THE SEMI-LINEAR CREDIBILITY MODEL

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Lucrarea prezintă şi analizează estimatorii parametrilor structurali din modelul de credibilitate semi-liniară, implicând proprietăţi matematice complicate ale valorilor medii condiţionate şi ale covarianţelor condiţionte.

Deci pentru a putea folosi rezultatele superioare de credibilitate semi-liniară, obţinute in acest model, vom oferi estimatori utili ai parametrilor de structură.

Din punct de vedere practic, este evidenţiată proprietatea atractivă de nedeplasare a acestor estimatori.

The paper presents and analyses the estimators of the structural parameter, in the semi-linear credibility model involving complicated mathematical properties of conditional expectations and of conditional covariances.

Thus, to be able to use the better semi-linear credibility results obtained in this model, we will provide useful estimators for the structure parameters.

From the practical point of view, the attractive property of the unbiasedness of these estimators is highlighted.

Key words: contracts, unbiased estimators, structure parameters, semi-linear credibility theory.

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1. Introduction

In this article we first give the semi-linear credibility model (see **Section 1.**), which involves only one isolated contract.

We derive the optimal linearized credibility estimate for the risk premium for this case.

It turns out that this procedure does not provide us with a statistic computable from the observations, since the result involves unknown parameters of the structure function.

To obtain estimates for these structure parameters, for the semi-linear credibility model we embed the contract in a collective of contracts, all providing independent information on the structure distribution (see **Section 2.**).

The usefulness of the approximation (to $f_0(X_{t+1})$ or to $\mu_0(\theta)$ based on prescribed approximating functions $f_1,...,f_n$) is that it is easy to apply, since it is sufficient to know estimates for the parameters a_{pq}, b_{pq} appearing in the credibility factors z_p .

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Section 1.
The description of the semi-linear credibility model

Consider a finite sequence $\theta, X_1,..., X_t, X_{t+1}$ of random variables. Assume that for fixed θ the variables $X_1,..., X_{t+1}$ are conditionally independent and identically distributed (conditionally i.i.d) with known common distribution function $F_{X|\theta}(x,\theta)$. The variables $X_1,.., X_t$ are observable and θ is the structure variable. The structure distribution function is $U(\theta) = P(\theta \leq \theta)$. The variable X_{t+1} is considered as being not (yet) observable.

We assume that $f_p(X_r), p = \overline{0,n}; r = \overline{1,t+1}$ have finite variance. For f_0, we take the function of X_{t+1} we want to forecast.

We use the notation:

$$\mu_p(\theta) = E\lfloor f_p(X_r) \mid \theta \rfloor \quad (1.1)$$

$\left(p = \overline{0,n}; r = \overline{1,t+1}\right)$

This expression does not depend on r.

For this model we define the following structure parameters:

$$m_p = E\lfloor \mu_p(\theta) \rfloor = E\{E\lfloor f_p(X_r) \mid \theta \rfloor\} = E\lfloor f_p(X_r) \rfloor \quad (1.2),$$

$$a_{pq} = E\{Cov\lfloor f_p(X_r), f_q(X_r) \mid \theta \rfloor\} \quad (1.3),$$

$$b_{pq} = Cov\lfloor \mu_p(\theta), \mu_q(\theta) \rfloor \quad (1.4),$$

$$c_{pq} = Cov\lfloor f_p(X_r), f_q(X_r) \rfloor \quad (1.5),$$

$$d_{pq} = Cov\lfloor f_p(X_r), \mu_q(\theta) \rfloor \quad (1.6),$$

for $p, q = \overline{0,n}$. These expressions do not depend on $r = \overline{1,t+1}$.

The structure parameters are connected by the following relations:

$$c_{pq} = a_{pq} + b_{pq} \quad (1.7),$$

$$d_{pq} = b_{pq} \quad (1.8),$$

for $p, q = \overline{0,n}$.

This follows from the covariance relations obtained in the probability theory where they are very well-known.

Just as in the case of considering linear combinations of the observable variables themselves, we can also obtain non-homogeneous credibility estimates, taking as estimators the class of linear combinations of given functions of the observable variables, as shown in the following theorem:

Theorem 1.1 (*Optimal non-homogeneous linearized estimators*)

The linear combination of 1 and the random variables $f_p(X_r), p=\overline{1,n}; r=\overline{1,t}$ closest to $\mu_0(\theta)=E[f_0(X_{t+1})\mid\theta]$ and to $f_0(X_{t+1})$ in the least squares sense equals:

$$M=\sum_{p=1}^{n} z_p \sum_{r=1}^{t} \frac{1}{t} f_p(X_r)+m_0-\sum_{p=1}^{n} z_p m_p \qquad (1.9),$$

where $z_1, z_2, ..., z_n$ is a solution to the linear system of equations:

$$\sum_{p=1}^{n}\left[c_{pq}+(t-1)d_{pq}\right]z_p = td_{0q} \;(q=\overline{1,n}) \qquad (1.10),$$

, or to the equivalent linear system of equations:

$$\sum_{p=1}^{n}\left(a_{pq}+tb_{pq}\right)z_p = tb_{0q} \;(q=\overline{1,n}) \qquad (1.11)$$

For the special case when n=1, *Theorem 1.1* reads:

Theorem 1.2 (*Optimal non-homogeneous linearized estimator, n = 1*)

The linear combination of 1 and the random variables $f_1(X_r)$ ($r=\overline{1,t}$) closest to $\mu_0(\theta)$ and to $f_0(X_{t+1})$ in the least squares sense equals:

$$M=z\sum_{r=1}^{t}\frac{1}{t}f_1(X_r)+m_0-zm_1 \qquad (1.12)$$

where: $m_1 = E[f_1(X_r)]$,

$z = td_{01}/\{c_{11}+(t-1)d_{11}\}$ with:

$d_{01} = Cov[f_0(X_r), f_1(X_{r'})]$ }

$d_{11} = Cov[f_1(X_r), f_1(X_{r'})]$ } for $r \neq r'$

$c_{11} = Cov[f_1(X_r), f_1(X_r)]$

Proof:

For n = 1, the relation (1.4) implies:

$$m_1 = E[f_1(X_r)].$$

For n = 1, the relation (1.7) implies:

$$c_{11} = Cov[f_1(X_r), f_1(X_r)].$$

For n = 1, the linear system of equations (1.10) reads:

$$[c_{11} + (t-1)d_{11}]z = td_{01}$$

which is equivalent to the following equation:

$$z = td_{01} / \{c_{11} + (t-1)d_{11}\}.$$

Let $r, r' = \overline{1,t}$ with $r \neq r'$. From (1.1), (1.4), (1.8) we get:

$$\text{Cov}\left[f_p(X_r), f_q(X_{r'})\right]=E\left[f_p(X_r)f_q(X_{r'})\right]-E\left[f_p(X_r)\right]E\left[f_q(X_{r'})\right]=$$

$$=\text{E}\left\{E\left[f_p(X_r)f_q(X_{r'})\mid\theta\right]\right\}-E\left\{E\left[f_p(X_r)\mid\theta\right]\right\}E\left\{E\left[f_q(X_{r'})\mid\theta\right]\right\}=$$

$= \mathrm{E}\left\{E\lfloor f_p(X_r) \mid \theta \rfloor E\lfloor f_q(X_{r'}) \mid \theta \rfloor\right\} - E\lfloor \mu_p(\theta)\rfloor E\lfloor \mu_q(\theta)\rfloor = E\lfloor \mu_p(\theta)\mu_q(\theta)\rfloor$-

-E $\lfloor \mu_p(\theta)\rfloor E\lfloor \mu_q(\theta)\rfloor = Cov\lfloor \mu_p(\theta), \mu_q(\theta)\rfloor = b_{pq} = d_{pq}$ (1.13),

for all p,q = $\overline{1,n}$ .From the relation (1.13) one obtains for n = 1 that:

$$d_{01} = \mathrm{Cov}\,[f_0(X_r), f_1(X_{r'})],$$
$$d_{11} = \mathrm{Cov}\,[f_1(X_r), f_1(X_{r'})],$$

where $r \neq r'$. Finally, for n = 1, the relation (1.9) reads:

$$M = z\sum_{r=1}^{t}\frac{1}{t}f_1(X_r) + m_0 - zm_1 .$$

So the theorem is proven.

Remark 1.1 It should be noted that the solution (1.12) to the linearized credibility problem only yields a statistics computable from the observations, if the structure parameters are known.

Generally, however, the structure function $U(\cdot)$ is not known. Then the “estimator” as it stands is not a statistic.

Its interest is merely theoretical, but will be the basis for further results on semi-linear credibility.

In the following section we consider different contracts, each with the same structure parameters, so we can estimate these quantities using the statistics of the different contracts.

Section 2
Parameter estimation

Here and in the following we present the main results leaving the detailed computations to the reader.

The estimator obtained in the previous section contained structure parameters.

In this section we assume the structure parameters are unknown, so the expressions for these (pseudo-) estimators are no longer statistics. But since the contracts are embedded in a collective of identical contracts, we now have more than one observation available on the risk parameter θ, so we can replace the unknown structure parameters by estimates.

So now that we embedded the separate contract j in a collective of identical contracts, it is possible to give unbiased estimators of these quantities.

It should be noted that the approximation to $f_0(X_{t+1})$ or to $\mu_0(\theta)$ based on a unique optimal approximating function f is always better than the one furnished in Section 1.1 based on prescribed approximating functions $f_1, f_2, \ldots, f_n$.

The usefulness of the latter approximation is that it is easy to apply, since it is sufficient to know estimates for the parameters a_{pq}, b_{pq} appearing in the credibility factors z_p.

In this section we give some unbiased estimators for the parameters. For this purpose we consider k contracts, $j=\overline{1,k}$ and $k(\geq 2)$ independent and identically distributed random vectors $\left(\theta_j, \underline{X}_j^{'}\right)=\left(\theta_j, X_{j1}, ... X_{jt}\right)$, for $j=\overline{1,k}$. The contract indexed j is a random vector consisting of a random structure parameter θ_j and observations X_{j1}, X_{j2}, …, X_{jt}, where $j=\overline{1,k}$. For every contract $j=\overline{1,k}$ and for θ_j fixed, the variables X_{j1}, X_{j2}, …, X_{jt} are conditionally independent and identically distributed.

Here we will only derive estimators for the following parameters:

$$m_0 = E\left[f_0\left(X_{jr}\right)\right] \tag{2.1}$$

$$a_{01} = E\left\{Cov\left[f_0\left(X_{jr}\right), f_1\left(X_{jr}\right) \mid \theta_j\right]\right\} \tag{2.2}$$

$$b_{01} = Cov\left\{E\left[f_0\left(X_{jr}\right) \mid \theta_j\right], E\left[f_1\left(X_{jr}\right) \mid \theta_j\right]\right\} \tag{2.3}$$

One can prove the following theorem to hold.

Theorem 2.1 (Unbiased estimators for structure parameters)

Let:

$$\hat{m}_0 = \frac{1}{kt} X_{..}^0 = \frac{1}{kt}\sum_{j=1}^{k}\sum_{r=1}^{t} f_0(X_{jr}) \tag{2.4}$$

$$\hat{a}_{01} = \frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t}\left(X_{jr}^0 - \frac{1}{t}X_{j.}^0\right)\left(X_{jr}^1 - \frac{1}{t}X_{j.}^1\right) \tag{2.5}$$

$$\hat{b}_{01} = \frac{1}{k-1}\sum_{j=1}^{k}\left(\frac{1}{t}X_{j.}^0 - \frac{1}{kt}X_{..}^0\right)\left(\frac{1}{t}X_{j.}^1 - \frac{1}{kt}X_{..}^1\right) - \frac{\hat{a}_{01}}{t} \tag{2.6}$$

then:

$$E\left(\hat{m}_0\right) = m_0 \tag{2.7}$$

$$E\left(\hat{a}_{01}\right) = a_{01} \tag{2.8}$$

$$E\left(\hat{b}_{01}\right) = b_{01} \tag{2.9}$$

Proof

We have:

$$E\left(\hat{m}_{01}\right) = \frac{1}{kt}\sum_{j=1}^{k}\sum_{r=1}^{t} E\left[f_0\left(X_{jr}\right)\right] = \frac{1}{kt}\sum_{j=1}^{k}\sum_{r=1}^{t} m_0 = \frac{kt}{kt} m_0 = m_0 \tag{2.10}$$

(see (2.11).

So the verification of equality (2.7) is readily performed.

Remark 2.1 Note that the usual definitions of the structure parameters apply, with θ_j replacing θ and X_{jr} replacing X_r, so:

$$m_p = E\left[\mu_p(\theta_j)\right] = E\left\{E\left[f_p(X_{jr}) \mid \theta_j\right]\right\} = E\left[f_p(X_{jr})\right] \tag{2.11}$$

$$a_{pq} = E\left\{Cov\left[f_p(X_{jr}), f_q(X_{jr}) \mid \theta_j\right]\right\} \tag{2.12}$$

$$b_{pq} = Cov\left[\mu_p(\theta_j), \mu_q(\theta_j)\right] = Cov\left\{E\left[f_p(X_{jr}) \mid \theta_j\right], E\left[f_q(X_{jr}) \mid \theta_j\right]\right\} \tag{2.13}$$

$$c_{pq} = Cov\left[f_p(X_{jr}), f_q(X_{jr})\right] \tag{2.14}$$

Next:

$$E\left(\hat{a}_{01}\right) = \frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t} E\left[\left(X_{jr}^{0} - \frac{1}{t}X_{j.}^{0}\right)\left(X_{jr}^{1} - \frac{1}{t}X_{j.}^{1}\right)\right] =$$

$$= \frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t} E\left(X_{jr}^{0}X_{jr}^{1} - X_{jr}^{0}\frac{1}{t}X_{j.}^{1} - \frac{1}{t}X_{j.}^{0}X_{jr}^{1} + \frac{1}{t^2}X_{j.}^{0}X_{j.}^{1}\right) =$$

$$= \frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t}\left[E\left(X_{jr}^{0}X_{jr}^{1}\right) - E\left(X_{jr}^{0}\frac{1}{t}X_{j.}^{1}\right) - E\left(\frac{1}{t}X_{j.}^{0}X_{jr}^{1}\right) + E\left(\frac{1}{t}X_{j.}^{0}\frac{1}{t}X_{j.}^{1}\right)\right] =$$

$$\frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t}\left[Cov\left(X_{jr}^{0}, X_{jr}^{1}\right) + E\left(X_{jr}^{0}\right)E\left(X_{jr}^{1}\right) - Cov\left(X_{jr}^{0}, \frac{1}{t}X_{j.}^{1}\right) -\right.$$

$$- E\left(X_{jr}^{0}\right)E\left(\frac{1}{t}X_{j.}^{1}\right) - Cov\left(\frac{1}{t}X_{j.}^{0}, X_{jr}^{1}\right) - E\left(\frac{1}{t}X_{j.}^{0}\right)E\left(X_{jr}^{1}\right) +$$

$$\left. + Cov\left(\frac{1}{t}X_{j.}^{0}, \frac{1}{t}X_{j.}^{1}\right) + E\left(\frac{1}{t}X_{j.}^{0}\right)E\left(\frac{1}{t}X_{j.}^{1}\right)\right] \tag{2.15}$$

But:

$$Cov\left(X_{jr}^{0}, X_{jr'}^{1}\right) = Cov\left[f_0(X_{jr}), f_1(X_{jr'})\right] =$$

$$= \begin{cases} b_{01} & (see(2.16)) \quad r \neq r' \\ a_{01} + b_{01} & (see(2.17)) \quad r = r' \end{cases}, \text{ because:}$$

- for $r \neq r'$, we have:

$Cov[f_0(X_{jr}),f_1(X_{jr'})] = E[Cov(f_0(X_{jr}),f_1(X_{jr'})|\theta_j)]+Cov[E(f_0(X_{jr})|\theta_j),$
$,E(f_1(X_{jr'}|\theta_j)]=E\{E[f_0(X_{jr})f_1(X_{jr'})|\theta_j]-E[f_0(X_{jr})|\theta_j]E[f_1(X_{jr'})|\theta_j]\}+$
$+Cov[\mu_0(\theta_j), \mu_1(\theta_j)]=E\{E[f_0(X_{jr})|\theta_j]E[f_1(X_{jr'})|\theta_j]-E[f_0(X_{jr})|\theta_j]E[f_1(X_{jr'})|$
$|\theta_j]+b_{01}=b_{01}$ (2.16)

- for r = r’, we have:

$Cov[f_0(X_{jr}),f_1(X_{jr})]=E\{Cov[f_0(X_{jr}),f_1(X_{jr})|\theta_j]\}+Cov[E(f_0(X_{jr})|\theta_j),E(f_1(X_{jr})|$
$|\theta_j)]=a_{01}+b_{01}$ (2.17)

So:

$$\text{Cov}\left(X_{jr}^{0}, X_{jr'}^{1}\right) = \delta_{rr'}a_{01}+b_{01} \tag{2.18}$$

Next:

$$E(X_{jr}^{0}) = E[f_0(X_{jr})] = m_0 \tag{2.19}$$

$$E(X_{jr}^{1}) = E[f_1(X_{jr})] = m_1 \tag{2.20}$$

Also, we have:

$$\mathrm{Cov}(X_{jr}^{0},\frac{1}{t}X_{j.}^{1}) = \frac{1}{t}\mathrm{Cov}\left(X_{jr}^{0},\sum_{r'=1}^{t}X_{jr'}^{1}\right) = \frac{1}{t}\sum_{r'=1}^{t}Cov\left(X_{jr}^{0},X_{jr'}^{1}\right) =$$

$$= \frac{1}{t}\sum_{r'=1}^{t}\left(\delta_{rr'}a_{01}+b_{01}\right) = \frac{1}{t}\left[\delta_{rr}a_{01}+b_{01}+\sum_{r',r'\neq r}\left(\delta_{rr'}a_{01}+b_{01}\right)\right] =$$

$$= \frac{1}{t}\left[1\cdot a_{01}+b_{01}+\sum_{r',r'\neq r}\left(0\cdot a_{01}+b_{01}\right)\right] = \frac{1}{t}\left[a_{01}+b_{01}+(t-1)b_{01}\right] = \frac{1}{t}a_{01}+b_{01}$$

So:

$$\mathrm{Cov}\left(X_{jr}^{0},\frac{1}{t}X_{j.}^{1}\right) = \frac{1}{t}a_{01}+b_{01} \tag{2.21}$$

Next:

$$\mathrm{E}\left(\frac{1}{t}X_{j.}^{1}\right) = \frac{1}{t}\sum_{r=1}^{t}E\left(X_{jr}^{1}\right) = \frac{1}{t}\sum_{r=1}^{t}m_1 = \frac{1}{t}tm_1 = m_1 \tag{2.22}$$

$$\mathrm{Cov}\left(\frac{1}{t}X_{j.}^{0},X_{jr}^{1}\right) = \frac{1}{t}Cov\left(X_{j.}^{0},X_{jr}^{1}\right) = \frac{1}{t}Cov\left(\sum_{r'=1}^{t}X_{jr'}^{0},X_{jr}^{1}\right) =$$

$$= \frac{1}{t}\sum_{r'=1}^{t}Cov\left(X_{jr'}^{0},X_{jr}^{1}\right) = \frac{1}{t}\sum_{r'=1}^{t}\left(\delta_{rr'}a_{01}+b_{01}\right) = \ldots = \frac{1}{t}a_{01}+b_{01}$$

(see the calculations from (2.21)).

So:

$$Cov\left(\frac{1}{t}X_{j.}^{0},X_{jr}^{1}\right) = \frac{1}{t}a_{01}+b_{01} \tag{2.23}$$

Next:

$$\mathrm{E}\left(\frac{1}{t}X_{j.}^{0}\right) = \frac{1}{t}\sum_{r=1}^{t}E\left(X_{jr}^{0}\right) = \frac{1}{t}\sum_{r=1}^{t}m_0 = \frac{1}{t}tm_0 = m_0 \tag{2.24}$$

and:

$$Cov\left(\frac{1}{t}X_{j.}^{0},\frac{1}{t}X_{j.}^{1}\right) = \frac{1}{t^2}Cov\left(\sum_{r=1}^{t}X_{jr}^{0},\sum_{r'=1}^{t}X_{jr'}^{1}\right) = \frac{1}{t^2}\sum_{r=1}^{t}\sum_{r'=1}^{t}Cov\left(X_{jr}^{0},X_{jr'}^{1}\right) =$$

$$= \frac{1}{t^2}\sum_{r=1}^{t}\sum_{r'=1}^{t}\left(\delta_{rr'}a_{01}+b_{01}\right) = \frac{1}{t^2}\sum_{r=1}^{t}\left[\delta_{rr}a_{01}+b_{01}+\sum_{r',r'\neq r}\left(\delta_{rr'}a_{01}+b_{01}\right)\right] =$$

$$= \frac{1}{t^2}\sum_{r=1}^{t}\left[a_{01}+b_{01}+(t-1)b_{01}\right] = \frac{1}{t^2}\left[t\left(a_{01}+b_{01}\right)+t(t-1)b_{01}\right] =$$

$$= \frac{1}{t}\left(a_{01}+b_{01}+tb_{01}-b_{01}\right) = \frac{1}{t}a_{01}+b_{01}$$

So:

$$Cov\left(\frac{1}{t}X^0_{j.},\frac{1}{t}X^1_{j.}\right)=\frac{1}{t}a_{01}+b_{01} \tag{2.25}$$

Inserting (2.18), (2.19), (2.20), (2.21), (2.22), (2.23), (2.24) and (2.25) in (2.15) one obtains:

$$E\left(\hat{a}_{01}\right)=\frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t}\left[(a_{01}+b_{01})+m_0m_1-\left(\frac{1}{t}a_{01}+b_{01}\right)-m_0m_1-\right.$$
$$\left.-\left(\frac{1}{t}a_{01}+b_{01}\right)-m_0m_1+\left(\frac{1}{t}a_{01}+b_{01}\right)+m_0m_1\right]=\frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t}\left(a_{01}+b_{01}-\right.$$
$$\left.-\frac{1}{t}a_{01}-b_{01}\right)=\frac{1}{k(t-1)}kt\frac{(t-1)}{t}a_{01}=a_{01}$$

as was to be proven (see (2.8)).

Finally, we have:

$$E\left(\hat{b}_{01}\right)=\frac{1}{k-1}\sum_{j=1}^{k}E\left[\left(\frac{1}{t}X^0_{j.}-\frac{1}{kt}X^0_{..}\right)\left(\frac{1}{t}X^1_{j.}-\frac{1}{kt}X^1_{..}\right)\right]-\frac{1}{t}E\left(\hat{a}_{01}\right)=$$
$$=\frac{1}{k-1}\frac{1}{t^2}\sum_{j=1}^{k}E\left[\left(X^0_{j.}-\frac{1}{k}X^0_{..}\right)\left(X^1_{j.}-\frac{1}{k}X^1_{..}\right)\right]-\frac{a_{01}}{t}=$$
$$=\frac{1}{(k-1)t^2}\sum_{j=1}^{k}\left[E\left(X^0_{j.}X^1_{j.}\right)-\frac{1}{k}E\left(X^0_{j.}X^1_{..}\right)-\frac{1}{k}E\left(X^0_{..}X^1_{j.}\right)+\frac{1}{k^2}E\left(X^0_{..}X^1_{..}\right)\right]-$$
$$-\frac{a_{01}}{t}=\frac{1}{k-1}\sum_{j=1}^{k}\left[E\left(\frac{1}{t}X^0_{j.}\frac{1}{t}X^1_{j.}\right)-E\left(\frac{1}{t}X^0_{j.}\right)E\left(\frac{1}{t}X^1_{j.}\right)+E\left(\frac{1}{t}X^0_{j.}\right)E\left(\frac{1}{t}X^1_{j.}\right)-\right.$$
$$-E\left(\frac{1}{t}X^0_{j.}\frac{1}{kt}X^1_{..}\right)+E\left(\frac{1}{t}X^0_{j.}\right)E\left(\frac{1}{kt}X^1_{..}\right)-E\left(\frac{1}{t}X^0_{j.}\right)E\left(\frac{1}{kt}X^1_{..}\right)-E\left(\frac{1}{kt}X^0_{..}\cdot\right.$$
$$\left.\cdot\frac{1}{t}X^1_{j.}\right)+E\left(\frac{1}{kt}X^0_{..}\right)E\left(\frac{1}{t}X^1_{j.}\right)-E\left(\frac{1}{kt}X^0_{..}\right)E\left(\frac{1}{t}X^1_{j.}\right)+E\left(\frac{1}{kt}X^0_{..}\frac{1}{kt}X^1_{..}\right)-$$
$$\left.-E\left(\frac{1}{kt}X^0_{..}\right)E\left(\frac{1}{kt}X^1_{..}\right)+E\left(\frac{1}{kt}X^0_{..}\right)E\left(\frac{1}{kt}X^1_{..}\right)\right]-\frac{a_{01}}{t}=\frac{1}{k-1}\sum_{j=1}^{k}\left[Cov\left(\frac{1}{t}X^0_{j.},\right.\right.$$
$$\left.,\frac{1}{t}X^1_{j.}\right)+E\left(\frac{1}{t}X^0_{j.}\right)E\left(\frac{1}{t}X^1_{j.}\right)-Cov\left(\frac{1}{t}X^0_{j.},\frac{1}{kt}X^1_{..}\right)-E\left(\frac{1}{t}X^0_{j.}\right)E\left(\frac{1}{kt}X^1_{..}\right)-$$
$$-Cov\left(\frac{1}{kt}X^0_{..},\frac{1}{t}X^1_{j.}\right)-E\left(\frac{1}{kt}X^0_{..}\right)E\left(\frac{1}{t}X^1_{j.}\right)+Cov\left(\frac{1}{kt}X^0_{..},\frac{1}{kt}X^1_{..}\right)+E\left(\frac{1}{kt}X^0_{..}\right)\cdot$$
$$\left.\cdot E\left(\frac{1}{kt}X^1_{..}\right)\right]-\frac{a_{01}}{t} \tag{2.26}$$

But:

$$\text{Cov}\left(\frac{1}{t}X^0_{j.}, \frac{1}{t}X^1_{j.}\right) = \frac{1}{t}a_{01} + b_{01} \tag{2.27}$$

(see (2.25)); Next:

$$\text{Cov}\left(\frac{1}{t}X^0_{j.}, \frac{1}{kt}X^1_{..}\right) = \frac{1}{kt^2}Cov\left(\sum_{r=1}^{t}X^0_{jr}, \sum_{j'=1}^{k}\sum_{r'=1}^{t}X^1_{j'r'}\right) = \frac{1}{kt^2}\sum_{r=1}^{t}\sum_{j'=1}^{k}\sum_{r'=1}^{t}Cov(X^0_{jr},$$

$$,X^1_{j'r'}) = \frac{1}{kt^2}\sum_{r=1}^{t}\sum_{j'=1}^{k}\sum_{r'=1}^{t}\delta_{jj'}\left(\delta_{rr'}a_{01} + b_{01}\right) = \frac{1}{kt^2}\sum_{r=1}^{t}\sum_{j'=1}^{k}\delta_{jj'}\sum_{r'=1}^{t}\left(\delta_{rr'}a_{01} + b_{01}\right) = \frac{1}{kt^2}$$

$$\cdot\sum_{r=1}^{t}\sum_{j'=1}^{k}\delta_{jj'}\left[\delta_{rr}a_{01} + b_{01} + \sum_{r',r'\neq r}\left(\delta_{rr'}a_{01} + b_{01}\right)\right] = \frac{1}{kt^2}\sum_{r=1}^{t}\sum_{j'=1}^{k}\delta_{jj'}\left[a_{01} + b_{01} + (t-1)\cdot\right.$$

$$\left.\cdot b_{01}\right] = \frac{1}{kt^2}\sum_{r=1}^{t}\left\{\delta_{jj}\left[a_{01} + b_{01} + (t-1)b_{01}\right] + \sum_{j',j'\neq j}\delta_{jj'}\left[a_{01} + b_{01} + (t-1)b_{01}\right]\right\} = \frac{1}{kt^2}t\cdot$$

$$\cdot\left[a_{01} + b_{01} + (t-1)b_{01}\right] = \frac{1}{kt}a_{01} + \frac{1}{k}b_{01}\text{, because:}$$

$$Cov\left(X^0_{jr}, X^1_{j'r'}\right) = \begin{cases} Cov\left(X^0_{j'r}, X^1_{j'r'}\right) & j = j' \\ Cov\left(X^0_{jr}, X^1_{j'r'}\right) & j \neq j' \end{cases} =$$

$$= \begin{cases} \delta_{rr'}a_{01} + b_{01} & (see(2.18)) \quad j = j' \\ 0 & (see(2.29)) \quad j \neq j' \end{cases} = \delta_{jj'}\left(\delta_{rr'}a_{01} + b_{01}\right) \tag{2.28},$$

where:

$$Cov\left(X^0_{jr}, X^1_{j'r'}\right) = E\left[Cov\left(X^0_{jr}, X^1_{j'r'} \mid \theta_j\right)\right] + Cov\left[E\left(X^0_{jr} \mid \theta_j\right), E\left(X^1_{j'r'} \mid \theta_j\right)\right] =$$

$$= E\left[E\left(X^0_{jr}X^1_{j'r'} \mid \theta_j\right) - E\left(X^0_{jr} \mid \theta_j\right)E\left(X^1_{j'r'} \mid \theta_j\right)\right] + Cov\left[E\left(X^0_{jr} \mid \theta_j\right), E\left(X^1_{j'r'}\right)\right] =$$

$$= E\left[E\left(X^0_{jr} \mid \theta_j\right)E\left(X^1_{j'r'} \mid \theta_j\right) - E\left(X^0_{jr} \mid \theta_j\right)E\left(X^1_{j'r'}\right)\right] + 0 = E[E\left(X^0_{jr} \mid \theta_j\right)E\left(X^1_{j'r'}\right) -$$

$$E\left(X^0_{jr} \mid \theta_j\right)E\left(X^1_{j'r'}\right)\Big] = E(0) = 0\text{, if } j \neq j'.$$

So:

$$\text{Cov}\left(X^0_{jr}, X^1_{j'r'}\right) = 0 \;\; (j \neq j') \tag{2.29}$$

and:

$$\text{Cov}\left(\frac{1}{t}X^0_{j.}, \frac{1}{kt}X^1_{..}\right) = \frac{1}{kt}a_{01} + \frac{1}{k}b_{01} \tag{2.30}$$

Next:

$$\text{Cov}\left(\frac{1}{kt}X^0_{..}, \frac{1}{t}X^1_{j.}\right) = \frac{1}{kt^2}Cov\left(X^0_{..}, X^1_{j.}\right) = \frac{1}{kt^2}Cov\left(\sum_{j'=1}^{k}\sum_{r=1}^{t}X^0_{j'r}, \sum_{r'=1}^{t}X^1_{jr'}\right) =$$

$$= \frac{1}{kt^2}\sum_{j'=1}^{k}\sum_{r=1}^{t}\sum_{r'=1}^{t}Cov\left(X^0_{j'r}, X^1_{jr'}\right) = \frac{1}{kt^2}\sum_{j'=1}^{k}\sum_{r=1}^{t}\sum_{r'=1}^{t}\delta_{jj'}\left(\delta_{rr'}a_{01} + b_{01}\right) =$$

$$=\frac{1}{kt^2}\sum_{j'=1}^{k}\delta_{jj'}\sum_{r=1}^{t}\sum_{r'=1}^{t}\left(\delta_{rr'}a_{01}+b_{01}\right)=\frac{1}{kt^2}\sum_{j'=1}^{k}\delta_{jj'}\sum_{r=1}^{t}\left[\delta_{rr}a_{01}+b_{01}+\sum_{r',r'\neq r}\left(\delta_{rr'}a_{01}+\right.\right.$$

$$\left.\left.+b_{01}\right)\right]=\frac{1}{kt^2}\sum_{j'=1}^{k}\delta_{jj'}\sum_{r=1}^{t}\left[a_{01}+b_{01}+(t-1)b_{01}\right]=\frac{1}{kt^2}\sum_{j'=1}^{k}\delta_{jj'}\sum_{r=1}^{t}\left(a_{01}+tb_{01}\right)=$$

$$=\frac{1}{kt^2}\sum_{j'=1}^{k}\delta_{jj'}t\left(a_{01}+tb_{01}\right)=\frac{1}{kt^2}\left[\delta_{jj}t\left(a_{01}+tb_{01}\right)+\sum_{j',j'\neq j}\delta_{jj'}t\left(a_{01}+tb_{01}\right)\right]=$$

$$=\frac{1}{kt}a_{01}+\frac{1}{k}b_{01}$$

(see (2.28)). So:

$$Cov\left(\frac{1}{kt}X^{0}_{..},\frac{1}{t}X^{1}_{j.}\right)=\frac{1}{kt}a_{01}+\frac{1}{k}b_{01} \tag{2.31}$$

Next:

$$Cov\left(\frac{1}{kt}X^{0}_{..},\frac{1}{kt}X^{1}_{..}\right)=\frac{1}{k^2t^2}Cov\left(X^{0}_{..},X^{1}_{..}\right)=\frac{1}{k^2t^2}Cov\left(\sum_{j=1}^{k}\sum_{r=1}^{t}X^{0}_{jr},\right.$$

$$\left.,\sum_{j'=1}^{k}\sum_{r'=1}^{t}X^{1}_{j'r'}\right)=\frac{1}{k^2t^2}\sum_{j=1}^{k}\sum_{r=1}^{t}\sum_{j'=1}^{k}\sum_{r'=1}^{t}Cov\left(X^{0}_{jr},X^{1}_{j'r'}\right)=\frac{1}{k^2t^2}\sum_{j=1}^{k}\sum_{r=1}^{t}\sum_{j'=1}^{k}\sum_{r'=1}^{t}\delta_{jj'}\left(\delta_{rr'}\cdot\right.$$

$$\left.\cdot a_{01}+b_{01}\right)=\frac{1}{k^2t^2}\sum_{j=1}^{k}\sum_{r=1}^{t}\sum_{j'=1}^{k}\delta_{jj'}\left[\delta_{rr}a_{01}+b_{01}+\sum_{r',r'\neq r}\left(\delta_{rr'}a_{01}+b_{01}\right)\right]=\frac{1}{k^2t^2}\cdot$$

$$\cdot\sum_{j=1}^{k}\sum_{r=1}^{t}\sum_{j'=1}^{k}\delta_{jj'}\left[a_{01}+b_{01}+(t-1)b_{01}\right]=\frac{1}{k^2t^2}\sum_{j=1}^{k}\sum_{r=1}^{t}\sum_{j'=1}^{k}\delta_{jj'}\left(a_{01}+tb_{01}\right)=\frac{1}{k^2t^2}\cdot$$

$$\cdot\sum_{j=1}^{k}\sum_{r=1}^{t}\left[\delta_{jj}\left(a_{01}+tb_{01}\right)+\sum_{j',j'\neq j}\delta_{jj'}\left(a_{01}+tb_{01}\right)\right]=\frac{1}{k^2t^2}\sum_{j=1}^{k}\sum_{r=1}^{t}\left(a_{01}+tb_{01}\right)=$$

$$=\frac{1}{k^2t^2}kt\left(a_{01}+tb_{01}\right)=\frac{1}{kt}a_{01}+\frac{1}{k}b_{01}$$

So:

$$Cov\left(\frac{1}{kt}X^{0}_{..},\frac{1}{kt}X^{1}_{..}\right)=\frac{1}{kt}a_{01}+\frac{1}{k}b_{01} \tag{2.32}$$

Also, we have:

$$E\left(\frac{1}{t}X^{0}_{j.}\right)=m_0 \tag{2.33}$$

(see (2.24)).

$$E\left(\frac{1}{t}X^{1}_{j.}\right)=m_1 \tag{2.34}$$

(see (2.22)).

$$E\left(\frac{1}{kt}X_{..}^{0}\right)=\frac{1}{kt}E\left(\sum_{j=1}^{k}\sum_{r=1}^{t}X_{jr}^{0}\right)=\frac{1}{kt}\sum_{j,r}E\left(X_{jr}^{0}\right)=\frac{1}{kt}ktm_{0}=m_{0} \qquad (2.35)$$

(see (2.19)).

$$E\left(\frac{1}{kt}X_{..}^{1}\right)=\frac{1}{kt}E\left(\sum_{j=1}^{k}\sum_{r=1}^{t}X_{jr}^{1}\right)=\frac{1}{kt}\sum_{j,r}E\left(X_{jr}^{1}\right)=\frac{1}{kt}ktm_{1}=m_{1} \qquad (2.36)$$

(see (2.20)).

Inserting the values of the covariances and of the expectations (see (2.27), (2.30), (2.31), (2.32), (2.33), (2.34), (2.35) and ((2.36)) in (2.26), provides us with the desired results.

Indeed:

$$E\left(\hat{b}_{01}\right)=\frac{1}{k-1}\sum_{j=1}^{k}\left[\left(\frac{1}{t}a_{01}+b_{01}\right)+m_0m_1-\left(\frac{1}{kt}a_{01}+\frac{1}{k}b_{01}\right)-m_0m_1-\left(\frac{1}{kt}a_{01}+\right.\right.$$

$$\left.\left.+\frac{1}{k}b_{01}\right)-m_0m_1+\left(\frac{1}{kt}a_{01}+\frac{1}{k}b_{01}\right)+m_0m_1\right]-\frac{a_{01}}{t}=\frac{1}{k-1}\sum_{j=1}^{k}\left(\frac{1}{t}a_{01}+b_{01}-\frac{1}{kt}\cdot\right.$$

$$\left.\cdot a_{01}-\frac{1}{kt}b_{01}\right)-\frac{a_{01}}{t}=\frac{1}{k-1}k\frac{k-1}{k}b_{01}+\frac{1}{k-1}k\frac{k-1}{kt}a_{01}-\frac{a_{01}}{t}=b_{01}+\frac{a_{01}}{t}-$$

$$-\frac{a_{01}}{t}=b_{01}$$

as was to be proven (see (2.9)).

3. Conclusions

This paper completes the solution of the semi-linear credibility model in case of a non homogeneous linear estimator for $f_0(X_{j,t+1})$, or what amounts to the same, for $\mu_0(\theta_j)$.

In view of assumption about independence of the contracts, it might come as a surprise that the premium for contract j involves results from other contracts.

A closer look at this assumption reveals that this is so because the other contracts provide additional information on the structure distribution.

For this reason the claim figures of other contracts cannot be ignored when estimating the parameters appearing in the semi-linear credibility estimate for contract j.

In this article, the semi-linear credibility model is refined by the introduction of the isolated contract j in a collective of contracts, all providing independent information on the structure distribution.

But since the contracts are embedded in a collective of identical contracts, we now have more than one observation available on the risk parameter θ, so we can estimate these structural parameters in the semi-linear credibility model using the statistics of the different contracts.

The above two theorems show that it is possible to give unbiased estimators of these quantities (the portfolio characteristics), if we embed the separate contract j in a collective of identical contracts.

The article contains a description of the semi-linear credibility model, behind a heterogeneous portfolio, involving an underlying risk parameter for the individual risks.

Since these risks can now no longer be assumed to be independent, mathematical properties of conditional expectations and of conditional covariances become useful.

The original model involving only one contract contains the basics of all further semi-linear credibility models.

In the refined semi-linear credibility model a portfolio of contracts is studied, to be able to use the semi-linear credibility results.

Therefore, the main purpose of this paper is to get unbiased estimators for the portfolio characteristics.

The mathematical theory provides the means to calculate useful estimators for the structure parameters.

From the practical point of view, the property of unbiasedness of these estimators is very appealing and very attractive.

The fact that it is based on complicated mathematics, involving conditional expectations, conditional covariances and variational calculus, needs not bother the user more than it does when he applies statistical tools like discriminatory analysis, scoring models, SAS and GLIM.

These techniques can be applied by anybody on his own field of endeavor, be it economics, medicine, or insurance.

R E F E R E N C E S

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