∗-FRAMES IN HILBERT C^*-MODULES

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Certain facts about frames are extended for the new frames in Hilbert C^*-modules where they are called ∗-frames. It is shown that ∗-frames for Hilbert C^*-modules share several useful properties with frames for Hilbert C^*-modules. The paper studies also the operators associated to a given ∗-frame, ∗-frames for Hilbert C^*-modules over commutative unitary C^*-algebras, and the construction of new ∗-frames. The relations between frames and ∗-frames in Hilbert C^*-modules are considered. Moreover, ∗-frames in Hilbert C^*-modules over different C^*-algebras are compared, and some characterizations of ∗-frames in a Hilbert C^*-module with respect to another Hilbert C^*-module are presented. Finally, dual ∗-frames are characterized.

Keywords: : frame, ∗-frame, ∗-Bessel sequence, ∗frame operator, pre-∗-frame operator, ∗-frame representation, dual ∗-frame, C^*-algebra, Hilbert C^*-module.

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Introduction and Basic Definitions

Frames were first introduced in 1952 by Duffin and Schaeffer [7]. They abstracted the fundamental notion of Gabor [11] to study signal processing. It seems, however, that Duffin-Schaeffer ideas did not attract much interest outside the realm of nonharmonic Fourier series until the paper by Daubechies, Grassman and Mayer [6] was published in 1986.

The theory of frames was rapidly generalized and various generalizations consisting of different vectors in Hilbert spaces were developed [15, 20, 21]. In 2000, Frank-Larson [9] introduced the notion of frames in Hilbert C^*-modules as a generalization of frames in Hilbert spaces and Jing [12] continued to consider them. It is well known that Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a $C*$-algebra rather than in the field of complex numbers. Also, the theory of Hilbert C^*-modules has applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry, and KK-theory. There are some differences between Hilbert C^*-modules and Hilbert spaces. For example, we know that the Riesz representation theorem for continuous linear functionals on Hilbert spaces dose not extend to Hilbert C^*-modules [22] and there exist closed subspaces in Hilbert C^*-modules that have no orthogonal complement [17]. Moreover, we know that every bounded operator on a Hilbert space has an adjoint, while there are bounded operators on Hilbert C^*-modules which do not have any [18]. It is expected that problems about frames and ∗-frames for Hilbert C^*-modules to be more complicated than those for Hilbert spaces. This makes the study of the frames for Hilbert C^*-modules important and interesting. The main purposes of the present paper are to introduce the ∗-frames, to consider the relation between frames and ∗-frames in a given Hilbert C^*-module, to study the properties of them in Hilbert C^*-modules with different C^*-algebras and to characterize the dual ∗-frames.

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The paper is organized as follows. We continue this introductory section with a review of the basic definitions and notations of C^*-algebras, Hilbert C^*-modules, frames in Hilbert spaces and frames in Hilbert C^*-modules. Section 1 introduces $*$-frames and presents nontrivial examples of such $*$-frames. In what follows, we consider corresponding operators associated with a given $*$-frame, the relation between frames in Hilbert C^*-modules, and $*$-frames in Hilbert C^*-modules over commutative C^*-algebras. Following that, the new $*$-frames are constructed by a given $*$-frame in Section 2. One of the main results of the paper is included in Section 3, where $*$-frames in modular spaces with different C^*-algebras are studied and the final section states the dual $*$-frames as another new result.

Let us recall some definitions and basic properties of C^*-algebras and Hilbert C^*-modules that we need in the rest of the parer. We also introduce frames in Hilbert space and Hilbert C^*-modules. For more details, we refer the interested reader to [5, 8, 9, 16, 19, 22].

Let $\mathcal{A}$ be a unitary C^*-algebra and $a \in \mathcal{A}$. The nonzero element a is called strictly nonzero if zero doesn't belong to $\sigma(a)$, and a is said to be strictly positive if it is strictly nonzero and positive. If a is positive, then there is a positive element b in $\mathcal{A}$ such that $b^2 = a$. Moreover, b commutes with all the elements that commutes with a [3, Theorem 6.2.10]. We use the notation $\sqrt{a}$ or $a^{\frac{1}{2}}$ for b. The absolute value of a is defined by $|a| := (a^*a)^{\frac{1}{2}}$. The relation " $\leq$ " given by

$$a \leq b \ \ if \ \ and \ \ only \ \ if \ \ b - a \ \ is \ \ positive$$

defines a partial ordering on $\mathcal{A}$. Some elementary facts about "$\leq$" are given in the following statements for $a, b, c \in \mathcal{A}$.

(1) $a \leq \|a\|$.
(2) $0 \leq a \leq b$ implies $\|a\| \leq \|b\|$, $ab \geq 0$, $a + b \geq 0$, and $a^t \leq b^t$ for $t \in (0, 1)$.
(3) If $a \leq b$, then $cac^* \leq cbc^*$. Moreover, if c commutes with a and b, then $ca \leq cb$ for $c \geq 0$.
(4) If a and b are positive invertible elements and $a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$.

In this paper, the notation $a < b$ denotes $a \leq b$ and $a \neq b$. Now, let $\mathcal{B}$ be an another unitary C^*-algebra. The tensor product of the algebras $\mathcal{A}$ and $\mathcal{B}$ is the completion of $\mathcal{A} \otimes_{alg} \mathcal{B}$ with the spatial norm and the following operation and involution,

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb' \ \ , \ \ (a \otimes b)^* = a^* \otimes b^* \quad \forall a \otimes b, a' \otimes b' \in \mathcal{A} \otimes \mathcal{B}.$$

Hence $\mathcal{A} \otimes \mathcal{B}$ is a C^*-algebra such that $\|a \otimes b\| = \|a\|.\|b\|$ for $a \otimes b \in \mathcal{A} \otimes \mathcal{B}$. If $0 \leq a_1 \leq a_2$ in $\mathcal{A}$ and $0 \leq b_1 \leq b_2$ in $\mathcal{B}$, then $0 \leq a_1 \otimes b_1 \leq a_2 \otimes b_2$, see [16, Lemma 4.3].

The following proposition is a useful tool and will be used frequently in the rest of the paper.

Proposition 0.1. *[8, 13, 19] If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a $*$-homomorphism between C^*-algebras, then φ has the following properties.*

(1) $\varphi(1) = 1$.
(2) *If a is invertible, then so is $\varphi(a)$, and $\varphi(a^{-1}) = \varphi(a)^{-1}$.*
(3) *The $*$-homomorphism φ is positive and increasing, that is, $\varphi(\mathcal{A}^+) \subseteq \mathcal{B}^+$, and if $a_1 \leq a_2$, then $\varphi(a_1) \leq \varphi(a_2)$.*
(4) *For $a \in \mathcal{A}$, we have $\sigma(\varphi(a)) \subseteq \sigma(a)$, and if φ is injective, then $\sigma(\varphi(a)) = \sigma(a)$.*
(5) *If a is strictly positive, then so is $\varphi(a)$.*

Now, we are going to introduce some of the elementary definitions and the basic properties of Hilbert C^*-modules. Let $\mathcal{A}$ be a C^*-algebra. A pre-Hilbert C^*-module is a linear space and algebraic (left) $\mathcal{A}$-module $\mathcal{H}$ together with an $\mathcal{A}$-inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{A}$ that possesses the following properties,

(i) $\langle f, f \rangle \geq 0$, for any $f \in \mathcal{H}$.

(*ii*) $\langle f, f\rangle = 0$ if and only if $f = 0$.
(*iii*) $\langle f, g\rangle = \langle g, f\rangle^*$, for any $f, g \in \mathcal{H}$.
(*iv*) $\langle \lambda f, h\rangle = \lambda\langle f, h\rangle$, for any $\lambda \in \mathbb{C}$ and $f, h \in \mathcal{H}$.
(*v*) $\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle$, for any $a, b \in \mathcal{A}$ and $f, g, h \in \mathcal{H}$.

The action of $\mathcal{A}$ on $\mathcal{H}$ is $\mathbb{C}$- and $\mathcal{A}$-linear i.e., $\lambda(af) = (\lambda a)f = a(\lambda f)$ for every $\lambda \in \mathbb{C}, a \in \mathcal{A}$ and $f \in \mathcal{H}$. The map $f \longmapsto \|f\| = \|\langle f, f\rangle\|_{\mathcal{A}}^{\frac{1}{2}}$, defines a norm on $\mathcal{H}$. If a pre-Hilbert C^*-module $\mathcal{H}$ is complete with respect to this norm, then $(\mathcal{H}, \mathcal{A}, \langle\cdot,\cdot\rangle)$ is called a Hilbert C^*-module over $\mathcal{A}$ or, simply, a Hilbert $\mathcal{A}$-module. We write $\mathcal{H}$ or $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ instead of $(\mathcal{H}, \mathcal{A}, \langle\cdot,\cdot\rangle)$ when the $\mathcal{A}$-valued inner product and the C^*-algebra are well known. The Hilbert $\mathcal{A}$-module $\mathcal{H}$ is called to be a full Hilbert $\mathcal{A}$-module when the linear span of $\{\langle f, g\rangle : f, g \in \mathcal{H}\}$ is dense in $\mathcal{A}$.

The C^*-algebra $\mathcal{A}$ itself can be recognized as a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b\rangle = ab^*$. The standard Hilbert $\mathcal{A}$-module $l_2(\mathcal{A})$ is defined by

$$l_2(\mathcal{A}) := \{\{a_j\}_{j\in\mathbb{N}} \subseteq \mathcal{A} : \sum_{j\in\mathbb{N}} a_j a_j^* \ \ converges \ \ in \ \ \mathcal{A}\},$$

with $\mathcal{A}$-inner product $\langle \{a_j\}_{j\in\mathbb{N}}, \{b_j\}_{j\in\mathbb{N}}\rangle = \sum_{j\in\mathbb{N}} a_j b_j^*$. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_1)$ and $(\mathcal{K}, \langle\cdot,\cdot\rangle_2)$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ (not necessarily linear or bounded) is said to be adjointable (with respect to the $\mathcal{A}$-inner products $(\mathcal{H}, \langle\cdot,\cdot\rangle_1)$ and $(\mathcal{K}, \langle\cdot,\cdot\rangle_2)$), if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ satisfying $\langle Tf, g\rangle_2 = \langle f, T^*g\rangle_1$ whenever $f \in \mathcal{H}$, and $g \in \mathcal{K}$. The map T^* is called the adjoint of T [22]. The class of all adjointable maps from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $B_*(\mathcal{H}, \mathcal{K})$ and the class of all bounded $\mathcal{A}$-module maps from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $B_b(\mathcal{H}, \mathcal{K})$. It is known that $B_*(\mathcal{H}, \mathcal{K}) \subseteq B_b(\mathcal{H}, \mathcal{K})$. We denote $B_*(\mathcal{H}, \mathcal{H})$ and $B_b(\mathcal{H}, \mathcal{H})$ with $B_*(\mathcal{H})$ and $B_b(\mathcal{H})$, respectively. (We avoid the classical notation $B(\mathcal{H}, \mathcal{K})$ which is used for different notions by operator theorists and frame theorists.)
Let $(\mathcal{H}, \mathcal{A}, \langle\cdot,\cdot\rangle_{\mathcal{A}})$ and $(\mathcal{K}, \mathcal{B}, \langle,\rangle_{\mathcal{B}})$ be two Hilbert C^*-modules. Similarly to the tensor product of C^*-algebras, the tensor product of Hilbert C^*-modules $\mathcal{H}$ and $\mathcal{K}$, that is denoted by $\mathcal{H} \otimes \mathcal{K}$, is the completion of $\mathcal{H} \otimes_{alg} \mathcal{K}$ with the following operations,

$$\langle f_1 \otimes g_1, f_2 \otimes g_2\rangle = \langle f_1, f_2\rangle_{\mathcal{A}} \otimes \langle g_1, g_2\rangle_{\mathcal{B}} \quad , \quad (a \otimes b)(f \otimes g) = af \otimes bg,$$

for $f, f_1, f_2 \in \mathcal{H}$, $g, g_1, g_2 \in \mathcal{K}$ and $a \otimes b \in \mathcal{A} \otimes \mathcal{B}$. If U and V are two module maps on $\mathcal{H}$ and $\mathcal{K}$, respectively, then the tensor product $U \otimes V$ is defined by $U \otimes V(f \otimes g) = Uf \otimes Vg$ for $f \otimes g \in \mathcal{H} \otimes \mathcal{K}$.

Throughout the paper, we need the following lemma that it will illustrate lower and upper bounds of operators corresponding to a given operator T with respect to $\mathcal{A}$-valued inner products.

Lemma 0.1. *(see [2].) Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in B_*(\mathcal{H}, \mathcal{K})$. Then*
(*i*) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and $\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2$.*
(*ii*) *If T is surjective, then the adjointable map TT^* is invertible and $\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2$.*

The remainder of the section introduces frames in two spaces, Hilbert spaces and Hilbert C^*-modules. A frame for the Hilbert space H is a countable family $\{f_j\}_{j\in J}$ in H satisfying

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq B\|f\|^2,$$

for all $f \in H$ and some positive constants A and B independent of f.

The notion of frames for Hilbert spaces had been extended by Frank-Larson [10] to the notion of frames in a Hilbert $\mathcal{A}$-modules as a countable family $\{f_j\}_{j\in J}$ in a Hilbert

$\mathcal{A}$-module $\mathcal{H}$ satisfying

$$A\langle f,f\rangle \leq \sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle \leq B\langle f,f\rangle$$

for all $f\in\mathcal{H}$ and some positive constants A and B independent of f.

In the rest of this paper, we fix the notations $\mathcal{A}$ and J for a given unitary C^*-algebra and a finite or countably infinite index set, respectively. Also, the Hilbert $\mathcal{A}$-module $\mathcal{H}$ is assumed to be finitely or countably generated.

1. ∗-Frames and Their Corresponding Operators

In several spaces, frames can be a good candidate instead of basis. In this section, we extend the concept of Hilbert space frames to ∗-frames in Hilbert C^*-modules with $\mathcal{A}$-valued bounds. In Subsection 2.1, ∗-frames and frames are compared through some examples. Similar to the Hilbert frames case, operators corresponding to a ∗-frame play an important role in its characterization and investigation, and are given in Subsection 2.2. We illustrate ∗-frames in Hilbert C^*-modules over commutative C^*-algebras in the last subsection.

1.1.∗-Frames

∗-Frames are C^*-algebra version of frames. Actually, we need strictly positive elements of C^*-algebra $\mathcal{A}$ instead of positive real numbers.

Definition 1.1. *A sequence $\{f_j\in\mathcal{H}:j\in J\}$ is a ∗-frame for $\mathcal{H}$ if there exist two strictly nonzero elements A and B in $\mathcal{A}$ such that*

$$A\langle f,f\rangle A^* \leq \sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle \leq B\langle f,f\rangle B^*, \qquad \forall f\in\mathcal{H}. \tag{1.1}$$

The elements A and B are called the lower and upper ∗-frame bounds, respectively. Since $\mathcal{A}$ is not a partial ordered set, lower and upper ∗-frame bounds may not have order and the optimal bounds may not exist.

If $\lambda=A=B$, then the ∗-frame $\{f_j\}_{j\in J}$ is said to be a λ-tight ∗-frame. In the special case $A=B=1_{\mathcal{A}}$, it is called a Parseval ∗-frame or a normalized ∗-frame. Precisely, in a Hilbert $\mathcal{A}$-module, the set of all normalized ∗-frames and the set of all normalized frames are the same but this is not true in the tight case. (See Example 1.1 and Example 1.2.)

If $\{f_j\}_{j\in J}$ possesses an upper ∗-frame bound, but not necessarily a lower ∗-frame bound, we called it a ∗-Bessel sequence for $\mathcal{H}$ with ∗-Bessel bound B.

If the sum in the inequality (1.1) converges in norm, then the (normalized, tight) ∗-frames and ∗-Bessel sequences are called to be standard (normalized, tight) ∗-frames and standard ∗-Bessel sequences. In what follows, by (normalized, tight) ∗-frames and ∗-Bessel sequences, we mean standards ones.

We mentioned that the set of all of frames in Hilbert $\mathcal{A}$-modules can be considered as a subset of ∗-frames. To illustrate this, let $\{f_j\}_{j\in J}$ be a frame for Hilbert $\mathcal{A}$-module $\mathcal{H}$ with frame real bounds A and B. Note that for $f\in\mathcal{H}$,

$$(\sqrt{A})1_{\mathcal{A}}\langle f,f\rangle(\sqrt{A})1_{\mathcal{A}} \leq \sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle \leq (\sqrt{B})1_{\mathcal{A}}\langle f,f\rangle(\sqrt{B})1_{\mathcal{A}}.$$

Therefore, every frame for a Hilbert $\mathcal{A}$-module $\mathcal{H}$ with real bounds A and B is a ∗-frame for $\mathcal{H}$ with $\mathcal{A}$-valued ∗-frame bounds $(\sqrt{A})1_{\mathcal{A}}$ and $(\sqrt{B})1_{\mathcal{A}}$. In the following examples, we are going to illustrate some $*$-frames with $\mathcal{A}$-valued bounds. We will show that in some cases, $\mathcal{A}$-valued bounds are preferred to real-valued bounds.

Example 1.1. *Let l^∞ be the unitary C^*-algebra of all bounded complex-valued sequences with the following operations.*

$$uv = \{u_iv_i\}_{i\in\mathbb{N}}, \quad u^* = \{\overline{u_i}\}_{i\in\mathbb{N}}, \quad \|u\| = \sup_{i\in\mathbb{N}} |u_i|, \quad \forall u = \{u_i\}_{i\in\mathbb{N}}, v = \{v_i\}_{i\in\mathbb{N}} \in l^\infty.$$

Let c_0 be the set of all sequences converging to zero. Then c_0 is a Hilbert l^∞-module with l^∞-valued inner product $\langle u, v\rangle = \{u_i\overline{v_i}\}_{i\in\mathbb{N}}$, for $u, v \in c_0$. Let $J = \mathbb{N}$ and define $f_j \in c_0$ by

$$f_j = \{f_i^j\}_{i\in\mathbb{N}} \;\; such \;\; that \;\; f_i^j = \begin{cases} \frac{1}{2} + \frac{1}{i} & i = j \\ 0 & i \neq j \end{cases}, \qquad \forall j \in \mathbb{N}.$$

We observe that

$$\sum_{j\in J} \langle u, f_j\rangle\langle f_j, u\rangle = \{|u_i|^2(\frac{1}{2} + \frac{1}{i})^2\}_{i\in\mathbb{N}} = \{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbb{N}}\langle\{u_i\}_{i\in\mathbb{N}}, \{u_i\}_{i\in\mathbb{N}}\rangle\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbb{N}},$$

for $u = \{u_j\}_{j\in\mathbb{N}} \in c_0$. The sequence $\{f_j\}_{j\in\mathbb{N}}$ is a $\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathbb{N}}$-tight $*$-frame but it is not a tight frame for Hilbert l^∞-module c_0. Note that, $\{f_j\}_{j\in\mathbb{N}}$ is a frame for Hilbert l^∞-module c_0 with optimal lower and upper real bounds $\frac{1}{2}$ and $\frac{3}{2}$, respectively.

Example 1.2. *Let $\mathcal{A}$ be the C^*-algebra of the set of all diagonal matrices in $M_{2\times 2}(\mathbb{C})$ and suppose $\mathcal{A}$ is the Hilbert $\mathcal{A}-$module over itself. (Here, diagonal matrix means a 2×2 matrix (a_{ij}) such that $a_{11} = a$, $a_{22} = b$ and $a_{12} = a_{21} = 0$, for $a, b \in \mathbb{C}$.) Consider, $A_i = \begin{bmatrix} \frac{1}{2^i} & 0 \\ 0 & \frac{1}{3^i} \end{bmatrix}$, for all $i \in \mathbb{N}$. For $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathcal{A}$, we have*

$$\sum_{i\in\mathbb{N}} \langle A, A_i\rangle\langle A_i, A\rangle = \begin{bmatrix} \frac{|a|^2}{3} & 0 \\ 0 & \frac{|b|^2}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix} \langle A, A\rangle \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}.$$

Then $\{A_i\}_{i\in\mathbb{N}}$ is a $\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{bmatrix}$-tight $*$-frame for Hilbert $\mathcal{A}$-module $\mathcal{A}$ but this is a frame for $\mathcal{A}$ with optimal lower and upper real bounds $\frac{1}{\sqrt{8}}$ and $\frac{1}{\sqrt{3}}$, respectively.

In the special case, that $\mathcal{A}$ is the Hilbert C^*-module over itself, the interesting results are revealed. For example, $*$-frames in $\mathcal{A}$ are sequences in $l_2(\mathcal{A})$ but some elements of $l_2(\mathcal{A})$ are not $*$-frames for $\mathcal{A}$. The following proposition and example consider these facts.

Proposition 1.1. *Let $\mathcal{A}$ be a Hilbert C^*-module over itself. Then the set of all $*$-frames for $\mathcal{A}$ is a subset of $l_2(\mathcal{A})$.*

Proof. Assume that $\{f_j\}_{j\in J}$ is a $*$-frame for $\mathcal{A}$. For $f \in \mathcal{A}$, we have

$$\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle = f(\sum_{j\in J}|f_j|^2)f^*.$$

Then $\sum_{j\in J}|f_j|^2$ converges and it implies that $\{f_j\}_{j\in J} \in l_2(\mathcal{A})$. □

Example 1.3. *Let c_0 be the Hilbert l^∞-module the same as in Example 1.1. For $j \in J$, consider $f_j = \{f_i^j\}_{i\in\mathbb{N}}$ such that* $f_i^j = \begin{cases} \frac{1}{i} & i = j, \\ 0 & i \neq j. \end{cases}$ *If $u = \{u_i\}_{i\in\mathbb{N}}$ is a sequence in c_0, then we have*

$$\sum_{j\in J}\langle u, f_j\rangle\langle f_j, u\rangle = \{\frac{|u_i|^2}{i}\}_{i\in\mathbb{N}} = \{\frac{1}{i}\}_{i\in\mathbb{N}}\langle\{u_i\}_{i\in\mathbb{N}}, \{u_i\}_{i\in\mathbb{N}}\rangle\{\frac{1}{i}\}_{i\in\mathbb{N}}.$$

Since $\{\frac{1}{i}\}_{i\in\mathbb{N}}$ is not strictly nonzero in l^∞, the sequence $\{f_j\}_{j\in J}$ has not lower bound condition in l^∞ and then it is not a $*$-frame for c_0 but $\{f_j\}_{j\in J} \in l_2(l^\infty)$. On the other hand, $\{f_j\}_{j\in J}$ is a $*$-Bessel sequence with $*$-Bessel bound $\{1, \varepsilon + \frac{1}{i}\}_{i\geq 2}$ in l^∞ for $\varepsilon > 0$ and so this is a Bessel sequence for c_0 with optimal bound 1. It is interesting that $\{1, \varepsilon + \frac{1}{i}\}_{i\geq 2} < 1$, means that $l^\infty-$valued $*$-Bessel bound is less than real-valued Bessel bound.

1.2. Operators Corresponding to $*$-Frames

Similar to the ordinary frames, we introduce the pre-$*$-frame operator and $*$-frame operator for $*$-frames and state some of the important properties of them as follows.

Theorem 1.1. *Let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $\mathcal{H}$ with lower and upper $*$-frame bounds A and B, respectively. The $*$-frame transform or pre-$*$-frame operator $T : \mathcal{H} \longrightarrow l_2(\mathcal{A})$ defined by $T(f) = \{\langle f, f_j\rangle\}_{j\in J}$ is an injective and closed range adjointable $\mathcal{A}$-module map and $\|T\| \leq \|B\|$. The adjoint operator T^* is surjective and it is given by $T^*(e_j) = f_j$ for $j \in J$ where $\{e_j : j \in J\}$ is the standard basis for $l_2(\mathcal{A})$.*

Proof. By the definition of norm in $l_2(\mathcal{A})$,

$$\|Tf\|^2 = \|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\| \leq \|B\|^2\|\langle f, f\rangle\|, \quad \forall f \in \mathcal{H}.$$

This inequality implies that T is well defined and $\|T\| \leq \|B\|$. Clearly, T is a linear $\mathcal{A}$-module map. We now show that R_T is closed. Let $\{Tf_n\}_{n\in\mathbb{N}}$ be a sequence in R_T such that $Tf_n \longrightarrow g$ as $n \to \infty$. By (1.1), we have $\|A\langle f_n - f_m, f_n - f_m\rangle A^*\| \leq \|T(f_n - f_m)\|^2$. Since $\{Tf_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in $l_2(\mathcal{A})$, $\|A\langle f_n - f_m, f_n - f_m\rangle A^*\| \longrightarrow 0$ as $n, m \to \infty$. Note that for $n, m \in \mathbb{N}$,

$$\|\langle f_n - f_m, f_n - f_m\rangle\| \leq \|A^{-1}\|^2\|A\langle f_n - f_m, f_n - f_m\rangle A^*\|.$$

Therefore the sequence $\{f_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in $\mathcal{H}$ and hence there exists $f \in \mathcal{H}$ such that $f_n \longrightarrow f$ as $n \to \infty$. Again by the definition of $*$-frames, we obtain $\|T(f_n - f)\|^2 \leq \|B\|^2\|\langle f_n - f, f_n - f\rangle\|$. Thus $\|Tf_n - Tf\| \longrightarrow 0$ as $n \to \infty$ implies that $Tf = g$. It concludes that R_T is closed. In order to show that T is injective, suppose that $f \in \mathcal{H}$ and $Tf = 0$. By (1.1),

$$\|\langle f, f\rangle\| = \|A^{-1}A\langle f, f\rangle A^*(A^*)^{-1}\| \leq \|A^{-1}\|^2\|Tf\|^2.$$

Thus $f = 0$ and T is injective. To determine the adjoint operator T^*, consider the equalities $\langle Tf, e_k\rangle = \langle\{\langle f, f_j\rangle\}_{j\in J}, e_k\rangle = \langle f, f_k\rangle$, for all $k \in J$ and $f \in \mathcal{H}$. Now, given $f \in \mathcal{H}$ and $\{a_j\}_{j\in J} \in l_2(\mathcal{A})$,

$$\langle\{a_j\}_{j\in J}, Tf\rangle = \sum_{j\in J} a_j\langle f, f_j\rangle^* = \langle\sum_{j\in J} a_j f_j, f\rangle.$$

This implies that $\sum_{j\in J} a_j f_j$ converges in $\mathcal{H}$ and $T^*(\{a_j\}_{j\in J}) = \sum_{j\in J} a_j f_j$ for every $\{a_j\}_{j\in J} \in l_2(\mathcal{A})$. By injectivity of T,the operator T^* has closed range and $\mathcal{H} = R_{T^*}$, which completes the proof. □

Now, we are ready to define $*$-frame operator and compare its properties with ordinary case.

Definition 1.2. *Let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $\mathcal{H}$ with pre-$*$-frame operator T and lower and upper $*$-frame bounds A and B, respectively. The $*$-frame operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by $Sf = T^*Tf = \sum_{j\in J}\langle f, f_j\rangle f_j$.*

The $*$-frame operator has some similar properties with frame operator in ordinary frames, but the other properties are different. The main cause of differences is $\mathcal{A}$-valued bounds. However, the reconstruction formula is given from the $*$-frame operator.

Theorem 1.2. *Let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $\mathcal{H}$ with $*$-frame operator S and lower and upper $*$-frame bounds A and B, respectively. Then S is positive, invertible and adjointable. Also, the following inequality $\|A^{-1}\|^{-2} \leq \|S\| \leq \|B\|^2$ holds, and the reconstruction formula $f = \sum_{j\in J}\langle f, S^{-1}f_j\rangle f_j$ holds $\forall f \in \mathcal{H}$. Moreover, $\{f_j\}_{j\in J}$ is a set of module generators of $\mathcal{H}$.*

Proof. By Lemma 0.1 and Theorem 1.1, S is invertible. Clearly, S is positive adjointable map. The definition of $*$-frames concludes that $\langle f, f\rangle \leq A^{-1}\langle Sf, f\rangle(A^*)^{-1}$ and $\langle Sf, f\rangle \leq B\langle f, f\rangle B^*$, and then

$$\|A^{-1}\|^{-2}\|\langle f, f\rangle\| \leq \|\langle Sf, f\rangle\| \leq \|B\|^2\|\langle f, f\rangle\|, \qquad \forall f \in \mathcal{H}.$$

If we take supremum on all $f \in \mathcal{H}$, where $\|f\| \leq 1$, then $\|A^{-1}\|^{-2} \leq \|S\| \leq \|B\|^2$.

The reconstruction formula concludes by the invertibility of S similar to ordinary frames. □

Finding optimal bounds plays an important role in the study of frames and $*$-frames. As we saw in the previous examples, $\mathcal{A}$-valued bounds may be more suitable than real valued bounds for a $*$-frame. In addition, there are tight $*$-frames that are not tight frames. At the end of the subsection, we introduce lower and upper real bounds for every $*$-frame and we see that $*$-frames can be studied as frames with different bounds.

Corollary 1.1. *Let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $\mathcal{H}$ with pre-$*$-frame operator T and lower and upper $*$-frame bounds A and B, respectively. Then $\{f_j\}_{j\in J}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $\|(T^*T)^{-1}\|^{-1}$ and $\|T\|^2$, respectively.*

Proof. By Theorem 1.1, T is injective and has closed range and by Lemma 0.1,

$$\|(T^*T)^{-1}\|^{-1}\langle f, f\rangle \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq \|T\|^2\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

Then $\{f_j\}_{j\in J}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $\|(T^*T)^{-1}\|^{-1}$ and $\|T\|^2$, respectively. □

The given results in the next sections are valid for frames in Hilbert C^*-modules by Theorem 1.1.

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1.3. $*$-Frames on Commutative C^*-Algebras

In ordinary Hilbert spaces, their inner product has complex values and the set of complex numbers is a commutative C^*-algebra. From this point of view, it seems that $*$-frames in a Hilbert C^*-module over a unitary commutative C^*-algebra have properties closed to ordinary frames. Therefore, we are going to study the properties of $*$-frames in these Hilbert $\mathcal{A}$-modules. Throughout this section, let $\mathcal{A}$ be a unitary commutative C^*-algebra. The $*$-frames will appear in the following form.

The sequence $\{f_j \in \mathcal{H} : j \in J\}$ is a $*$-frame for $\mathcal{H}$ if there exist two strictly positive elements A and B in $\mathcal{A}$ such that,

$$A\langle f, f\rangle \leq \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

Then $\sqrt{A}$ and $\sqrt{B}$ are lower and upper $*$-frame bounds. If there exists an element $f \in \mathcal{H}$ such that $\langle f, f\rangle$ is invertible in $\mathcal{A}$, then the above inequality implies that $A \leq B$.

Let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $\mathcal{H}$ with $*$-frame operator S and lower and upper $*$-frame bounds A and B, respectively. Then $\{f_j\}_{j\in J}$ is uniformly norm bounded by $\sqrt{\|B\|}$, $\{\langle f_j, f_j\rangle\}_{j\in J}$ is a bounded sequence of positive elements in $\mathcal{A}$ with $\mathcal{A}$-valued bound B, and $A \leq S \leq B$. Moreover, if $\{f_j\}_{j\in J}$ is a $\sqrt{A}$-tight $*$-frame with $*$-frame operator S, then $S = AI$ and its canonical dual $*$-frame is $\{A^{-1}f_j\}_{j\in J}$. One of the interesting results about $*$-frames for Hilbert $\mathcal{A}$-module $\mathcal{A}$ is as follows.

Proposition 1.2. *Let $\mathcal{A}$ be a Hilbert C^*-module over itself. Every $*$-frame $\{f_j\}_{j\in J}$ is a tight $*$-frame for $\mathcal{A}$.*

Proof. Suppose that $\{f_j\}_{j\in J}$ is a $*$-frame for $\mathcal{A}$ with $*$-frame operator S. By the invertibility of S, we have

$$1_{\mathcal{A}} = SS^{-1}1_{\mathcal{A}} = \sum_{j\in J}\langle S^{-1}1_{\mathcal{A}}, f_j\rangle f_j = S^{-1}1_{\mathcal{A}}\sum_{j\in J}|f_j|^2.$$

This equality shows that $\sum_{j\in J}|f_j|^2$ is an invertible element in $\mathcal{A}$ and then $\sum_{j\in J}|f_j|^2$ is a strictly positive element in $\mathcal{A}$. So the middle sum in (1.1) is

$$\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle = \sum_{j\in J}|f_j|^2\langle f, f\rangle, \qquad \forall f\in\mathcal{A}.$$

Then $\{f_j\}_{j\in J}$ is a $\sqrt{\sum_{j\in J}|f_j|^2}$−tight $*$-frame and this completes the proof. □

Proposition 1.3. *Let $\{f_j\in\mathcal{H} : j\in J\}$ be a $*$-frame for $\mathcal{H}$ with $*$-frame operator S and lower and upper $*$-frame bounds $\sqrt{A}$ and $\sqrt{B}$, respectively. Suppose that α is a strictly positive element in $\mathcal{A}$. Then the sequence $\{\alpha f_j\}_{j\in J}$ is a $*$-frame for $\mathcal{H}$ with $*$-frame operator $|\alpha|^2S$ and lower and upper $*$-frame bounds $\alpha\sqrt{A}$ and $\alpha\sqrt{B}$, respectively.*

Proof. For $f\in\mathcal{H}$, we have $\sum_{j\in J}\langle f, \alpha f_j\rangle\langle\alpha f_j, f\rangle = |\alpha|^2\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle$. Therefore $\{\alpha f_j\}_{j\in J}$ is a $*$-frame for $\mathcal{H}$ with lower and upper $*$-frame bounds $\alpha\sqrt{A}$ and $\alpha\sqrt{B}$, respectively. If S_α is $*$-frame operator $\{\alpha f_j\}_{j\in J}$, then

$$S_\alpha f = \sum_{j\in J}\langle f, \alpha f_j\rangle\alpha f_j = |\alpha|^2\sum_{j\in J}\langle f, f_j\rangle f_j = |\alpha|^2 Sf, \qquad \forall f\in\mathcal{H}.$$

□

2. Construction of Some New $*$-Frames

In this Section, we are going to construct new $*$-frames for Hilbert $\mathcal{A}$-module $\mathcal{A}$ and for new Hilbert C^*-modules by given $*$-frames. We will also study a family of full Hilbert C^*-modules by using new $*$-frames. The next theorem presents a collection of $*$-frames for Hilbert $\mathcal{A}$-module $\mathcal{A}$ associated to a given $*$-frame.

Theorem 2.1. *Let $\{f_j\in\mathcal{H} : j\in J\}$ be a $*$-frame for $\mathcal{H}$ with $*$-frame operator S and lower and upper $*$-frame bounds A and B in the center of $\mathcal{A}$. Suppose that f is an element in $\mathcal{H}$ such that $\langle f, f\rangle$ is an invertible element in the center of $\mathcal{A}$. Then the sequence $\{\langle f_j, f\rangle\}_{j\in J}$ is a $*$-frame for Hilbert $\mathcal{A}$-module $\mathcal{A}$ with lower and upper $*$-frame bounds $A\sqrt{\langle f, f\rangle}$ and $B\sqrt{\langle f, f\rangle}$, respectively. And its $*$-frame operator is $S_f a = a\langle Sf, f\rangle$ for $a\in\mathcal{A}$.*

Proof. For $a\in\mathcal{A}$, by the definition of $*$-frame $\{f_j\}_{j\in J}$

$$aA\langle f, f\rangle A^*a^* \le a(\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle)a^* \le aB\langle f, f\rangle B^*a^*, \tag{2.1}$$

and we have

$$\sum_{j\in J}\langle a,\langle f_j,f\rangle\rangle\langle\langle f_j,f\rangle,a\rangle = a(\sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle)a^*. \tag{2.2}$$

Since A, B and $\sqrt{\langle f,f\rangle}$ are in the center of $\mathcal{A}$ ([3, Theorem 6.2.10]) and by (2.1) and (2.2), the following inequalities are valid for all $a\in\mathcal{A}$

$$\begin{aligned} A\sqrt{\langle f,f\rangle}\langle a,a\rangle(A\sqrt{\langle f,f\rangle})^* &\le \sum_{j\in J}\langle a,\langle f_j,f\rangle\rangle\langle\langle f_j,f\rangle,a\rangle \\ &\le B\sqrt{\langle f,f\rangle}\langle a,a\rangle(B\sqrt{\langle f,f\rangle})^*. \end{aligned}$$

The last inequality shows that $\{\langle f_j,f\rangle\}_{j\in J}$ is a $*$-frame for Hilbert $\mathcal{A}$-module $\mathcal{A}$ with lower and upper $*$-frame bounds $A\sqrt{\langle f,f\rangle}$ and $B\sqrt{\langle f,f\rangle}$, respectively. To see S_f, let $a\in\mathcal{A}$. Then $S_f a=\sum_{j\in J}\langle a,\langle f_j,f\rangle\rangle\langle f_j,f\rangle = a\langle Sf,f\rangle$. □

Remark 2.1. *When $\mathcal{A}$ is commutative, $\{\langle f_j,f\rangle\}_{j\in J}$ is a $*$-frame if $\{f_j\}_{j\in J}$ is a $*$-frame and $\langle f,f\rangle$ is invertible.*

In this suitable situation, we find a necessary condition that $\mathcal{H}$ is a full Hilbert C^*-module.

Corollary 2.1. *Let $f\in\mathcal{H}$ and $\langle f,f\rangle$ be invertible in the center of $\mathcal{A}$. Then $\mathcal{H}$ is a full Hilbert $\mathcal{A}$-module.*

Proof. Applying Theorem 3.2 of [9], then there is a frame $\{f_j\}_{j\in J}$ for $\mathcal{H}$. It follows that $\overline{span}\{\langle f_j,f\rangle\}_{j\in J}=\mathcal{A}$ by Theorem 2.1. □

The following example shows that the above necessary condition is not sufficient, i.e., there is a full Hilbert $\mathcal{A}$-module $\mathcal{H}$ such that $\langle f,f\rangle$ is not invertible for all $f\in\mathcal{H}$.

Example 2.1. *Assume that c_0 is the same Hilbert l^∞-module as in Example 1.1 and let $\{e_j\}_{j\in\mathbb{N}}$ be the standard basis for c_0. The Hilbert l^∞-module c_0 is full because of*

$$\overline{span}\{\langle e_j,e_j\rangle\}_{j\in\mathbb{N}}=\overline{span}\{e_j\}_{j\in\mathbb{N}}=l^\infty$$

. Assume that $a=\{a_j\}_{j\in\mathbb{N}}\in c_0$, then $\langle a,a\rangle\in c_0$ and $\lim_{j\to\infty}|a_j|^2=0$. If $\langle a,a\rangle^{-1}$ exists, then $\lim_{j\to\infty}\frac{1}{|a_j|^2}=\infty$ and $\langle a,a\rangle^{-1}$ doesn't belong to l^∞. It shows that c_0 has not any element $\{a_j\}_{j\in\mathbb{N}}$ such that $\{\langle a_j,a_j\rangle\}_{j\in\mathbb{N}}$ is invertible.

In [14], the authors have shown that a tensor product of frames in Hilbert spaces is also a frame. We study the subject in Hilbert C^*-modules.

Theorem 2.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules over unitary C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\mathcal{F} = \{f_j \in \mathcal{H} : j \in J\}$ and $\mathcal{G} = \{g_j \in \mathcal{K} : j \in J\}$ be two $*$-frames for $\mathcal{H}$ and $\mathcal{K}$ with $*$-frame operators $S_{\mathcal{F}}$ and $S_{\mathcal{G}}$ and $*$-frame bounds (A, B) and (C, D), respectively. Then $\{f_i \otimes g_j\}_{i,j\in J}$ is a $*$-frame for Hilbert $\mathcal{A} \otimes \mathcal{B}-$module $\mathcal{H} \otimes \mathcal{K}$ with $*$-frame operator $S_{\mathcal{F}} \otimes S_{\mathcal{G}}$ and lower and upper $*$-frame bounds $A \otimes C$ and $B \otimes D$, respectively.*

Proof. By the definition of $*$-frames $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$, one obtains that

$$\begin{aligned} A\langle f, f\rangle A^* \otimes C\langle g, g\rangle C^* &\le \sum_{i\in J}\langle f, f_i\rangle\langle f_i, f\rangle \otimes \sum_{j\in J}\langle g, g_j\rangle\langle g_j, g\rangle \\ &\le B\langle f, f\rangle B^* \otimes D\langle g, g\rangle D^*, \qquad \forall f \in \mathcal{H}, \forall g \in \mathcal{K}. \end{aligned} \tag{2.3}$$

Then for $f \otimes g \in \mathcal{H} \otimes \mathcal{K}$, one gets

$$\begin{aligned} (A \otimes C)\langle f \otimes g, f \otimes g\rangle(A \otimes C)^* &\le \sum_{i\in J}\sum_{j\in J}\langle f \otimes g, f_i \otimes g_j\rangle\langle f_i \otimes g_j, f \otimes g\rangle \\ &\le (B \otimes D)\langle f \otimes g, f \otimes g\rangle(B \otimes D)^*. \end{aligned} \tag{2.4}$$

Moreover, the inequality (2.4) is satisfied for every finite sum of elements in $\mathcal{H} \otimes_{alg} \mathcal{K}$ and then (2.4) is satisfied for all $z \in \mathcal{H} \otimes \mathcal{K}$. It shows that $\{f_i \otimes g_j\}_{i,j\in J}$ is a $*$-frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper $*$-frame bounds $A \otimes C$ and $B \otimes D$, respectively. Now, to see the form of the $*$-frame operator for $\mathcal{F} \otimes \mathcal{G}$, let $S_{\mathcal{F}\otimes\mathcal{G}}$ be the $*$-frame operator $\mathcal{F} \otimes \mathcal{G}$. We compute

$$\begin{aligned} S_{\mathcal{F}\otimes\mathcal{G}}(f \otimes g) &= \sum_{i,j\in J} \langle f \otimes g, f_i \otimes g_j\rangle f_i \otimes g_j \\ &= \sum_{i\in J}\langle f, f_i\rangle f_i \otimes \sum_{j\in J}\langle g, g_j\rangle g_j = (S_{\mathcal{F}} \otimes S_{\mathcal{G}})(f \otimes g), \end{aligned}$$

for $f \otimes g \in \mathcal{H} \otimes \mathcal{K}$. So $S_{\mathcal{F}\otimes\mathcal{G}} = S_{\mathcal{F}} \otimes S_{\mathcal{G}}$ and this completes the proof of the theorem. □

Let $\theta \in B_*(\mathcal{H})$. We are going to give some necessary and sufficient conditions which provided the $*$-frame-preserving property of operator θ. The following theorem generalizes the results in [1] in the case of $*$-frames.

Theorem 2.3. *Let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $\mathcal{H}$ with $*$-frame operator S and lower and upper $*$-frame bounds A and B, respectively. Then $\theta \in B_*(\mathcal{H})$ is surjective if and only if $\{\theta f_j \in \mathcal{H} : j \in J\}$ is a $*$-frame for $\mathcal{H}$. In this case, $S_\theta := \theta S\theta^*$, $A\|(\theta\theta^*)^{-1}\|^{-\frac{1}{2}}$, and $B\|\theta\|$ are $*$-frame operator and lower and upper $*$-frame bounds for $\{\theta f_j\}_{j\in J}$, respectively.*

Proof. First, let θ be surjective. By the definition of $*$-frames, for all $f \in \mathcal{H}$, we have

$$A\langle\theta^* f, \theta^* f\rangle A^* \le \sum_{j\in J}\langle\theta^* f, f_j\rangle\langle f_j, \theta^* f\rangle \le B\langle\theta^* f, \theta^* f\rangle B^*,$$

and then

$$A\langle\theta\theta^* f, f\rangle A^* \leq \sum_{j\in J}\langle f, \theta f_j\rangle\langle\theta f_j, f\rangle \leq B\langle\theta\theta^* f, f\rangle B^*. \tag{2.5}$$

Surjectivity of θ and Lemma 0.1 conclude that

$$\|(\theta\theta^*)^{-1}\|^{-1}\langle f, f\rangle \leq \langle\theta\theta^* f, f\rangle \leq \|\theta\|^2\langle f, f\rangle, \quad \forall f \in \mathcal{H}.$$

Multiplying with A, A^* and B, B^* on left and right parts of the last inequality, respectively, we obtain

$$A\|(\theta\theta^*)^{-1}\|^{-1}\langle f, f\rangle A^* \leq A\langle\theta\theta^* f, f\rangle A^*, \ \ \textit{and} \ \ B\langle\theta\theta^* f, f\rangle B^* \leq B\|\theta\|\langle f, f\rangle B^*\|\theta\|.$$

Using (2.5), it follows that

$$A\|(\theta\theta^*)^{-1}\|^{-\frac{1}{2}}\langle f, f\rangle(A\|(\theta\theta^*)^{-1}\|^{-\frac{1}{2}})^* \leq \sum_{j\in J}\langle f, \theta f_j\rangle\langle\theta f_j, f\rangle \leq B\|\theta\|\langle f, f\rangle(B\|\theta\|)^*,$$

holds for every $f \in \mathcal{H}$. Thus $\{\theta f_j\}_{j\in J}$ is a $*$-frame for $\mathcal{H}$. The proof of the rest of the theorem is similar to the Hilbert space case [5, Proposition 5.3.1]. □

Let S be a positive and invertible operator in the C^*-algebra $B_*(\mathcal{H})$. For $t \in \mathbb{R}$, the map $f(\lambda) = \lambda^t$ is continuous on $(0, \infty)$. Since S is positive and invertible, $\sigma(S) \subseteq (0, \infty)$. Using the Spectral Mapping theorem and the fact that $f \in C(\sigma(S))$, we have $f(S) \in B_*(\mathcal{H})$. Now, $f(S)$ is denoted by S^t and we are ready to extend Theorem 3.1. [4] to the next corollary.

Corollary 2.2. *Let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $\mathcal{H}$ with $*$-frame operator S and lower and upper $*$-frame bounds A and B , respectively. For $t \in \mathbb{R}$, the sequence $\{S^{\frac{t-1}{2}} f_j\}_{j\in J}$ is a $*$-frame for $\mathcal{H}$ with lower and upper $*$-frame bounds $A\|S^{1-t}\|^{-\frac{1}{2}}$ and $B\|S^{\frac{t-1}{2}}\|$, respectively. Moreover, S^t is its $*$-frame operator.*

Proof. For the proof we use the functional calculus for the selfadjoint element S of the C^*-algebra $B_*(\mathcal{H})$ to write $S = S^{(t-1)/2}S^{(3-t)/2}$, and apply the previous theorem for $\theta = S^{(t-1)}$. □

3. $*$-Frames in Modular Spaces with Different C^*-Algebras

Studying frames in Hilbert C^*-modules with different inner products is interesting and important. Frank-Larson [10] studied frames in Hilbert $\mathcal{A}$-modules constructed by two different $\mathcal{A}$-valued inner products. However, we study $*$-frames in two Hilbert C^*-modules with different C^*-algebras. Throughout this section, assume that $\mathcal{A}$ and $\mathcal{B}$ are two unitary C^*-algebras and let $(\mathcal{H}, \mathcal{A}, \langle\cdot,\cdot\rangle_{\mathcal{A}})$ and $(\mathcal{H}, \mathcal{B}, \langle\cdot,\cdot\rangle_{\mathcal{B}})$ be two Hilbert C^*-modules.

First, we are going to modify the proof the result [10] to show that the theorem remains valid under slightly weaker conditions.

Proposition 3.1. *Suppose that $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_1)$ is a Hilbert $\mathcal{A}$-module and let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_1)$ with $*$-frame operator S_1. Then $\{f_j\}_{j \in J}$ is a $*$-frame for $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_2)$, that has an equivalent norm to the given one, with $*$-frame operator S_2 if and only if there exists an injective and adjointable operator θ with closed range on $\mathcal{H}$ such that $\langle f, g \rangle_1 = \langle \theta f, \theta g \rangle_2$ for all $f, g \in \mathcal{H}$. Furthermore, the given operator θ is self-adjoint with respect to both inner products, and the equality $\langle f, S_2^{-1} g \rangle_2 = \langle f, S_1^{-1} g \rangle_1$ holds for all $f, g \in \mathcal{H}$, the $*$-frame operator S_1 commutes with the operator S_2^{-1}, and the $*$-frame operators S_1 and S_2 are self-adjoint with respect to both inner products.*

Proof. Proof of the 'if' part follows that [10]. For the converse, let $\theta \in B_*(H)$ be injective with closed range on $\mathcal{H}$ such that

$$\langle f, g \rangle_1 = \langle \theta f, \theta g \rangle_2, \quad \forall f, g \in \mathcal{H}.$$

The operator $\theta^* \theta$ is invertible by Lemma 0.1 and then for $f, h \in \mathcal{H}$, we obtain

$$\langle f, (\theta^* \theta)^{-1} h \rangle_1 = \langle f, h \rangle_2, \quad \forall f, h \in \mathcal{H}. \tag{3.1}$$

Now, we can give the result by Proposition 0.1, Lemma 0.1 and (3.1). □

Now, we consider $*$-frames in two Hilbert C^*-modules with different C^*-algebras and the same vector spaces.

Theorem 3.1. *Let $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_\mathcal{A})$ and $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_\mathcal{B})$ be two Hilbert C^*-modules and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $*$-homomorphism and θ be a map on $\mathcal{H}$ such that $\langle \theta f, \theta g \rangle_\mathcal{B} = \varphi(\langle f, g \rangle_\mathcal{A})$ for all $f, g \in \mathcal{H}$. Also, suppose that $\{f_j \in \mathcal{H} : j \in J\}$ is a $*$-frame for $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_\mathcal{A})$ with $*$-frame operator $S_\mathcal{A}$ and lower and upper $*$-frame bounds α_1, α_2, respectively. If θ is surjective, then $\{\theta f_j\}_{j \in J}$ is a $*$-frame for $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_\mathcal{B})$ with $*$-frame operator $S_\mathcal{B}$ and lower and upper $*$-frame bounds $\varphi(\alpha_1), \varphi(\alpha_2)$, respectively, and*

$$\langle S_\mathcal{B} \theta f, \theta g \rangle_\mathcal{B} = \varphi(\langle S_\mathcal{A} f, g \rangle_\mathcal{A}), \qquad \forall f \in \mathcal{H}. \tag{3.2}$$

Moreover, the map θ is surjective if the following conditions are valid.

(1) *φ is surjective;*

(2) *$\{\theta f_j\}_{j \in J}$ is a $*$-frame for $\mathcal{H}$; and*

(3) *$\theta(af) = \varphi(a) \theta f$, for all $a \in \mathcal{A}$, $f \in \mathcal{H}$.*

Proof. Assume that θ is surjective. Using Proposition 0.1, we have that

$$\sum_{j\in J}\langle \theta f,\theta f_j\rangle_{\mathcal{B}}\langle \theta f_j,\theta f\rangle_{\mathcal{B}}=\sum_{j\in J}\varphi(\langle f,f_j\rangle_{\mathcal{A}}\langle f_j,f\rangle_{\mathcal{A}})$$
$$\leq \varphi(\alpha_2\langle f,f\rangle_{\mathcal{A}}\alpha_2^*)=\varphi(\alpha_2)\langle \theta f,\theta f\rangle_{\mathcal{B}}\varphi(\alpha_2)^*,\quad \forall f\in\mathcal{H},$$

and $\varphi(\alpha_2)$ is a strictly nonzero element of $\mathcal{B}$. Then the sequence $\{\theta f_j\}_{j\in J}$ has upper $*$-frame bound $\varphi(\alpha_2)$. Similarly, $\varphi(\alpha_1)$ is a lower $*$-frame bound for $\{\theta f_j\}_{j\in J}$ and then $\{\theta f_j\}_{j\in J}$ is a $*$-frame for $(\mathcal{H},\mathcal{B},\langle\cdot,\cdot\rangle_{\mathcal{B}})$. The equation (3.2) follows from

$$\sum_{j\in J}\langle \theta f,\theta f_j\rangle_{\mathcal{B}}\langle \theta f_j,\theta g\rangle_{\mathcal{B}}=\varphi(\sum_{j\in J}\langle f,f_j\rangle_{\mathcal{A}}\langle f_j,g\rangle_{\mathcal{A}}),\qquad \forall f,g\in\mathcal{H}.$$

For the rest of the proof, let φ be surjective and $\theta(af)=\varphi(a)\theta f$, for all $a\in\mathcal{A}$ and $f\in\mathcal{H}$. By applying the reconstruction formula for $*$-frame $\{\theta f_j\}_{j\in J}$, we have $g=\sum_{j\in J}\langle g,S_{\mathcal{B}}^{-1}\theta f_j\rangle_{\mathcal{B}}\theta f_j$ for $g\in\mathcal{H}$. Since φ is surjective, $\varphi(a_j)=\langle g,S_{\mathcal{B}}^{-1}\theta f_j\rangle_{\mathcal{B}}$ for some $a_j\in\mathcal{A}$ and for all $j\in J$. Observe that $g=\sum_{j\in J}\varphi(a_j)\theta f_j=\sum_{j\in J}\theta(a_jf_j)=\theta(\sum_{j\in J}a_jf_j)$. This shows that θ is surjective and the proof is complete. □

Corollary 3.1. *Let $(\mathcal{H},\mathcal{A},\langle\cdot,\cdot\rangle_1)$ and $(\mathcal{H},\mathcal{A},\langle\cdot,\cdot\rangle_2)$ be two Hilbert $\mathcal{A}$-modules which have equivalent norms and let $\varphi:\mathcal{A}\longrightarrow\mathcal{A}$ be a $*$-homomorphism such that $\varphi(\langle f,g\rangle_1)=\langle f,g\rangle_2$ for all $f,g\in\mathcal{H}$. Then $\langle f,g\rangle_1=\langle f,g\rangle_2$ for all $f,g\in\mathcal{H}$. Therefore, if $(\mathcal{H},\mathcal{A},\langle\cdot,\cdot\rangle_1)$ is full, then φ is the identity map on $\mathcal{A}$.*

Proof. By [9], $(\mathcal{H},\mathcal{A},\langle\cdot,\cdot\rangle_1)$ contains a $*$-frame with $*$-frame operator S_1. In Theorem 3.1, set $\theta=I_{\mathcal{H}}$. Then $\{f_j\}_{j\in J}$ is also a $*$-frame for $(\mathcal{H},\mathcal{A},\langle\cdot,\cdot\rangle_2)$ with $*$-frame operator S_2. By Theorem 3.1 and Proposition 3.1, we conclude that $\langle S_2f,g\rangle_2=\varphi(\langle S_1f,g\rangle_1)=\langle S_1f,g\rangle_2$, for all $f,g\in\mathcal{H}$. Then $S_1=S_2$ and the two $\mathcal{A}$-valued inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are the same by Proposition 3.1. Now, assume that $(\mathcal{H},\mathcal{A},\langle\cdot,\cdot\rangle_1)$ is full. Thus $\overline{span}\{\langle f,g\rangle_1:f,g\in\mathcal{H}\}=\mathcal{A}$ and by the properties of φ, φ is the identity. □

Corollary 3.2. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{H}$, $\{f_j\}_{j\in J}$ and φ be as in Theorem 3.1. Also, let θ be a $\mathcal{B}$-module map on $\mathcal{H}$ such that $\varphi(\langle f,g\rangle_{\mathcal{A}})=\langle\theta f,\theta g\rangle_{\mathcal{B}}$. Then θ is surjective if and only if $\{\theta f_j\}_{j\in J}$ is a $*$-frame for $(\mathcal{H},\mathcal{B},\langle\cdot,\cdot\rangle_{\mathcal{B}})$.*

Proof. Proof of the 'if part' is similar to the proof of Theorem 3.1. For the converse, since θ is $\mathcal{B}$-module map, $g=\sum_{j\in J}\langle g,S_{\mathcal{B}}^{-1}\theta f_j\rangle_{\mathcal{B}}\theta f_j=\theta(\sum_{j\in J}\langle g,S_{\mathcal{B}}^{-1}\theta f_j\rangle_{\mathcal{B}}f_j)$, for $g\in\mathcal{H}$, and it completes the proof. □

We are ready to characterize the set of $*$-frames for $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ with respect to $*$-frames in $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$. The following propositions illustrate this fact.

Proposition 3.2. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{H}$ be the same in Theorem 3.1. If φ is a $*$-isomorphism and θ is surjective map on $\mathcal{H}$ such that $\varphi(\langle f, g \rangle_{\mathcal{A}}) = \langle \theta f, \theta g \rangle_{\mathcal{B}}$, then the set of all $*$-frames for $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ is precisely $\{\theta f_j\}_{j \in J}$ where $\{f_j\}_{j \in J}$ is a $*$-frame for $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$.*

Proof. Theorem 3.1 concludes that the sequence $\{\theta f_j\}_{j \in J}$ is a $*$-frame for $(\mathcal{H}, \mathcal{B}\ , \langle \cdot, \cdot \rangle_{\mathcal{B}})$ if $\{f_j\}_{j \in J}$ is a $*$-frame for $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$. Now, assume that $\{g_j\}_{j \in J}$ is a $*$-frame for $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ with lower and upper $*$-frame bounds β_1 and β_2. By the properties of θ, and Proposition 0.1, there exist the sequence $\{f_j\}_{j \in J}$ in $\mathcal{H}$ and two elements α_1, α_2 in $\mathcal{A}$ such that $g_j = \theta f_j$ for $j \in J$, $\varphi(\alpha_1) = \beta_1$, and $\varphi(\alpha_2) = \beta_2$. The elements α_1 and α_2 are strictly nonzero by Proposition 0.1. Using the definition of the $*$-frame $\{g_j\}_{j \in J}$, we have

$$\begin{aligned}\varphi(\sum_{j \in J} \langle f, f_j \rangle_{\mathcal{A}} \langle f_j, f \rangle_{\mathcal{A}}) &= \sum_{j \in J} \langle \theta f, \theta f_j \rangle_{\mathcal{B}} \langle \theta f_j, \theta f \rangle_{\mathcal{B}} \\ &\leq \beta_2 \langle \theta f, \theta f \rangle_{\mathcal{B}} \beta_2^* = \varphi(\alpha_2 \langle f, f \rangle_{\mathcal{A}} \alpha_2^*), \qquad \forall f \in \mathcal{H}.\end{aligned}$$

We apply Proposition 0.1 again, $\sum_{j \in J} \langle f, f_j \rangle_{\mathcal{A}} \langle f_j, f \rangle_{\mathcal{A}} \leq \alpha_2 \langle f, f \rangle_{\mathcal{A}} \alpha_2^*$, for $f \in \mathcal{H}$. Similarly, α_1 is a lower $*$-frame bound for $\{f_j\}_{j \in J}$. This shows that every $*$-frame in $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ is obtained by the action of θ on a $*$-frame in $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$. □

Also, we can characterize all $*$-frames in the Hilbert $\mathcal{B}$-module $\mathcal{B}$ with respect to all $*$-frames in the Hilbert $\mathcal{A}$-module $\mathcal{A}$ and obtain some relations between their operators.

Proposition 3.3. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $*$-isomorphism. The set of all of $*$-frames for the Hilbert $\mathcal{B}$-module $\mathcal{B}$ is precisely $\{\varphi(a_j)\}_{j \in J}$, where $\{a_j\}_{j \in J}$ is a $*$-frame for the Hilbert $\mathcal{A}$-module $\mathcal{A}$. Moreover, if $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ are $*$-frame operators for $\{a_j\}_{j \in J}$ and $\{\varphi(a_j)\}_{j \in J}$, respectively, then $\varphi o S_{\mathcal{A}} = S_{\mathcal{B}} o \varphi$.*

Proof. For a sequence $\{a_j\}_{j \in J}$ in $\mathcal{A}$, we have

$$\sum_{j \in J} \langle \varphi(a), \varphi(a_j) \rangle_{\mathcal{B}} \langle \varphi(a_j), \varphi(a) \rangle_{\mathcal{B}} = \varphi(\sum_{j \in J} \langle a, a_j \rangle_{\mathcal{A}} \langle a_j, a \rangle_{\mathcal{A}}), \quad \forall a \in \mathcal{A}. \tag{3.3}$$

Proposition 0.1 and the above equalities imply that $\{\varphi(a_j)\}_{j \in J}$ is a $*$-frame for $\mathcal{B}$ if $\{a_j\}_{j \in J}$ is a $*$-frame for $\mathcal{A}$. Now, suppose $\{b_j\}_{j \in J}$ is a $*$-frame for $\mathcal{B}$. Since φ is surjective, there exists a sequence $\{a_j\}_{j \in J}$ in $\mathcal{A}$ such that $b_j = \varphi(a_j)$ for $j \in J$. Also, applying Proposition 0.1 and (3.3), we obtain that $\{a_j\}_{j \in J}$ is a $*$-frame for $\mathcal{A}$. For the rest of the proof, let $S_{\mathcal{A}}$

and $S_{\mathcal{B}}$ be $*$-frame operators for $\{a_j\}_{j\in J}$ and $\{\varphi(a_j)\}_{j\in J}$, respectively. Then $\varphi S_{\mathcal{A}}(a) = \varphi(\sum_{j\in J} aa_j^* a_j) = S_{\mathcal{B}}\varphi(a)$, for all $a \in \mathcal{A}$, and $\varphi o S_{\mathcal{A}} = S_{\mathcal{B}} o \varphi$. □

4. The dual $*$-frames

The dual frames play an important role to study of frames. In this section, we introduce dual $*$-frames and extend the characterization of dual frames [5] to dual $*$-frames associated to a given $*$-frame.

Definition 4.1. *Let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $\mathcal{H}$ with $*$-frame operator S. If there exists a $*$-frame $\{g_j \in \mathcal{H} : j \in J\}$ for $\mathcal{H}$ such that $f = \sum_{j\in J}\langle f, g_j\rangle f_j$ for $f \in \mathcal{H}$, then the $*$-frame $\{g_j\}_{j\in J}$ is called the dual $*$-frame of $\{f_j\}_{j\in J}$. The spacial dual $*$-frame $\{S^{-1}f_j\}_{j\in J}$ is said to be the canonical dual $*$-frame of $\{f_j\}_{j\in J}$.*

It is well known, that if T and V are pre-$*$-frame operators of two $*$-Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$, respectively, then $f = \sum_{j\in J}\langle f, g_j\rangle f_j$ for $f \in \mathcal{H}$ if and only if $T^*V = id_{\mathcal{H}}$. The following lemma shows that the roles of two $*$-Bessel sequences can be changed and obtains a relation between bounds of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$.

Lemma 4.1. *Let $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be $*$-Bessel sequences for $\mathcal{H}$ with pre-$*$-frame operators T and V, respectively. Then for $f \in \mathcal{H}$ the following statements are equivalent.*

i. $f = \sum_{j\in J}\langle f, g_j\rangle f_j$.

ii. $f = \sum_{j\in J}\langle f, f_j\rangle g_j$.

In the case that one of the above equalities is satisfied, $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are dual $*$-frames. Moreover, if B is an upper $*$-frame bound for $\{f_j\}_{j\in J}$ and S is its $*$-frame operator, then $B\|S^{-1}\|^{-\frac{1}{2}}\|T\|^{-1}$ is a lower $*$-frame bound for $\{g_j\}_{j\in J}$.

Proof. The proof of the equivalency of the two conditions is similar to the proof of [5, Lemma 5.6.2]. Now, suppose that the conditions i and ii are valid. By i, we have $T^*V = id_{\mathcal{H}}$ and T^* is surjective. Then it follows that the sequence $\{f_j\}_{j\in J}$ is a $*$-frame [12]. Similarly, the $*$-Bessel sequence $\{g_j\}_{j\in J}$ is a $*$-frame.

Finally, let B be an upper $*$-frame bound for $\{f_j\}_{j\in J}$. By the definition of $*$-frames $\{f_j\}_{j\in J}$ and $T^*V = id_{\mathcal{H}}$, we can write $\langle Tf, Tf\rangle \le B\langle T^*Vf, T^*Vf\rangle B^*$, for $f \in \mathcal{H}$. Using Lemma 0.1, we have $\|(T^*T)^{-1}\|^{-1}\langle f, f\rangle \le \langle Tf, Tf\rangle$, for $f \in \mathcal{H}$. It follows that $B^{-1}\|S^{-1}\|^{-\frac{1}{2}}\|T\|^{-1}\langle f, f\rangle(B^{-1}\|S^{-1}\|^{-\frac{1}{2}}\|T\|^{-1})^* \le \langle Vf, Vf\rangle$, for $f \in \mathcal{H}$. Therefore, $B\|S^{-1}\|^{-\frac{1}{2}}\|T\|^{-1}$ is a lower $*$-frame bound for $\{g_j\}_{j\in J}$ and the proposition follows. □

Proposition 4.1. *Let $\{f_j \in \mathcal{H} : j \in J\}$ be a $*$-frame for $\mathcal{H}$ with pre-$*$-frame operator T. The set of all of dual $*$-frames for $\{f_j\}_{j\in J}$ is precisely the set of the families $\{g_j\}_{j\in J} = \{V^*(e_j)\}_{j\in J}$, where $V : \mathcal{H} \longrightarrow l_2(\mathcal{A})$ is an adjointable right-inverse of T^* and the sequence $\{e_j\}_{j\in J}$ is the standard basis for $l_2(\mathcal{A})$.*

Proof. By Lemma 4.1 and [12, Proposition 3.11], the proof is clear. □

Now, we can characterize all dual $*$-frames for a given $*$-frame $\{f_j \in \mathcal{H} : j \in J\}$ with respect to $*$-Bessel sequences, similar to [5]. First, it follows from Proposition 4.1 that every right- inverse of pre-$*$-frame operator T of $\{f_j\}_{j\in J}$ has the form $TS^{-1} + (I - TS^{-1}T^*)U$, where S is the $*$-frame operator and U is an adjointable operator from $\mathcal{H}$ into $l_2(\mathcal{A})$. In the end, all of dual $*$-frames of $\{f_j\}_{j\in J}$ are precisely the families

$$\{g_j\}_{j\in J} = \{S^{-1}f_j + h_j - \sum_{i\in J}\langle S^{-1}f_j, f_i\rangle h_i\}_{j\in J}$$

where $\{h_j\}_{j\in J}$ is a $*$-Bessel sequence for $\mathcal{H}$.

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