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# ON CERTAIN GEOMETRIC PROPERTIES OF NORMALIZED MITTAG-LEFFLER FUNCTIONS

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*The main aim of this paper is to study some geometric properties of Mittag-Leffler functions. We focus on close-to-convexity, starlikeness, strongly starlikeness, strongly convexity and unifomly convexity of normalized Mittag-Leffler functions.*

**Keywords:** Analytic functions, Normalized Mittag-Leffler Functions, Strongly convexity, Strongly starlikeness, Uniformly convex functions, Close-to-convex

**MSC 2010:** 30C45, 33E12.

## 1. Introduction

The one parameter Mittag-Leffler function $\mathbb{E}_\alpha$ defined by

$$\mathbb{E}_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1}$$

was introduced by Mittag-Leffler [7]. It is an entire function of complex variables $z$. The series in (1) converges in the whole complex plane $\mathbb{C}$ when $Re(\alpha) > 0$. The function $\mathbb{E}_{\alpha,\beta}$ is a generalization of $\mathbb{E}_\alpha$. It is given as

$$\mathbb{E}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \alpha, \ \beta \in \mathbb{C}, \ Re(\alpha) > 0, \ Re(\beta) > 0, \ z \in \mathbb{C}. \tag{2}$$

For different values of parameters, the conditions of convergence vary. When $\alpha$ and $\beta$ are positive and real, the above given series converges in the entire complex plane. The Mittag-Leffler functions defined in (2) were first appeared in the work of Wiman [13]. Later on, these functions were studied by Agarwal [2]. In some recent years, researchers has shown great interest in the geometric properties and applications of the Mittag-Leffler functions. For some details see [3, 6, 9, 12, 14].

Let $\mathcal{A}$ be the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{3}$$

analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Let $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $\mathcal{K}(\alpha)$, $\widetilde{\mathcal{S}}^*(\alpha)$ and $\widetilde{\mathcal{C}}(\alpha)$ denote the classes of

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starlike, convex, close-to-convex and strongly starlike functions of order $\alpha$ respectively and are analytically defined as

$$\begin{aligned}
\mathcal{S}^*(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathcal{U}, \, \alpha \in [0,1) \right\}, \\
\mathcal{C}(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } \; zf'(z) \in \mathcal{S}^*(\alpha), \quad z \in \mathcal{U}, \, \alpha \in [0,1) \right\}, \\
\mathcal{K}(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } \; Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \quad z \in \mathcal{U}, \, \alpha \in [0,1), \; g \in \mathcal{S}^* \right\}, \\
\widetilde{\mathcal{S}^*}(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } \left| \arg\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\alpha\pi}{2}, \quad z \in \mathcal{U}, \, \alpha \in (0,1] \right\},
\end{aligned}$$

and

$$\widetilde{\mathcal{C}}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } zf'(z) \in \widetilde{\mathcal{S}^*}(\alpha) \quad z \in \mathcal{U}, \, \alpha \in (0,1] \right\}.$$

It is clear that $\widetilde{\mathcal{S}^*}(1) = \mathcal{S}^*(0) = \mathcal{S}^*$, $\mathcal{C}(0) = \widetilde{\mathcal{C}}(1) = \mathcal{C}$ and $\mathcal{K}(0) = \mathcal{K}$, where $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ are usual classes of starlike, convex and close-to-convex functions respectively. Rosy *et al.* [10] defined the subclass $\mathcal{USD}(\mu)$ of normalized univalent functions as

$$\mathcal{USD}(\mu) = \left\{ f \in \mathcal{A} : Re\left(f'(z)\right) \geq \mu \left| zf''(z) \right|, \; \mu > 0, \, z \in \mathcal{U} \right\}.$$

Let $f$ and $g$ be analytic functions. Then the function $f$ is said to be subordinate to $g$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w$ which is analytic with $w(0) = 0$ and $|w| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g$ is univalent in $\mathcal{U}$, then $f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad$ and $\quad f(\mathcal{U}) \subset g(\mathcal{U})$.

The function $\mathbb{E}_{\alpha,\beta}$ does not belong to the class $\mathcal{A}$. We consider the following normalized form of Mittag-Leffler functions

$$E_{\alpha,\beta}(z) = \Gamma(\beta) \, z\mathbb{E}_{\alpha,\beta}(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)} z^{n+1}, \tag{4}$$

where $\alpha, \; \beta \in \mathbb{C}, \; Re(\alpha) > 0, \; \beta \neq -1, -2, \cdots$. The function defined in (4) was introduced by Bansal and Prajapat [3]. In this paper, we restrict ourself on the parameters $\alpha, \; \beta \in \mathbb{R}$.

To prove our main results, we need the following lemmas.

**Lemma 1.1.** [10] *A function $f$ of the form* (3) *is in the class $\mathcal{USD}(\mu)$, if*

$$\sum_{n=2}^{\infty} \left[ n(1-\mu) + n^2\mu \right] |a_n| \leq 1.$$

**Lemma 1.2.** [8] *Let $\{a_n\}_{n=1}^{\infty}$ be the sequence such that $a_n \in \mathbb{R}^+$ and $a_1 = 1$. Suppose that, $a_1 \geq 8a_2$ and $(n-1)a_{n-1} \geq (n+1)a_{n+1}$, for all $n \geq 2$. Then $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ is close-to-convex with respect to $z/1 - z^2$. Note that $z/1 - z^2$ is a starlike function.*

**Lemma 1.3.** [8] *Let $\{a_n\}_{n=1}^{\infty}$ be the sequence such that $a_n \in \mathbb{R}^+$ and $a_1 = 1$. If for $0 \leq \mu < 1$,*

(1) $(1-\mu)a_1 \geq (2-\mu)a_2 \geq 2^{\mu+1}(3-\mu)a_3$,

(2) $(n-1-\mu)(n-\mu)a_n \geq n(n+1-\mu)a_{n+1}$, *for all* $n \geq 3$,

*then $f \in S^*(\mu)$.*

**Lemma 1.4.** [5] *Let $G(z)$ be convex and univalent in the open unit disc with condition $G(0) = 1$. Let $F(z)$ be analytic in the open unit disc with condition $F(0) = 1$ and $F \prec G$ in the open unit disc. Then $\forall \; n \in \mathbb{N} \cup \{0\}$, we obtain*

$$(n+1)z^{-1-n} \int_0^z t^n F(t)dt \prec (n+1)z^{-1-n} \int_0^z t^n G(t)dt.$$

**Lemma 1.5.** [11] *If $f \in \mathcal{A}$ satisfies* $\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}$, *then* $f \in \mathcal{UCV}$.

## 2. Main Results

**Theorem 2.1.** *Let* $0 \leq \gamma \leq 1, \mu > 0$. *Then* $(1-\gamma) E_{\alpha,\beta}(z) + \gamma z E'_{\alpha,\beta}(z) = h_\gamma(z) \in \mathcal{USD}(\mu)$, *if*

$$\mu\gamma E'''_{\alpha,\beta}(1) + (\mu+\gamma-5\mu\gamma) E''_{\alpha,\beta}(1) + (1+\gamma) E'_{\alpha,\beta}(1) - \gamma \leq 2. \tag{5}$$

*Proof.* It is clear that

$$h_\gamma(z) = z + \sum_{n=2}^{\infty} \frac{(1+n\gamma-\gamma)\Gamma(\beta) z^n}{\Gamma(\alpha(n-1)+\beta)}, \quad 0 \leq \gamma \leq 1. \tag{6}$$

Let

$$E_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n.$$

Then

$$E'_{\alpha,\beta}(1) - 1 = \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}. \tag{7}$$

Also, we obtain

$$E''_{\alpha,\beta}(1) = \sum_{n=2}^{\infty} \frac{n(n-1)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \quad E'''_{\alpha,\beta}(1) = \sum_{n=2}^{\infty} \frac{n(n-1)(n-2)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}. \tag{8}$$

Since, we are to show that $h_\gamma(z) \in \mathcal{USD}(\mu)$, therefore, by Lemma 1.1, it is enough to prove that

$$\sum_{n=2}^{\infty} n(1-\mu+n\mu)(1+n\gamma-\gamma) \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \leq 1.$$

Now let

$$\begin{aligned} &\sum_{n=2}^{\infty} n(1-\mu+n\mu)(1+\gamma\mu-\gamma) \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ = \;& \mu\gamma \sum_{n=2}^{\infty} \frac{n^3\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} + (\mu+\gamma-2\mu\gamma) \sum_{n=2}^{\infty} \frac{n^2\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ & + (1+\mu\gamma-\mu) \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}. \end{aligned}$$

Then writing $n^3 = n(n-1)(n-2) - 3n(n-1) + n$ and $n^2 = n(n-1) + n$, we have

$$\begin{aligned} &\sum_{n=2}^{\infty} n(1-\mu+n\mu)(1+\gamma\mu-\gamma) \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ = \;& \mu\gamma \sum_{n=2}^{\infty} \frac{n(n-1)(n-2)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} + (\mu+\gamma-5\mu\gamma) \sum_{n=2}^{\infty} \frac{n(n-1)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ + \;& (1+\gamma) \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}. \end{aligned}$$

From (7) and (8), we get

$$\begin{aligned} &\sum_{n=2}^{\infty} n(1-\mu+n\mu)(1+\gamma\mu-\gamma) \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ = \;& \mu\gamma E'''_{\alpha,\beta}(1) + (\mu+\gamma-5\mu\gamma) E''_{\alpha,\beta}(1) + (1+\gamma) E'_{\alpha,\beta}(1) - (1+\gamma). \end{aligned}$$

The last expression is bounded above by 1 if (5) holds. This completes the proof. □

**Theorem 2.2.** *Let* $\beta = 1$ *and* $\alpha \gtrapprox 3.23\cdots$. *Then* $E_{\alpha,\beta}(z)$ *is close-to-convex with respect to the starlike function* $\frac{z}{1-z^2}$.

*Proof.* It is enough to verify that $E_{\alpha,\beta}(z)$ satisfies conditions of Lemma 1.2. Consider $E_{\alpha,\beta}(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, where $\{a_n\}_{n=1}^{\infty}$ satisfies

$$a_1 = 1, \quad a_2 = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad a_{n+1} = \frac{\Gamma(\alpha(n-1)+\beta)}{\Gamma(\alpha n+\beta)} a_n. \tag{9}$$

Now

$$a_1 \gtrapprox 8a_2, \quad \Gamma(\alpha+\beta) \gtrapprox 8\Gamma(\beta), \quad for \;\; \beta = 1 \;\; and \;\; \alpha \gtrapprox 3.23....$$

Again

$$\begin{aligned} & (n-1)a_n - (n+1)a_{n+1} \\ = \; & (n-1)a_n - (n+1)\frac{\Gamma(\alpha(n-1)+\beta)}{\Gamma(\alpha n+\beta)} a_n \\ = \; & \frac{a_n}{\Gamma(\alpha n+\beta)}\left[(n-1)\Gamma(\alpha n+\beta) - (n+1)\Gamma(\alpha(n-1)+\beta)\right] \\ = \; & A(n)M(n), \end{aligned}$$

where

$$\begin{aligned} A(n) &= \frac{a_n}{\Gamma(\alpha n+\beta)} \quad and \\ M(n) &= (n-1)\Gamma(\alpha n+\beta) - (n+1)\Gamma(\alpha(n-1)+\beta) \\ &= (n-3)\Gamma(\alpha n+\beta) + 2\Gamma(\alpha n+\beta) \\ &\quad - (n-3)\Gamma(\alpha(n-1)+\beta) + 2\Gamma(\alpha(n-1)+\beta) \\ &\geq 0, \quad for \quad \beta = 1 \;\; and \;\; \alpha = \frac{5+2\sqrt{17}}{4}. \end{aligned}$$

$M(n)$ is increasing function for $n \geq 3$. Also $M(3) > 0$ implies that $(n-1)a_n \geq (n+1)a_{n+1}$, for all $n \geq 3$. This verifies the fact that $\{a_n\}$ satisfies the hypothesis of Lemma 1.2 and the proof is complete. □

**Theorem 2.3.** *Let* $\alpha \geq 1.67$ *and* $\beta = 1$. *If* $\Gamma(\alpha+\beta) \geq \Gamma(\beta)$, *then* $E_{\alpha,\beta}(z)$ *is starlike of order* $\mu$ *in* $\mathcal{U}$.

*Proof.* It is sufficient to demonstrate that $E_{\alpha,\beta}(z)$ satisfies the conditions of Lemma 1.3. Consider $E_{\alpha,\beta}(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, where $\{a_n\}_{n=1}^{\infty}$ satisfies (9). Simple computation yields

$$\begin{aligned} (1-\mu)a_1 &\geq (2-\mu)a_2 \\ (1-\mu) &\geq (2-\mu)\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ (1-\mu)\Gamma(\alpha+\beta) &\geq (2-\mu)\Gamma(\beta) \end{aligned}$$

$$\Rightarrow (1-\mu)\Gamma(\alpha+\beta) - (2-\mu)\Gamma(\beta) \geq 0, \quad \text{for} \quad \beta = 1 \;\text{ and }\; \alpha \geq 1. \tag{10}$$

and

$$\begin{aligned}
& (2-\mu)\, b_2 - 2^{\mu+1}(3-\mu)\, b_3 \\
= \; & (2-\mu)\, b_2 - 2^{\mu+1}(3-\mu)\, \frac{\Gamma(\alpha+\beta)}{\Gamma(2\alpha+\beta)} b_2 \\
= \; & \frac{b_2}{\Gamma(2\alpha+\beta)} \left[(2-\mu)\,\Gamma(2\alpha+\beta) - 2^{\mu+1}(3-\mu)\,\Gamma(\alpha+\beta)\right] \\
\geq \; & 0, \quad for \quad \beta = 1 \text{ and } \alpha \geq \frac{3}{2}.
\end{aligned}$$

Now

$$\begin{aligned}
& (n-1-\mu)(n-\mu)\, a_n - n(n+1-\mu)\, a_{n+1} \\
= \; & \frac{a_n}{\Gamma(\alpha n+\beta)} \left[(n-1-\mu)(n-\mu)\,\Gamma(\alpha n+\beta) - n(n+1-\mu)\,\Gamma(\alpha(n-1)+\beta)\right] \\
= \; & A(n)\, M(n),
\end{aligned}$$

where

$$\begin{aligned}
A(n) &= \frac{a_n}{\Gamma(\alpha n+\beta)} \quad and \\
M(n) &= (n-1-\mu)(n-\mu)\,\Gamma(\alpha n+\beta) - n(n+1-\mu)\,\Gamma(\alpha(n-1)+\beta) \\
&= \{(2-\mu)\,\Gamma(\alpha n+\beta) - \Gamma(\alpha(n-1)+\beta)\}(n-3)^2 + \\
&\quad \{(5-2\mu)\,\Gamma(\alpha n+\beta) - (7-\mu)\,\Gamma(\alpha(n-1)+\beta)\}(n-3) \\
&\quad + \{(2-\mu)(3-\mu)\,\Gamma(\alpha n+\beta) - 3(4-\mu)\,\Gamma(\alpha(n-1)+\beta)\} \\
&= \sum_{i=0}^{2} T_{i+1}(n-3)^i,
\end{aligned}$$

where for $\beta = 1$,

$$\begin{aligned}
T_1 &= (2-\mu)(3-\mu)\,\Gamma(\alpha n+\beta) - 3(4-\mu)\,\Gamma(\alpha(n-1)+\beta) \geq 0, \quad \alpha \geq 1.67, \\
T_2 &= (5-2\mu)\,\Gamma(\alpha n+\beta) - (7-\mu)\,\Gamma(\alpha(n-1)+\beta) \geq 0, \quad \alpha \geq 1.35, \\
T_3 &= (2-\mu)\,\Gamma(\alpha n+\beta) - \Gamma(\alpha(n-1)+\beta) \geq 0, \quad \alpha \geq 1.
\end{aligned}$$

These inequalities holds for $\beta = 1$. Also $M(n)$ is increasing for $n \geq 3$. Further $M(3) > 0$ implies that $(n-1-\mu)(n-\mu)\, a_n \geq n(n+1-\mu)\, a_{n+1}$, for all $n \geq 3$. This shows $\{a_n\}$ satisfies the conditions of Lemma 1.3. This completes the proof. □

**Theorem 2.4.** *If $\alpha \geq 1$ and $\beta \geq 3.21431974337$, then $E_{\alpha,\beta} \in \widetilde{\mathcal{S}^*}(\alpha)$, where*

$$\alpha = \frac{2}{\pi} \arcsin\left(\kappa\sqrt{1-\frac{\kappa^2}{4}} + \frac{\kappa}{2}\sqrt{1-\kappa^2}\right), \tag{11}$$

*and $\kappa = \frac{2\beta^2+\beta-1}{\beta^3-\beta^2}$.*

*Proof.* By using the well-known triangle inequality and with the inequality $n \leq 2^{n-1}$, $\forall$ $n \geq 1$, we obtain

$$\begin{aligned}
\left|E'_{\alpha,\beta}(z) - 1\right| &\leq \sum_{n=1}^{\infty} \frac{n+1}{(\beta)_n} \\
&\leq \frac{1}{\beta} \sum_{n=1}^{\infty} \left(\frac{2}{\beta+1}\right)^{n-1} + \frac{1}{\beta} \sum_{n=1}^{\infty} \left(\frac{1}{\beta+1}\right)^{n-1} \\
&= \frac{2\beta^2+\beta-1}{\beta^3-\beta^2} = \kappa.
\end{aligned} \tag{12}$$

For $0<\kappa\leq 1$ and from $(12)$, we concluded that

$$E'_{\alpha,\beta}\prec 1+\kappa z \quad\Rightarrow\quad \left|\arg\left(E'_{\alpha,\beta}(z)\right)\right|<\arcsin\kappa. \tag{13}$$

With the help of Lemma 1.4, take $n=0$ with $F(z)=E'_{\alpha,\beta}(z)$ and $G(z)=1+\kappa z$, we get

$$\frac{E_{\alpha,\beta}}{z}\prec 1+\frac{\kappa}{2}z. \tag{14}$$

As a result

$$\left|\arg\left(\frac{E_{\alpha,\beta}(z)}{z}\right)\right|<\arcsin\frac{\kappa}{2}. \tag{15}$$

By using $(13)$ and $(14)$, we obtain

$$\begin{aligned}\left|\arg\left(\frac{zE'_{\alpha,\beta}(z)}{E_{\alpha,\beta}(z)}\right)\right| &\leq \left|\arg\left(\frac{z}{E_{\alpha,\beta}(z)}\right)\right|+\left|\arg\left(E'_{\alpha,\beta}(z)\right)\right|\\ &< \arcsin\frac{\kappa}{2}+\arcsin\kappa=\arcsin\left(\kappa\sqrt{1-\frac{\kappa^2}{4}}+\frac{\kappa}{2}\sqrt{1-\kappa^2}\right).\end{aligned}$$

Which implies that $E_{\alpha,\beta}\in\widetilde{\mathcal{S}}^*(\alpha)$ for $\alpha=\frac{2}{\pi}\arcsin\left(\kappa\sqrt{1-\frac{\kappa^2}{4}}+\frac{\kappa}{2}\sqrt{1-\kappa^2}\right)$. □

**Remark 2.1.** *The condition obtained in the above theorem is better than the result of* [4].

**Theorem 2.5.** *If $\alpha\geq 1$ and $\beta\geq 5.34936...$, then $E_{\alpha,\beta}\in\widetilde{\mathcal{C}}(\alpha)$, where*

$$\alpha=\frac{2}{\pi}\arcsin\left(\kappa\sqrt{1-\frac{\kappa^2}{4}}+\frac{\kappa}{2}\sqrt{1-\kappa^2}\right), \tag{16}$$

*and $\kappa=\frac{3\beta^3-5\beta^2-5\beta+3}{\beta^4-4\beta^3+3\beta^2}$.*

*Proof.* By using the well-known triangle inequality with the inequalities $n^2\leq 4^{n-1}$, $n\leq 2^{n-1}\quad\forall\, n\in\mathbb{N}$, we obtain

$$\begin{aligned}\left|\left(zE'_{\alpha,\beta}(z)\right)'-1\right| &\leq \sum_{n=1}^{\infty}\frac{(n+1)^2}{(\beta)_n}\\ &\leq \frac{1}{\beta}\sum_{n=1}^{\infty}\left(\frac{4}{\beta+1}\right)^{n-1}+\frac{1}{\beta}\sum_{n=1}^{\infty}\left(\frac{2}{\beta+1}\right)^{n-1}+\frac{1}{\beta}\sum_{n=1}^{\infty}\left(\frac{1}{\beta+1}\right)^{n-1}\\ &= \frac{3\beta^3-5\beta^2-5\beta+3}{\beta^4-4\beta^3+3\beta^2}=\kappa.\end{aligned} \tag{17}$$

For $0<\kappa\leq 1$ and from $(17)$, we concluded that

$$\left(zE'_{\alpha,\beta}(z)\right)'\prec 1+\kappa z \quad\Rightarrow\quad \left|\arg\left(zE'_{\alpha,\beta}(z)\right)'\right|<\arcsin\kappa. \tag{18}$$

With the help of Lemma 1.4, take $n=0$ with $F(z)=\left(zE'_{\alpha,\beta}(z)\right)'$ and $G(z)=1+\kappa z$, we get

$$E'_{\alpha,\beta}(z)\prec 1+\frac{\kappa}{2}z. \tag{19}$$

As a result

$$\left|\arg E'_{\alpha,\beta}(z)\right|<\arcsin\frac{\kappa}{2}. \tag{20}$$

By using (18) and (20), we obtain

$$\begin{aligned}\left|\arg\left(\frac{\left(zE'_{\alpha,\beta}(z)\right)'}{E'_{\alpha,\beta}(z)}\right)\right| &\leq \left|\arg\left(zE'_{\alpha,\beta}(z)\right)'\right| + \left|\arg\left(E'_{\alpha,\beta}(z)\right)\right| \\ &< \arcsin\frac{\kappa}{2} + \arcsin\kappa = \arcsin\left(\kappa\sqrt{1-\frac{\kappa^2}{4}} + \frac{\kappa}{2}\sqrt{1-\kappa^2}\right).\end{aligned}$$

Which implies that $E_{\alpha,\beta} \in \widetilde{\mathcal{C}}(\alpha)$ for $\alpha = \frac{2}{\pi}\arcsin\left(\kappa\sqrt{1-\frac{\kappa^2}{4}} + \frac{\kappa}{2}\sqrt{1-\kappa^2}\right)$. □

**Theorem 2.6.** *Let* $\alpha \geq 1$ *and* $\beta \geq 9.1112597744$, *then* $E_{\alpha,\beta} \in \mathcal{UCV}$.

*Proof.* Since

$$\left|\frac{zE''_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)}\right| \leq \frac{2\left(\beta^2-1\right)}{(\beta-3)\left(\beta^2-3\beta+2\right)}.$$

By using Lemma 1.5, we have

$$\left|\frac{zE''_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)}\right| < \frac{1}{2},$$

This implies that $\beta^3 - 10\beta^2 + 7\beta + 10 > 0$, which is positive for $\beta > 9.1112597744$. Hence the required result. □

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