M).$$

Let A_x and B_x satisfy (6) and (7) respectively. It follows from Lemma 2.2 that A_x satisfies (5). By the definition of Tx, we get that

$$
\begin{aligned}
& 0 \le (Tx)(t) \\
= \ & B_x + \int_0^t \phi^{-1}\left(\frac{p(1)}{p(s)}\phi(A_x) + \frac{\int_s^1 q(u)f(u,x(u),x'(u))du}{p(s)}\right) ds \\
= \ & \frac{1}{\alpha - \sum_{i=1}^m a_i} \times \\
& \left[\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{p(1)}{p(s)}\phi(A_x) + \frac{\int_s^1 q(u)f(u,x(u),x'(u))du}{p(s)}\right) ds\right. \\
& \left. +\beta\phi^{-1}\left(\frac{1}{p(0)}p(1)\phi(A_x) + \frac{\int_0^1 q(u)f(u,x(u),x'(u))du}{p(0)}\right)\right] \\
& + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_x) + \frac{\int_s^1 q(u)f(u,x(u),x'(u))du}{p(s)}\right) ds \\
\le \ & \frac{1}{\alpha - \sum_{i=1}^m a_i} \times \\
& \left[\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{\delta p(s)}\int_0^1 q(u)\phi(M)du + \frac{\int_s^1 q(u)\phi(M)du}{p(s)}\right) ds\right. \\
& \left. +\beta\phi^{-1}\left(\frac{1}{\delta p(0)}\int_0^1 q(u)\phi(M)du + \frac{\int_0^1 q(u)\phi(M)du}{p(0)}\right)\right] \\
& + \int_0^t \phi^{-1}\left(\frac{1}{\delta p(s)}\int_0^1 q(u)\phi(M)du + \frac{\int_s^1 q(u)\phi(M)du}{p(s)}\right) ds
\end{aligned}
$$

$$
\begin{aligned}
\le \ & M\left[\frac{\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1+\delta}{\delta p(s)}\int_0^1 q(u)du\right) ds}{\alpha - \sum_{i=1}^m a_i}\right. \\
& \left. + \frac{\beta\phi^{-1}\left(\frac{1+\delta}{\delta p(0)}\int_0^1 q(u)du\right)}{\alpha - \sum_{i=1}^m a_i} + \phi^{-1}\left(\frac{1+\delta}{\delta \min_{t\in[0,1]} p(t)}\int_0^1 q(u)du\right)\right] \\
= \ & M(a_1 + b_1) + Md_1 \le 2A,
\end{aligned}
$$

and

$$\begin{aligned}0 &\le (Tx)'(t)\\ &= \phi^{-1}\left(\frac{p(1)}{p(t)}\phi(A_x)+\frac{1}{p(t)}\int_t^1 q(u)f(u,x(u),x'(u))du\right)\\ &\le \phi^{-1}\left(\frac{1}{\delta p(t)}\int_0^1 q(u)\phi(aN)du+\frac{1}{p(t)}\int_t^1 q(u)\phi(M)du\right)\\ &\le M\phi^{-1}\left(\frac{1+\delta}{\delta\min_{t\in[0,1]}p(t)}\int_0^1 q(u)du\right)\\ &= Md_1\le A\le 2A.\end{aligned}$$

It follows that $Tx\in\overline{P}_{2A}$.

By the definitions of u_0 and v_0, we have

$$u_0(t)\le v_0(t),\ \ u_0'(t)\le v_0'(t),\ \ t\in[0,1].$$

Then $u_0\ll v_0$. Then Lemma 2.4(v) implies that $u_n\ll v_n$ for all $n=1,2,3,\cdots$.

Now, we prove that $u_{n-1}\ll u_n$. It suffices to prove that $u_0\ll u_1$. First, one has

$$\begin{aligned}&u_1(t)=(Tu_0)(t)\\ =\ &\frac{\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{p(1)}{p(s)}\phi(A_{u_0})+\frac{1}{p(s)}\int_s^1 q(u)f(u,a(1-u),-a)du\right)ds}{\alpha-\sum_{i=1}^m a_i}\\ &+\frac{\beta\phi^{-1}\left(\frac{1}{p(0)}p(1)\phi(A_{u_0})+\frac{1}{p(0)}\int_0^1 q(u)f(u,0,0)du\right)}{\alpha-\sum_{i=1}^m a_i}\\ &+\int_0^t\phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_{u_0})+\frac{1}{p(s)}\int_s^1 q(u)f(u,0,0)du\right)ds\\ \ge\ &0=u_0(t).\end{aligned}$$

Second, we have

$$\begin{aligned}u_1'(t) &= (Tu_0)'(t)\\ &= \phi^{-1}\left(\frac{1}{p(t)}p(1)\phi(A_{u_0})+\frac{1}{p(t)}\int_t^1 q(u)f(u,0,0)du\right)\\ &\ge 0=u_0'(t).\end{aligned}$$

It follows that $u_1(t)\ge u_0(t)$ and $u_1'(t)\ge u_0'(t)$ for all $t\in[0,1]$. Then $u_0\ll u_1$. Hence one has that

$$u_0\ll u_1\ll u_2\ll\cdots\ll u_n\ll\cdots. \tag{12}$$

Now, we prove that $v_n\ll v_{n-1}$. It suffices to prove that $v_1\le v_0$. First, (9) implies that

$$0\le A_{v_0}\ \le\ \phi^{-1}\left(\frac{\int_0^1 q(u)f(u,v_0(u),v_0'(u))du}{\delta p(1)}\right).$$

Sice $\max_{t\in[0,1]}f(t,2A,2A)\le\phi(M),\quad t\in[0,1]$, we get

$$f(t,v_0(t),v_0'(t))=f(t,A+At,A)\le f(t,2A,2A)\le\phi(M),\ \ t\in[0,1].$$

Then

$$
\begin{aligned}
& v_1(t) = (Tv_0)(t) \\
= \ & \frac{\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{p(1)}{p(s)}\phi(A_{v_0}) + \frac{1}{p(s)}\int_s^1 q(u)f(u,bu+a,b)du\right)ds}{\alpha - \sum_{i=1}^m a_i} \\
& + \frac{\beta\phi^{-1}\left(\frac{1}{p(0)}p(1)\phi(A_{v_0}) + \frac{1}{p(0)}\int_0^1 q(u)f(u,bu+a,b)du\right)}{\alpha - \sum_{i=1}^m a_i} \\
& + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_{v_0}) + \frac{1}{p(s)}\int_s^1 q(u)f(u,bu+a,b)du\right)ds \\
\leq \ & \frac{\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1+\delta}{\delta p(s)}\int_0^1 q(u)f(u,v_0(u),v_0'(u))du\right)ds}{\alpha - \sum_{i=1}^m a_i} \\
& + \frac{\beta\phi^{-1}\left(\frac{1+\delta}{\delta p(0)}\int_0^1 q(u)f(u,v_0(u),v_0'(u))du\right)}{\alpha - \sum_{i=1}^m a_i} \\
& + \int_0^t \phi^{-1}\left(\frac{1+\delta}{\delta p(s)}\int_0^1 q(u)f(u,v_0(u),v_0'(u))du\right)ds \\
\leq \ & M\left[\frac{\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1+\delta}{\delta p(s)}\int_0^1 q(u)du\right)ds}{\alpha - \sum_{i=1}^m a_i}\right. \\
& \left. + \frac{\beta\phi^{-1}\left(\frac{1+\delta}{\delta p(0)}\int_0^1 q(u)du\right)}{\alpha - \sum_{i=1}^m a_i} + \int_0^t \phi^{-1}\left(\frac{1+\delta}{\delta p(s)}\int_0^1 q(u)du\right)ds\right] \\
\leq \ & M\left[\frac{\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1+\delta}{\delta p(s)}\int_0^1 q(u)du\right)ds}{\alpha - \sum_{i=1}^m a_i}\right. \\
& \left. + \frac{\beta\phi^{-1}\left(\frac{1+\delta}{\delta p(0)}\int_0^1 q(u)du\right)}{\alpha - \sum_{i=1}^m a_i} + t\phi^{-1}\left(\frac{1+\delta}{\delta \min_{t\in[0,1]} p(t)}\int_0^1 q(u)du\right)\right] \\
= \ & M(a_1+b_1) + Md_1 t \\
\leq \ & A + At = v_0(t).
\end{aligned}
$$

Second, we have

$$
\begin{aligned}
& v_1'(t) = (Tv_0)'(t) \\
= \ & \phi^{-1}\left(\frac{1}{p(t)}p(1)\phi(A_{v_0}) + \frac{1}{p(t)}\int_t^1 q(u)f(u,v_0(u),v_0'(u))du\right) \\
\leq \ & \phi^{-1}\left(\frac{1+\delta}{\delta \min_{t\in[0,1]} p(t)}\int_0^1 q(u)\phi(M)du\right) \\
= \ & M\phi^{-1}\left(\frac{1+\delta}{\delta \min_{t\in[0,1]} p(t)}\int_0^1 q(u)du\right) \\
= \ & Md_1 \leq A = v_0'(t).
\end{aligned}
$$

It follows that $v_1(t) \le v_0(t)$ and $v_1'(t) \le v_0'(t)$ for all $t \in [0,1]$. Then $v_0 \ge v_1$. Hence one has that

$$v_0 \ge v_1 \ge v_2 \ge \cdots \ge v_n \le \cdots . \quad (13)$$

It follows from (12) and (13) that

$$0 = u_0 \ll u_1 \ll \cdots \ll u_n \ll \cdots \ll v_n \ll \cdots \ll v_1 \ll v_0 = At + A.$$

Since u_n is a uniformly increasing sequence, it is easy to show that u_n is equicontinuous. Then $\lim_{n\to\infty} u_n = u^* \in P$ satisfies

$$u_0(t) = a(1-t) \le u^*(t) \le bt + a,\ Tu^* = u^*.$$

Then $x = u^*$ is a solution of BVP(3). Similarly to above discussion,

$$\lim_{n\to\infty} v_n = v^* \in P$$

satisfies

$$u_0(t) = a(1-t) \le v^*(t) \le bt + a,\ Tv^* = v^*.$$

Then $x = v^*$ is a solution of BVP(3). It is easy to see that BVP(3) has unique solution x in $\{x \in P : 0 \le x \le At + A\}$ if $u^* = v^*$. BVP(3) has multiple solutions if $u^* \ne v^*$. The proof is complete.

Remark 2.1 The quantity A_x is given implicitly, as a root of equation (6), it must be determined in every step of iteration.

Theorem 2.2. Suppose that $(A1) - (A3)$ hold. Furthermore, suppose that f satisfies

$$\limsup_{x\to 0} \sup_{t\in[0,1]} \frac{f(t,2A,2A)}{\phi(A)} < \phi\left(\min\left\{\frac{1}{a_1+b_1}, \frac{1}{d_1}\right\}\right). \quad (14)$$

Then BVP(3) has at least one positive solution $x \in P$.

Proof. It follows from (14) that there exists a constant $A > 0$ such that

$$f(t,2A,2A) \le \phi(A)\phi\left(\min\left\{\frac{1}{a_1+b_1}, \frac{1}{d_1}\right\}\right).$$

The remainder of the proof is similar to that of the proof of Theorem 2.1 and is omitted.

3. **An example**

Now, we present a boundary value problem to which our results can readily apply, whereas the known results in the current literature do not cover.

Example 3.1. Consider the following BVP

$$\begin{cases} [(x'(t))^3]' + f(t,x(t),x'(t)) = 0, \quad t \in (0,1), \\ x(0) - x'(0) = \frac{1}{2}x(1/2), \\ x'(1) = \frac{1}{4}x'(1/2). \end{cases} \quad (15)$$

Corresponding to BVP(3), one sees that $\phi(x) = x^3$ with $\phi^{-1}(x) = x^{\frac{1}{3}}$, $\alpha = 1, \beta = 1$, $\xi_1 = 1/2$, $a_1 = 1/2, b_1 = 1/4$, $q(t) \equiv 1, t \in [0,1]$, $p(t) = 1$,

$$f(t,u,v) = \frac{t}{1000000} + \frac{1}{24}x^3 + \frac{1}{24}y^3$$

is nonnegative and continuous.

One finds that

$$\begin{aligned}\delta &= \phi\left(\frac{1}{\sum_{i=1}^m b_i\phi^{-1}\left(\frac{1}{p(\xi_i)}p(1)\right)}\right)-1=63,\\ a_1 &= \frac{\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1+\delta}{\delta p(s)}\int_0^1 q(u)du\right)ds}{\alpha-\sum_{i=1}^m a_i}=\frac{2}{\sqrt[3]{63}},\\ b_1 &= \frac{\beta\phi^{-1}\left(\frac{1+\delta}{\delta p(0)}\int_0^1 q(u)du\right)}{\alpha-\sum_{i=1}^m a_i}=\frac{8}{\sqrt[3]{63}},\\ d_1 &= \phi^{-1}\left(\frac{1+\delta}{\delta\min_{t\in[0,1]}p(t)}\int_0^1 q(u)du\right)=\frac{4}{\sqrt[3]{63}},\end{aligned}$$

and

$$M=\min\left\{\frac{A}{\frac{2}{\sqrt[3]{63}}+\frac{8}{\sqrt[3]{63}}},\ \frac{A}{\frac{4}{\sqrt[3]{63}}}\right\}=\frac{\sqrt[3]{63}A}{4}. \tag{16}$$

It is easy to check that there exists a constant $A>0$ such that

$$\frac{f(t,2A,2A)}{\phi(A)}=\frac{\frac{t}{1000000}+\frac{1}{3}A^3+\frac{1}{3}A^3}{A^3}\le\frac{63}{64},\ t\in[0,1].$$

One can check that $(A1),(A2),(A3)$ hold. Then Theorem 2.1 implies that BVP(15) has at least one positive solution x with $x=\lim_{n\to\infty}u_n$ or $x=\lim_{n\to\infty}v_n$, where $u_0(t)=0,\quad v_0(t)=A+At$ and $u_n(t)=(Tu_{n-1})(t),\quad v_n(t)=(Tv_{n-1})(t)$, where T is defined by

$$\begin{aligned}(Tx)(t) &= 2\left[\frac{1}{2}\int_0^{1/2}\left(A_x^3+\int_s^1)f(u,x(u),x'(u))du\right)^{1/3}ds\right.\\ &\quad\left.+\left(A_x^3+\int_0^1 f(u,x(u),x'(u))du\right)^{1/3}\right]\\ &\quad+\int_0^t\left(A_x^3+\int_s^1 f(u,x(u),x'(u))du\right)^{1/3}ds\end{aligned}$$

and A_x satisfies

$$A_x=\frac{1}{4}\left(A_x^3+\int_{1/2}^1 f(u,x(u),x'(u))du\right)^{1/3}$$

for $x\in C^1[0,1]$ with $x(t)\ge 0, t\in[0,1]$.

Remark 3.1. BVP(15) can not be solved by theorems obtained in [11-21].

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