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$B(K)$-LINEAR OPERATORS AND THEIR OPERATOR-VALUED SPECTRUM

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For separable Hilbert spaces H and K, the operator-valued spectrum of an operator on the Hilbert C^*-module $B(H,K)$ is introduced. It is shown that the newly defined spectrum contains the ordinary (scaler-valued) spectrum and is a (not necessarily compact) closed subset of $B(K)$. In case that H and K are finite-dimensional, we establish a one-to-one correspondence between $B(K)$-linear operators on $B(H,K)$ and the ordinary linear operators on $B(H)$, which helps us to characterize the operator-valued spectrum of $B(K)-$linear operators on $B(H,K)$.

Keywords: Adjoint, Operator-valued spectrum, Hilbert C^*-module, Operator-valued eigenvalue, Dual space.

1. Introduction

The notion of the spectrum of an operator has been generalized by Ernest [5]. He developed an analogue of the spectral theorem for all operators on separable Hilbert spaces. Authors in [15] defined an $n \times n$ matrix spectrum. These spectral generalizations of an operator were constructed by use of existence of representations of C^*-algebras generated by the operator and the identity operator. After that, Hadwin in [7] introduced reducing operator spectra. These spectra are based on geometric rather than algebraic considerations and the unifying feature of them is their relation to the closure of the unitary equivalence class of an operator with respect to different operator topologies [8].

A sequence of bounded linear operators between two Hilbert spaces is denoted by Sun [17] as generalized frames or g-frames. To modify and determine g-frames [1] we need to introduce a new generalization of spectrum of a $B(K)$-linear operator S on the Hilbert C^*-module $B(H,K)$, where H and K are two separable Hilbert spaces. In this regards, the definition of spectrum of S is based on the noninvertibility of $S - \Lambda I$, where the spectrum Λ is an operator on K and I is the identity operator on $B(H,K)$. This chracteristic of spectra hasn't appeared in the previous operator-valued spectra generalizations of operators. The aim of this paper is to introduce the new type of operator-valued spectrum of module operators on Hilbert C^*-modules, $B(H,K)$.

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The paper is organized as follows. We continue this introductory section with a review of the basic definitions and notations of Hilbert C^*-modules. In section 2, we introduce the notion of operator-valued spectrum and prove some properties of it. The main results of the paper are included in Section 3, where we study the module operators on $B(H,K)$ and we find operator-valued spectrum of them, when H and K are finite- dimensional.

Let us recall the definition of a Hilbert C^*-module and the set of bounded linear operators between two Hilbert spaces as a Hilbert C^*-module. For more details, we refer the interested reader to [11, 12, 14, 16, 18]. Also, the concept of module operators on a Hilbert C^*-module and their adjoint has appeared in [13, 14]. Let A be a C^*-algebra, M be a (left) A-module and $\lambda(ax) = a(\lambda x)$ for every $\lambda \in \mathbb{C}, a \in A$ and $x \in M$. If there exists a mapping $\langle .,. \rangle : M \times M \longrightarrow A$ with the following properties

i) $\langle x, x \rangle \geq 0$ for every $x \in M$,
ii) $\langle x, x \rangle = 0$ if and only if $x = 0$,
iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for every x,$y \in M$,
iv) $\langle ax, y \rangle = a \langle x, y \rangle$ for every $a \in A$ and $x, y \in M$,
v) $\langle x + y, z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for every $x, y, z \in M$,

such that M is complete with respect to the norm $\|x\| = \| \langle x, x \rangle \|^{\frac{1}{2}}$, then the pair $\{M, \langle .,. \rangle\}$ is called a (left) Hilbert C^*-module over A.

For two Hilbert spaces H and K, let $B(H,K)$ be the set of all bounded linear operators from H into K. The set $B(H,K)$ is easily seen to be a Hilbert C^*-module over $B(K)$, with $B(K)$-inner product $\langle T, S \rangle = TS^*$, for all $T, S \in B(H,K)$ and the linear operation on $B(K)$ define by $T_1 T = T_1 \circ T$ for all $T_1 \in B(K)$ and $T \in B(H,K)$. The $B(K)$-module $B(H,K)$ plays a crucial role in the study of frames and g-frames [17]. Such frames have applications in pure [4, 9] and applied mathematics [3], harmonic analysis [6], and even quantum communication [2].

Throughout this paper, we consider H and K as separable Hilbert spaces and $B(B(H,K))$ as the set of all bounded $B(K)$-linear operators (module operators) on $B(H,K)$.

2. **Operator-valued spectrum**

For S in $B(B(H,K))$, a subset of complex numbers that is called the spectrum of S is defined by $\sigma(S) = \{\lambda \in \mathbf{C} : \lambda I - S \ \ is \ \ not \ \ invertible\}$, and the resolvent set of S, $\rho(S)$, is defined by the complement of $\sigma(S)$ in $\mathbb{C}$.
We are going to extend the spectrum of S from complex numbers to operators.

Definition 2.1. Let $S \in B(B(H,K))$ and set

$$\rho_{ov}(S) = \{\Lambda \in B(K) : \Lambda I - S \ \ is \ \ invertible\},$$

where I is the identity operator on $B(H,K)$. We define by $\rho_{ov}(S)$ the operator-valued resolvent set of the bounded operator S, and the operator-valued spectrum of S, $\sigma_{ov}(S)$, define by $B(K) \setminus \rho_{ov}(S)$. An operator $\Lambda \in B(K)$ is said to be an operator-valued eigenvalue for S if $ST = \Lambda T$ for some nonzero T in $B(H,K)$, and the subspace $\{T \in B(H,K) : ST = \Lambda T\}$ is called operator-valued eigenspace corresponding to Λ.

Proposition 2.2. *Let $S \in B(B(H,K))$. Then $\sigma_{ov}(S) \neq \emptyset$.*

Proof. Let I_K be the identity operator on K and I be the identity operator on $B(H,K)$. Since $I_K I = I_K \circ I = I$, we derive

$$\lambda I - S = \lambda(I_K I) - S = (\lambda I_K)I - S.$$

Therefore, if $\lambda \in \sigma(S)$, then $\lambda I_K \in \sigma_{ov}(S)$ and $\sigma_{ov}(S) \neq \emptyset$. □

Remark 2.3. $\sigma(S)$ is isometrically embeded in $\sigma_{ov}(S)$ by $\lambda \mapsto \lambda I_K$. Therefore, we consider $\sigma(S)$ as a subset of $\sigma_{ov}(S)$. In the following we point out that $\sigma(S) \neq \sigma_{ov}(S)$.

Example 2.4. *Let H and K be Hilbert spaces over a field F such that dimH=dimK=2 . Then $B(H,K) \simeq M_2(F)$. Define $S : M_2(F) \longrightarrow M_2(F)$, by*

$$S\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix},$$

and consider $\Lambda \in B(K) \simeq M_2(F)$, by $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. If $I : M_2(F) \longrightarrow M_2(F)$ is the identity operator, then $\Lambda I - S$ is not invertible and hence $\Lambda \in \sigma_{ov}(S)$. On the other hand Λ hasn't form λI for any $\lambda \in \sigma(S)$. Therefore $\sigma(S) \neq \sigma_{ov}(S)$.

Theorem 2.5. *Let $S \in B(B(H,K))$. Then $\sigma_{ov}(S)$ is a closed subset of $B(K)$.*

Proof. Let $\Lambda, \Lambda_0 \in B(K)$ and Λ_0 be a limit point of $\sigma_{ov}(S)$ that is not in $\sigma_{ov}(S)$. Since

$$\begin{aligned} S - \Lambda I &= (S - \Lambda_0 I)[I + (S - \Lambda_0 I)^{-1}(S - \Lambda I - (S - \Lambda_0 I))] \\ &= (S - \Lambda_0 I)[I - (\Lambda - \Lambda_0)(S - \Lambda_0 I)^{-1}], \end{aligned}$$

$S - \Lambda I$ is invertible if $\|(\Lambda - \Lambda_0)(S - \Lambda_0 I)^{-1}\| \leq 1$ or $\|\Lambda - \Lambda_0\| < \frac{1}{\|(S-\Lambda_0 I)^{-1}\|}$. This means, if the distance between Λ and Λ_0 is less than $\frac{1}{\|(S-\Lambda_0 I)^{-1}\|}$, then Λ is not in $\sigma_{ov}(S)$ and this is a contradiction because Λ_0 is a limit point of $\sigma_{ov}(S)$. Therefore $\sigma_{ov}(S)$ is closed. □

Corollary 2.6. *Let $S \in B(B(H,K))$ and $\Lambda \in \rho_{ov}(S)$. If $d(\Lambda)$ is the distance between Λ and $\sigma_{ov}(S)$, then $\|(\Lambda I - S)^{-1}\| \geq \frac{1}{d(\Lambda)}$.*

It is well-known that $\sigma(S)$ is a compact set. In the following example we give an operator for which $\sigma_{ov}(S)$ is not a compact subset of $B(K)$.

Example 2.7. *Let H and K be Hilbert spaces such that dimH=dimK=2. Then $B(H,K) \simeq M_2(F)$. Define $S : M_2(F) \longrightarrow M_2(F)$, by*

$$S\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix},$$

and consider $\Lambda_n \in B(K) \simeq M_2(F)$, by $\Lambda_n = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$. If $I : M_2(F) \longrightarrow M_2(F)$ is the identity operator, then $\Lambda_n I - S$ is not invertible, and hence $\Lambda_n \in \sigma_{ov}(S)$ for each $n \in \mathbb{N}$. On the other hand $\|\Lambda_n\| \to \infty$ as $n \to \infty$. Therefore, $\sigma_{ov}(S)$ is not compact.

It is well-known that the dual space of a Banach space X is the set of all bounded linear functionals from X to $\mathbf{C}$ and is denoted by X^*. Therefore, $B(H,K)^* =$

$\{f|f : B(H,K) \longrightarrow \mathbf{C}$ *is a bounded linear map*$\}$. The set of all operator-valued functionals instead of scalar valued functionals is defined by Olsen [18] for a Hilbert C^*-module. In this way, the set of all bounded operator-valued functionals on $B(H,K)$, called $B(K)$-dual of $B(H,K)$, is defined by $B(H,K)^\sharp = \{f|f: B(H,K) \longrightarrow B(K)$ *is a bounded* $B(K)-$ *linear map*$\}$. Since $B(K)$ is a W^*-algebra, $B(H,K)^\sharp$ is a Hilbert C^*-module [14].

By the following theorem, we shall associate with each $S \in B(B(H,K))$ its adjoint, an operator $S^* \in B(B(H,K)^\sharp)$, and will see how certain properties of S are reflected in the behavior of S^*.

We write $\langle X, F\rangle$ for the value of the function F at the point X.

Theorem 2.8. *If* $S \in B(B(H,K))$, *then there exists a unique element* $S^* \in B(B(H,K)^\sharp)$ *such that* $\langle SU, W\rangle = \langle U, S^*W\rangle \,\forall U \in B(H,K)$ *and* $\forall W \in B(H,K)^\sharp$.

Proof. Let $S \in B(B(H,K))$, $U \in B(H,K)$ and $W \in B(H,K)^\sharp$. Define $S^* : B(H,K)^\sharp \longrightarrow B(H,K)^\sharp$, $W \mapsto W \circ S$. Since S and W are bounded and $B(K)$-linear, $S^*W \in B(H,K)^\sharp$. Also $\langle U, S^*W\rangle = (S^*W)(U) = (W \circ S)(U) = W(SU) = \langle SU, W\rangle$. Obviously S^* is unique. It remains to show that S^* is $B(K)$-linear. Let $\Lambda \in B(K)$, then we have

$$\begin{aligned}\langle U, S^*(\Lambda W)\rangle &= \langle SU, \Lambda W\rangle = (\Lambda W)(SU) = \Lambda \circ (W(SU)) = \Lambda \circ ((S^*W)(U))\\ &= (\Lambda(S^*W))(U) = \langle U, \Lambda(S^*W)\rangle\,.\end{aligned}$$

Therefore, $S^*(\Lambda W) = \Lambda(S^*W)$ and S^* is $B(K)$-linear. □

Note. We are going to show that the operator-valued spectrum of an operator in $B(B(H,K))$ and its adjoint in $B(B(H,K)^\sharp)$ are the same. First, we state some facts about $B(K)$-duals and adjointable elements of $B(B(H,K))$.

Lemma 2.9.

a) B(H,K) is a Hilbert C^*-module on which $B(H,K)^\sharp$ separates points.

b) $B(H,K)$ can be imbeded in $B(H,K)^{\sharp\sharp}$ as a closed subset, where $B(H,K)^{\sharp\sharp}$ is the $B(K)$-dual space of $B(H,K)^\sharp$.

c) $(SR)^*{=}R^*S^*$ for all $S, R \in B(B(H,K))$.

d) $(\Lambda I)^*{=}\Lambda I^*$, where I is the identity map on $B(H,K)$ and $\Lambda \in B(K)$.

e) If V and W are two subspaces of $B(H,K)$ with $V \subseteq W$, then $W^\perp \subseteq V^\perp$, where $V^\perp = \{f \in B(H,K)^\sharp : f(u) = 0 \quad \forall u \in V\}$.

f) S^{**} is an extension of S on $B(H,K)^{\sharp\sharp}$.

Proof. (a) Let $0 \neq T_0 \in B(H,K)$. Define $S : B(H,K) \longrightarrow B(K)$ by $ST = TT_0^*$. It is clear that, $S \in B(H,K)^\sharp$ and $ST_0 \neq 0$.
(b) Define the function $F : B(H,K) \longrightarrow B(H,K)^{\sharp\sharp}$, by $F(T)f = f(T)$, where $T \in B(H,K)$, $f \in B(H,K)^\sharp$. By applying (a) F is one to one.

(c) Let $S, R \in B(B(H,K))$, $T \in B(H,K)$ and $V \in B(H,K)^\sharp$. The relations, $\langle T, R^*S^*V\rangle = \langle (SR)T, V\rangle = \langle T, (SR)^*V\rangle$, implies that $(SR)^*{=}R^*S^*$.
(d) Let $U, V \in B(H,K)^\sharp$. We have

$$\begin{aligned}&\langle U, (\Lambda I)^*V\rangle = \langle (\Lambda I)U, V\rangle = V(\Lambda IU) = \Lambda V(IU) =\\ &\quad \Lambda\langle IU, V\rangle = \Lambda\langle U, I^*V\rangle = \Lambda(I^*V)(U) = (\Lambda I^*)(V)(U) = \langle U, (\Lambda I^*)V\rangle\,.\end{aligned}$$

Then $(\Lambda I)^* = \Lambda I^*$.
(e) and (f) are obvious. □

Theorem 2.10. *Let $S \in B(B(H,K))$. Then*

$$\sigma_{ov}(S) = \sigma_{ov}(S^*)$$

Proof. Let $\Lambda \notin \sigma_{ov}(S)$ and put $W = \Lambda I - S$. Since W^{-1} exists in $B(B(H,K))$, Theorem 2.8 implies that $(W^{-1})^*$ exists in $B(B(H,K)^\sharp)$. Using lemma 2.9, we conclude that $I^* = (WW^{-1})^* = (W^{-1})^* W^*$ and $W^* = \Lambda I^* - S^*$. Therefore $\Lambda I^* - S^*$ is invertible and $\Lambda \notin \sigma_{ov}(S^*)$.
On the other hand, let $\Lambda \notin \sigma_{ov}(S^*)$ and put again $W = \Lambda I - S$. We show that W is invertible. Since W^* is invertible by what has already been proved, $(W^{**})^{-1}$ exists in $B(B(H,K)^{\sharp\sharp})$. As we have seen in lemma 2.9, W is the restriction of W^{**} on $B(H,K)$ and hence it is one to one. It only remains to show that W is onto. Since W^{**} is invertible, it is a homeomorphism operator on $B(B(H,K)^{\sharp\sharp})$ and hence $W(B(H,K))$ is closed. Now let $W(B(H,K)) \subsetneqq B(H,K)$. Lemma 2.9 and the closeness of $W(B(H,K))$ implies that

$$0 = B(H,K)^\perp \subsetneqq W(B(H,K))^\perp.$$

Therefore there is $T \in W(B(H,K))^\perp$ such that $T \neq 0$. Then for each $U \in B(H,K)$

$$0 = \langle T, WU \rangle = \langle W^*T, U \rangle .$$

This means $W^*T = 0$, that is cotradicting the assumption that W^* is one to one. Therefore $\Lambda I - S$ is invertible and $\Lambda \notin \sigma_{ov}(S)$, which the proof is completed. □

3. Characterization of $B(B(H,K))$ and operator eigenvalued

In this section we find an orthogonal generator for the Hilbert $B(K)$-module $B(H,K)$ and we characterize the $B(K)$-linear Schmidt operators on $B(H,K)$. Afterwards, operator-valued eigenvalues and operator-valued eigenvectors of members $B(B(H,K))$ in finite dimensional case will be studied. Throughout this section suppose that I is a countable set.

In the following proposition, we show that the cardinal of generators of $B(H,K)$ is the same with the cardinal of generators of H.

Proposition 3.1. *Let $\{e_i : i \in I\}$ be an orthonormal basis for H and u be an element of K such that $\|u\| = 1$. Define $T_i : H \longrightarrow K$ by $x \mapsto \langle x, e_i \rangle u$. Then*

1) $T_k \in B(H,K)$ and $\|T_k\| = 1$,

2) $T_k^* y = \langle y, u \rangle e_k$,

3) $\langle T_i, T_j \rangle = 0$,

4) $T = \sum_{i \in I} \langle T, T_i \rangle T_i$ for all $T \in B(H,K)$, where the summation converges in strong operator topology.

Proof. For all $x \in H$ and $y \in K$, we have $\langle T_k x, y \rangle = \langle \langle x, e_k \rangle u, y \rangle = \langle x, e_k \rangle \langle u, y \rangle = \langle x, \langle y, u \rangle e_k \rangle$. This means that $T_k^* y = \langle y, u \rangle e_k$, and by an easy computation similar to the one above, parts 1 and 3 follow.

For $T \in B(H,K)$ and $i \in I$, define $\Lambda_i \in B(K)$ by $\Lambda_i = TT_i^* = \langle T, T_i \rangle$. Then $\Lambda_i x = \langle x, u \rangle Te_i$ and $(\sum_{i \in I} \langle T, T_i \rangle T_i) e_k = \sum_{i \in I} \Lambda_i T_i e_k = \sum_{i \in I} \Lambda_i \delta_{ik} u = \Lambda_k u = \langle u, u \rangle Te_k = Te_k$. This means that $\sum_{i \in I} \Lambda_i T_i$ converges to T in strong operator topology and the proof is completed. □

Definition 3.2. Let $\{e_i : i \in I\}$, $\{T_i : i \in I\}$ and u be as in the proposition above. We say $S \in B(B(H, K))$ is Hilbert C^*-Schmidt operator if $\sum_{i\in I}\sum_{l\in I} |\langle(ST_i)e_l, u\rangle|^2 < \infty$.

By the following theorem we characterize all Hilbert C^*-Schmidt operators in $B(B(H, K))$ with respect to Hilbert Schmidt operators in $B(H)$.

Theorem 3.3. *Let S be an operator on $B(H, K)$. Then S is $B(K)$-linear and Hilbert C^*-Schmidt operator if and only if there is a unique Hilbert Schmidt operator $P \in B(H)$ such that $ST = TP$, for all $T \in B(H)$.*

Proof. Suppose that $\{e_i : i \in I\}$, $\{T_i : i \in I\}$ and u be as in the proposition above. Let $P \in B(H)$ be a Hilbert Schmidt operator and $ST = TP$, for all $T \in B(H, K)$. Then $S(\Lambda T) = (\Lambda T)P = \Lambda(TP) = \Lambda(ST)$, for all $\Lambda \in B(K)$, and hence S is $B(K)$-linear. On the other hand

$$\begin{aligned}\sum_{i\in I}\sum_{l\in I} |\langle(ST_i)e_l, u\rangle|^2 &= \sum_{i\in I}\sum_{l\in I} |\langle(T_iP)e_l, u\rangle|^2 = \sum_{i\in I}\sum_{l\in I} |\langle\langle Pe_l, e_i\rangle u, u\rangle|^2 \\ &= \sum_{i\in I}\sum_{l\in I} |\langle Pe_l, e_i\rangle|^2 = \sum_{l\in I}\|Pe_i\|^2 = \|P\|^2_{Hs},\end{aligned}$$

and hence S is a Hilbert C^*-Schmidt operator. Conversely, let S be $B(K)$-linear. Define $P \in B(H)$ such that $Pe_l = \sum \langle(ST_k)e_l, u\rangle e_k$, $l \in I$. Now for every $l \in I$ and $T \in B(H, K)$,

$$\begin{aligned} STe_l &= S(\sum_{k\in I}\Lambda_k T_k)e_l = \sum_{k\in I}\Lambda_k(ST_k)e_l \\ &= \sum \langle(ST_k)e_l, u\rangle Te_k = T\sum\langle(ST_k)e_l, u\rangle e_k = T(Pe_l) = (TP)e_l.\end{aligned}$$

The above statements show that $P \in B(H)$, $ST = TP$ and P is Hilbert Schmidt. □

Note that when H and K are finite dimensional, the case we consider in the rest of the section, all operators in $B(B(H, K))$ are Hilbert C^*-Schmidt. Now we find a relation between a subset of the vector spectrum of an operator in $B(B(H, K))$ and the spectrum of its corresponding operator in $B(H)$.

Proposition 3.4. *Let $S \in B(B(H, K))$ and $P \in B(H)$ be the corresponding operator. Then $\lambda \in \sigma(P)$ if and only if $\lambda I_K \in \sigma_{ov}(S)$, where I_K is the identity operator on K.*

Proof. Let $\lambda \in \sigma(P)$, V_λ be the eigenspace corresponding to λ and T be the orthogonal projection on V_λ. Since $V_\lambda \neq 0$, T is nonzero. Due to the fact that V_λ and ${V_\lambda}^\perp$ are invariant under P, $x \in V_\lambda$ implies $T\lambda x = TPx$ and $x \in V_\lambda^\perp$ implies $T\lambda x = TPx = 0$. Then $(((\lambda I_K)I - S)T)(x) = (\lambda I_K T)x - (ST)x = T\lambda x - TPx = 0$ for all $x \in H$. Therefore $\lambda I_K \in \sigma_{ov}(S)$ and the proof is completed.

Conversely let $\Lambda = \lambda I_K \in \sigma_{ov}(S)$. Definition of $\sigma_{ov}(S)$ in the finite dimensional case implies that there exists a nonzero $T \in B(H, K)$ such that $ST = \Lambda T$. By applying Theorem 3.3, we have $ST = TP$ and hence

$$0 = (\Lambda IT - TP)(x) = T(\lambda I_H x - Px) = T(\lambda I_H - P)(x) \qquad \forall x \in H.$$

Now if $\lambda I_H - P$ is invertible, then $\{(\lambda I_H - P)(x) : x \in H\} = H$ and this means that $T = 0$. This is a contradiction to assumption that T is nonzero. Therefore $\lambda I_H - P$ is not invertible and the proof is completed. □

Recall that in the finite dimensional case all operators in $B(B(H,K))$ are Hilbert C^*-Schmidt and hence Theorem 3.3 and Proposition 3.4 are satisfied for all $S \in B(B(H,K))$.

Lemma 3.5. *Let $S \in B(B(H,K))$, $\Lambda \in B(K), dimH = n$ and $dimK = m$. If I, I_n and I_m are identity operators on $B(H,K), H$ and K, respectively and $E = \{E_{i,j}\}$ is the standard basis for $B(H,K)$, then the matrix representation of $\Lambda I - S$ with respect to E has the form*

$$[\Lambda I - S]_E = \Lambda \otimes I_n - I_m \otimes B^t,$$

where $B \in B(H)$ is the corresponding operator to S, i.e. $ST = TB$ for all $T \in B(H,K)$.

The notation $\otimes$ is the canonical tensor product in matrix theory and $E_{i,j}$ is a $m \times n$ matrix such that $(E_{i,j})_{k,l} = \delta_{k,l}$ for $1 \leqslant k \leqslant m$ and $1 \leqslant l \leqslant n$.

Proof. Since S is $B(K)$-linear, it is easy to see that the matrix representation for S respect to the standard basis E is of the form

$$\begin{pmatrix} B^t & . & . & . & . & 0 \\ & B^t & . & . & . & . \\ & & & \ddots & \ddots & \\ & 0 & & & B^t & . \\ & & & & & B^t \end{pmatrix}, \tag{3.0.1}$$

and by a routine computation

$$[\Lambda I - S]_E = \Lambda \otimes I_n - I_m \otimes B^t.$$

□

Theorem 3.6. *Let $dimH = n$, $dimK = m$, $S \in B(B(H,K))$ and $B \in B(H)$ be the corresponding operator to S. Then*

$$\Lambda \in \sigma_{ov}(S) \iff \sigma(\Lambda) \cap \sigma(B) \neq \emptyset.$$

Proof. By Lemma 3.4 we have, $[\Lambda I - S]_E = \Lambda \otimes I_n - I_m \otimes B^t$. If $\sigma(\Lambda) = \{\lambda_i : \ i = 1, ..., m\}$ and $\sigma(B) = \{\mu_j : \ j = 1, ..., n\}$, then $\sigma(\Lambda \otimes I_n - I_m \otimes B^t) = \{\lambda_i - \mu_j : \ i = 1, ..., m, j = 1, ..., n\}$ (including algebraic multiplicities in all three cases) and thus, $det(S - \Lambda I) = \Pi_{i,j}(\lambda_i - \mu_j)$ [10]. This means $det(S - \Lambda I) = 0$ if and only if for some i, j, $\lambda_i = \mu_j$. Thus $\Lambda \in \sigma(S)$ if and only if $\sigma(\Lambda) \cap \sigma(B) \neq \emptyset$. □

Now, let Λ be an operator-valued eigenvalue for S. We are going to determine the operator-valued eigenspace corresponding to Λ. If T is an operator-valued eigenvector for S corresponding to Λ, then $ST - \Lambda T = 0$ or $BT - \Lambda T = 0$. Therefore the problem of finding operator-valued eigenvectors for S corresponding to Λ is equivalent to solving the matrix equation $BT - \Lambda T = 0$. This subject is studied in [10]. In what follows, we mention some results of [10] without proof that we use to determine the operator-valued eigenvectors.

Remark 3.7. i) Let $J_r(0) \in M_r$ and $J_s(0) \in M_s$ be singular blocks. Then $T \in M_{r,s}$ is a solution of $J_r(0)T - TJ_s(0) = 0$ if and only if

$$T = (0 \ \ U), U \in M_r, 0 \in M_{r,s-r} \ \ if \ \ r \leq s, or$$

$$(3.0.2) \qquad T = \begin{pmatrix} U \\ 0 \end{pmatrix}, U \in M_s, 0 \in M_{r-s,s} \quad if \quad r \geq s,$$

where

$$(3.0.3) \qquad U \equiv \begin{pmatrix} a_0 & a_1 & a_2 & . & . & . \\ & a_0 & a_1 & . & . & . \\ & & a_0 & a_1 & & \\ & & & \ddots & \ddots & \\ & 0 & & & a_0 & a_1 \\ & & & & & a_0 \end{pmatrix} = [u_{i,j}]$$

is, in either case, an arbitrary upper Toeplitz matrix with $u_{i,j} = a_{i-j}$. The dimension of the nullspace of the linear transformation $T \to J_r(0)T - TJ_s(0)$ is $min\{r, s\}$.

ii) Let $S \in B(B(H, K))$, $\Lambda \in \sigma_{ov}(S)$, $\lambda_i \in \sigma(\Lambda)$ for $i = 1, 2, ..., p$ and $\mu_j \in \sigma(B)$ for $j = 1, 2, ..., q$. Then T is an operator-valued eigenvector for S corresponding to Λ if and only if

$$(3.0.4) \qquad T = \begin{pmatrix} T_{11} & . & . & . & T_{1q} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ T_{p1} & . & . & . & T_{pq} \end{pmatrix}$$

where $T_{i,j} \in M_{n_i,m_j}, i = 1, 2, ..., p, j = 1, 2, ..., q$ is a solution of the equation $J_{n_i}(\lambda_i)X_{i,j} - X_{i,j}J_{m_j}(\mu_j) = 0$. The dimension of the eigenspace corresponding to Λ is $\sum_i \sum_j t_{i,j}$ where $t_{i,j} = 0$ if $\lambda_i \neq \mu_j$ and $t_{i,j} = min\{n_i, m_j\}$ if $\lambda_i = \mu_j$.

Example 3.8. *Let $dimH = 4, dimK = 3$, $S \in B(B(H, K))$ and*

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

be the matrix corresponding to S. The eigenvalues of B are $\mu_1 = 1, \mu_2 = 2$ with multiplicity 2 . By Theorem 3.6

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is an operator valued eigenvalue of S. The eigenvalues of Λ are $\lambda_1 = -1, \lambda_2 = 1$ The Jordan forms of B and Λ are, respectively,

$$J_B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$J_\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, with notation of Remark 3.7: $n_1 = 1, n_2 = 2, m_1 = 2, m_2 = 2$ *and*

$$J_{m_1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$J_{m_2} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$J_{n_1} = \begin{pmatrix} -1 \end{pmatrix}$$

$$J_{n_2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Therefore T *is an operator valued eigenvector of* S *corresponding to* Λ *if and only if* T *is as follows :*

$$T_{11} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad , \quad T_{12} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad , \quad T_{21} = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} \quad , \quad T_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(3.**0**.5)
$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_0 & a_1 & 0 & 0 \\ 0 & a_0 & 0 & 0 \end{pmatrix},$$

where a_0 *and* a_1 *are arbitrary. By Remark 3.7, the dimension of the eigenspace corresponding to* Λ *is* 2.

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