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DUAL JET TIME-DEPENDENT HAMILTON GEOMETRY AND THE LEAST SQUARES VARIATIONAL METHOD

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In this paper we geometrize on the 1-jet space $J^{1*}(\mathbb{R}, M)$ the time-dependent Hamiltonians, in the sense of canonical nonlinear connections, Cartan N-linear connections, d-torsions and d-curvatures. Some time-dependent Hamiltonian field-like geometrical models (electromagnetic-like and gravitational-like) depending on momenta are also constructed. An application related to the time-dependent Hamiltonian of the least squares variational method is also studied.

Keywords: dual 1-jet spaces; Cartan N-linear connection; $d-$torsions; $d-$curvatures; momentum electromagnetic-like geometry; momentum gravitational-like geometry.

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1. **Introduction**

We further confine to the opinion expressed by Peter Olver in his celebrated work [13], which says that 1-jet spaces and their duals are appropriate fundamental ambient mathematical spaces used to model classical and quantum field theories. In such a physical and geometrical context, suggested by the cotangent bundle framework of Atanasiu ([1, 2]) and Miron et al. (see, e.g., [6, 8]), followed papers like [10] and [12] which are devoted to developing the *time-dependent covariant Hamilton geometry on dual* 1*-jet spaces* (in the sense of $d-$tensors, time-dependent semisprays of momenta, nonlinear connections, N-linear connections, $d-$torsions and $d-$curvatures), which is a natural dual jet extension of the Hamilton geometry on the cotangent bundle. The geometrical study from the papers [10] and [12] is realized on the *dual* 1*-jet vector bundle* $J^{1*}(\mathbb{R}, M) \equiv \mathbb{R}\times T^*M \to \mathbb{R}\times M$, whose local coordinates are denoted by (t, x^i, p^1_i). Here M^n is a smooth real manifold of dimension n, whose local coordinates are $(x^i)_{i=\overline{1,n}}$. The coordinates p^1_i are called *momenta*, and

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the dual 1-jet space $J^{1*}(\mathbb{R}, M)$ is called the *time-dependent phase space of momenta*. The transformations of coordinates $(t, x^i, p_i^1) \longleftrightarrow (\tilde{t}, \tilde{x}^i, \tilde{p}_i^1)$, induced from $\mathbb{R} \times M$ on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$, have the expressions

$$\begin{cases} \tilde{t} = \tilde{t}\,(t) \\ \tilde{x}^i = \tilde{x}^i\,(x^j) \\ \tilde{p}_i^1 = \dfrac{\partial x^j}{\partial \tilde{x}^i}\dfrac{d\tilde{t}}{dt}p_j^1, \end{cases} \tag{1}$$

where $d\tilde{t}/dt \neq 0$ and $\det(\partial \tilde{x}^i/\partial x^j) \neq 0$. Consequently, in our dual jet geometrical approach, we use a "*relativistic*" time t. As an example, in the Hamiltonian approach from the monograph [8], the authors use the trivial bundle $\mathbb{R} \times T^*M$ over the base cotangent space T^*M, whose coordinates induced by T^*M are (t, x^i, p_i). The changes of coordinates on the trivial bundle $\mathbb{R} \times T^*M \to T^*M$ are

$$\begin{cases} \tilde{t} = t \\ \tilde{x}^i = \tilde{x}^i\,(x^j) \\ \tilde{p}_i = \dfrac{\partial x^j}{\partial \tilde{x}^i}p_j, \end{cases}$$

pointing out the *absolute* character of the time variable t.

2. Time-dependent Hamiltonians of momenta

Let us start with a time-dependent Hamiltonian $H : E^* = J^{1*}(\mathbb{R}, M) \to \mathbb{R}$, locally expressed by

$$E^* \ni (t, x^i, p_i^1) \to H(t, x^i, p_i^1) \in \mathbb{R},$$

whose *fundamental vertical metrical d−tensor* is given by

$$G^{(i)(j)}_{(1)(1)} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i^1 \partial p_j^1}.$$

Let $h = (h_{11}(t))$ be a semi-Riemannian metric on the time manifold $\mathbb{R}$, together with a $d-$tensor $g^{ij}(t, x^k, p_k^1)$ on the dual 1-jet space E^*, which is symmetric, has the rank $n = \dim M$ and has a constant signature.

Definition 2.1. *A time-dependent Hamiltonian $H : E^* \to \mathbb{R}$, having the fundamental vertical metrical d−tensor of the form*

$$G^{(i)(j)}_{(1)(1)}(t, x^k, p_k^1) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i^1 \partial p_j^1} = h_{11}(t)g^{ij}(t, x^k, p_k^1), \tag{2}$$

is called a Kronecker h-regular time-dependent Hamiltonian function.

In this geometrical context, we can introduce the following notion:

Definition 2.2. *A pair of mathematical objects $H^n = (E^*, H)$, consisting of the dual 1-jet space $E^* = J^{1*}(\mathbb{R}, M)$ and a Kronecker h-regular time-dependent Hamiltonian $H : E^* \to \mathbb{R}$, is called a* time-dependent Hamilton space.

Example 2.1. *If $h_{11}(t)$ (respectively $\varphi_{ij}(x)$) is a semi-Riemannian metric on the time (respectively spatial) manifold $\mathbb{R}$ (respectively M) having the physical meaning of* gravitational potentials*, while m, c and e are the well-known constants from Theoretical Physics representing the* mass of the test body, speed of light *and* electric charge*, then let us consider the Kronecker h-regular time-dependent Hamiltonian $H_1 : E^* \to \mathbb{R}$, defined by*

$$H_1 = \frac{1}{4mc} h_{11}(t)\varphi^{ij}(x)p_i^1 p_j^1 - \frac{e}{m^2 c} A^{(i)}_{(1)}(x)p_i^1 + \frac{e^2}{m^3 c} F(t,x) - \mathsf{P}(t,x), \qquad (3)$$

where $A^{(i)}_{(1)}(x)$ is a $d-$tensor on E^* having the physical meaning of a potential $d-$tensor of an electromagnetic field*, $\mathsf{P}(t,x)$ is a* potential function *and the function $F(t,x)$ is given by*

$$F(t,x) = h^{11}(t)\varphi_{ij}(x)A^{(i)}_{(1)}(x)A^{(j)}_{(1)}(x).$$

Then the Hamilton space $\mathcal{EDH}^n = (E^*, H_1)$ defined by the time-dependent Hamiltonian (3) is called the time-dependent Hamilton space of electrodynamics of autonomous type*. This is natural, since in the particular case of the metric $h = \delta = 1$, we recover the classical Hamilton space of electrodynamics studied in the monograph* [8]*. The non-dynamical character (i.e., the independence on the temporal coordinate t) of the spatial gravitational potentials $\varphi_{ij}(x)$ motivated us to use the term* "autonomous"*.*

Example 2.2. *More generally, if we take on E^* a symmetric $d-$tensor field $g_{ij}(t,x)$ having the rank n and a constant signature, we can define the Kronecker h-regular time-dependent Hamiltonian $H_2 : E^* \to \mathbb{R}$, by putting*

$$H_2 = h_{11}(t)g^{ij}(t,x)p_i^1 p_j^1 + U^{(i)}_{(1)}(t,x)p_i^1 + \mathcal{F}(t,x), \qquad (4)$$

where $U^{(i)}_{(1)}(t,x)$ is a $d-$tensor field on E^* and $\mathcal{F}(t,x)$ is a function on E^*. Then the Hamilton space $\mathcal{NEDH}^n = (E^*, H_2)$ defined by the affine quadratic time-dependent Hamiltonian (4) is called the non-autonomous time-dependent Hamilton space of electrodynamics*. The dynamical character (i.e., the dependence on the temporal coordinate t) of the gravitational potentials $g_{ij}(t,x)$ motivated us to use the word* "non-autonomous"*.*

3. Canonical nonlinear connections on H^n-spaces

In the sequel, following the geometrical ideas from (Miron, [6]), we will prove that any Kronecker h-regular time-dependent Hamiltonian H produces a natural nonlinear connection on the dual 1-jet bundle E^*, which is determined by H alone. In order to do that, let us consider a Kronecker h-regular time-dependent Hamiltonian H, whose fundamental vertical metrical $d-$tensor is given by (2). Also, let us introduce the *generalized Christoffel symbols* of the inverse spatial metrical $d-$tensor $g_{ij}(t,x^k,p_k^1)$, where $g^{ij}(t,x^k,p_k^1) =$

$h^{11}(t)G^{(i)(j)}_{(1)(1)}(t,x^k,p^1_k)$, via the formulas

$$\Gamma^k_{ij}=\frac{g^{kl}}{2}\left(\frac{\partial g_{li}}{\partial x^j}+\frac{\partial g_{lj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\right).$$

In this context, by using the notations from above, we can state the following result:

Theorem 3.1. *The pair of local functions* $N=\left(\underset{1}{N}{}^{(1)}_{(i)1},\underset{2}{N}{}^{(1)}_{(i)j}\right)$ *on* E^*, *where*

$$\begin{aligned}\underset{1}{N}{}^{(1)}_{(i)1}&=H^1_{11}p^1_i=(h^{11}/2)(dh_{11}/dt)p^1_i,\\ \underset{2}{N}{}^{(1)}_{(i)j}&=\frac{h^{11}}{4}\left[\frac{\partial g_{ij}}{\partial x^k}\frac{\partial H}{\partial p^1_k}-\frac{\partial g_{ij}}{\partial p^1_k}\frac{\partial H}{\partial x^k}+g_{ik}\frac{\partial^2 H}{\partial x^j\partial p^1_k}+g_{jk}\frac{\partial^2 H}{\partial x^i\partial p^1_k}\right],\end{aligned}\tag{5}$$

represents a nonlinear connection on E^*, *which is called* the canonical nonlinear connection of the time-dependent Hamilton space $H^n=(E^*,H)$.

Proof. Taking into account the transformation rule of the Christoffel symbol H^1_{11} of the temporal semi-Riemannian metric h_{11}, by direct local computations, we deduce that the temporal components $\underset{1}{N}{}^{(1)}_{(i)1}$ from (5) verify the transformation rules of a temporal nonlinear connection (see [10] or [12, p.100]).

The spatial components from (5) become (except the multiplication factor h^{11}) exactly the canonical nonlinear connection from the classical Hamilton geometry (see [6] or [8, p.127]). □

4. Cartan canonical connection in H^n-spaces

Let $H^n=(E^*=J^{1*}(\mathbb{R},M),H)$ be a time-dependent Hamilton space, whose fundamental vertical metrical $d-$tensor is given by (2). Let

$$N=\left(\underset{1}{N}{}^{(1)}_{(i)1},\ \underset{2}{N}{}^{(1)}_{(i)j}\right)$$

be the canonical nonlinear connection of the time-dependent Hamilton space H^n, given by (5).

Theorem 4.1 (the Cartan canonical N-linear connection)**.** *On the time-dependent Hamilton space* $H^n=(E^*,H)$ *endowed with the canonical nonlinear connection (5), there exists a unique h-normal N-linear connection*

$$C\Gamma(N)=\left(H^1_{11},\ A^i_{j1},\ H^i_{jk},\ C^{i(k)}_{j(1)}\right),\tag{6}$$

having the following metrical properties:

(i) $g_{ij|k}=0,\quad g^{ij}|^{(k)}_{(1)}=0,$

(ii) $A^i_{j1}=\dfrac{g^{il}}{2}\dfrac{\delta g_{lj}}{\delta t},\quad H^i_{jk}=H^i_{kj},\quad C^{i(k)}_{j(1)}=C^{k(i)}_{j(1)},$

where "$_{/1}$", "$_{|k}$" *and* "$|^{(k)}_{(1)}$" *represent the local covariant derivatives induced by the h-normal N-linear connection* $C\Gamma(N)$.

Proof. Let $C\Gamma(N) = \left(H^1_{11},\ A^i_{j1},\ H^i_{jk},\ C^{i(k)}_{j(1)}\right)$ be an h-normal N-linear connection, whose local coefficients are defined by the relations

$$A^1_{11} = H^1_{11} = \frac{h^{11}}{2}\frac{dh_{11}}{dt}, \qquad A^i_{j1} = \frac{g^{il}}{2}\frac{\delta g_{lj}}{\delta t},$$
$$H^i_{jk} = \frac{g^{ir}}{2}\left(\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r}\right), \quad C^{j(k)}_{i(1)} = -\frac{g_{ir}}{2}\left(\frac{\partial g^{jr}}{\partial p^1_k} + \frac{\partial g^{kr}}{\partial p^1_j} - \frac{\partial g^{jk}}{\partial p^1_r}\right).$$

Taking into account the local expressions of the local covariant derivatives induced by the h-normal N-linear connection $C\Gamma(N)$, by local computations, we infer that $C\Gamma(N)$ satisfies conditions (i) and (ii).

Conversely, let us consider an h-normal N-linear connection

$$\tilde{C}\Gamma(N) = \left(\tilde{A}^1_{11},\ \tilde{A}^i_{j1},\ \tilde{H}^i_{jk},\ \tilde{C}^{i(k)}_{j(1)}\right),$$

which satisfies conditions (i) and (ii). It follows that we have

$$\tilde{A}^1_{11} = H^1_{11}, \quad \tilde{A}^i_{j1} = \frac{g^{il}}{2}\frac{\delta g_{lj}}{\delta t}.$$

Moreover, the metrical condition $g_{ij|k} = 0$ is equivalent with

$$\frac{\delta g_{ij}}{\delta x^k} = g_{rj}\tilde{H}^r_{ik} + g_{ir}\tilde{H}^r_{jk}.$$

Applying now a Christoffel process to the indices $\{i, j, k\}$, we get

$$\tilde{H}^i_{jk} = \frac{g^{ir}}{2}\left(\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r}\right).$$

By analogy, using the relations $C^{i(k)}_{j(1)} = C^{k(i)}_{j(1)}$ and $g^{ij}|^{(k)}_{(1)} = 0$, together with a Christoffel process applied to the indices $\{i, j, k\}$, we find

$$\tilde{C}^{j(k)}_{i(1)} = -\frac{g_{ir}}{2}\left(\frac{\partial g^{jr}}{\partial p^1_k} + \frac{\partial g^{kr}}{\partial p^1_j} - \frac{\partial g^{jk}}{\partial p^1_r}\right).$$

In conclusion, the uniqueness of the *Cartan canonical connection* $C\Gamma(N)$ on the dual 1-jet space $E^* = J^{1*}(\mathbb{R}, M)$ is obvious. □

Remark 4.1. *The Cartan canonical connection $C\Gamma(N)$ of the time-dependent Hamilton space H^n also verifies the metrical properties*

$$h_{11/1} = h_{11|k} = h_{11}|^{(k)}_{(1)} = 0, \quad g_{ij/1} = 0.$$

5. $d-$Torsions and $d-$curvatures

By applying the formulas of the local $d-$torsions and $d-$curvatures of an h-normal N-linear connection $D\Gamma(N)$ (see Tables and formulas from [5]) to the Cartan canonical connection $C\Gamma(N)$, we get the following important geometrical results:

Theorem 5.1. *The torsion tensor* $\mathbf{T}$ *of the Cartan canonical connection* $C\Gamma(N)$ *of the time-dependent Hamilton space* H^n *is determined by the local d−components*

	$h_{\mathbb{R}}$	h_M	v
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_M h_{\mathbb{R}}$	0	T^r_{1j}	$R^{(1)}_{(r)1j}$
$vh_{\mathbb{R}}$	0	0	$P^{(1)\ (j)}_{(r)1(1)}$
$h_M h_M$	0	0	$R^{(1)}_{(r)ij}$
vh_M	0	$P^{r(j)}_{i(1)}$	$P^{(1)\ (j)}_{(r)i(1)}$
vv	0	0	0

where $T^r_{1j} = -A^r_{j1}$, $P^{r(j)}_{i(1)} = C^{r(j)}_{i(1)}$,

$$P^{(1)\ (j)}_{(r)1(1)} = \frac{\partial \underset{1}{N}{}^{(1)}_{(r)1}}{\partial p^1_j} + A^j_{r1} - \delta^j_r H^1_{11}, \quad P^{(1)\ (j)}_{(r)i(1)} = \frac{\partial \underset{2}{N}{}^{(1)}_{(r)i}}{\partial p^1_j} + H^j_{ri},$$

$$R^{(1)}_{(r)1j} = \frac{\delta \underset{1}{N}{}^{(1)}_{(r)1}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(1)}_{(r)j}}{\delta t}, \quad R^{(1)}_{(r)ij} = \frac{\delta \underset{2}{N}{}^{(1)}_{(r)i}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(1)}_{(r)j}}{\delta x^i}.$$

Theorem 5.2. *The curvature tensor* $\mathbf{R}$ *of the Cartan canonical connection* $C\Gamma(N)$ *of the time-dependent Hamilton space* H^n *is determined by the following adapted local curvature d−tensors:*

	$h_{\mathbb{R}}$	h_M	v
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_M h_{\mathbb{R}}$	0	R^l_{i1k}	$-R^{(1)(l)}_{(i)(1)1k} = -R^l_{i1k}$
$vh_{\mathbb{R}}$	0	$P^{l\ (k)}_{i1(1)}$	$-P^{(1)(l)\ (k)}_{(i)(1)1(1)} = -P^{l\ (k)}_{i1(1)}$
$h_M h_M$	0	R^l_{ijk}	$-R^{(1)(l)}_{(i)(1)jk} = -R^l_{ijk}$
vh_M	0	$P^{l\ (k)}_{ij(1)}$	$-P^{(1)(l)\ (k)}_{(i)(1)j(1)} = -P^{l\ (k)}_{ij(1)}$
vv	0	$S^{l(j)(k)}_{i(1)(1)}$	$-S^{(1)(l)(j)(k)}_{(i)(1)(1)(1)} = -S^{l(j)(k)}_{i(1)(1)}$

where

$$R^l_{i1k} = \frac{\delta A^l_{i1}}{\delta x^k} - \frac{\delta H^l_{ik}}{\delta t} + A^r_{i1}H^l_{rk} - H^r_{ik}A^l_{r1} + C^{l(r)}_{i(1)}R^{(1)}_{(r)1k},$$

$$R^l_{ijk} = \frac{\delta H^l_{ij}}{\delta x^k} - \frac{\delta H^l_{ik}}{\delta x^j} + H^r_{ij}H^l_{rk} - H^r_{ik}H^l_{rj} + C^{l(r)}_{i(1)}R^{(1)}_{(r)jk},$$

$$P^{l\ (k)}_{i1(1)} = \frac{\partial A^l_{i1}}{\partial p^1_k} - C^{l(k)}_{i(1)/1} + C^{l(r)}_{i(1)}P^{(1)\ (k)}_{(r)1(1)},$$

$$P^{l\ (k)}_{ij(1)} = \frac{\partial H^l_{ij}}{\partial p^1_k} - C^{l(k)}_{i(1)|j} + C^{l(r)}_{i(1)}P^{(1)\ (k)}_{(r)j(1)},$$

$$S_{i(1)(1)}^{l(j)(k)} = \frac{\partial C_{i(1)}^{l(j)}}{\partial p_k^1} - \frac{\partial C_{i(1)}^{l(k)}}{\partial p_j^1} + C_{i(1)}^{r(j)} C_{r(1)}^{l(k)} - C_{i(1)}^{r(k)} C_{r(1)}^{l(j)}.$$

6. Momentum field-like geometrical models

In what follows, we create a large geometrical framework on the dual 1-jet space E^* for a time-dependent Hamiltonian approach of the electromagnetic and gravitational physical fields. Our geometric-physical construction is achieved starting only from a given time-dependent Hamiltonian function H, which naturally produces a canonical nonlinear connection N, a canonical Cartan N-linear connection $C\Gamma(N)$ and their corresponding local $d-$torsions and curvatures. In this context, we construct some geometrical time-dependent Hamiltonian electromagnetic-like and gravitational-like field theories, governed by some natural geometrical momentum Maxwell-like and Einstein-like equations.

6.1. Geometrical momentum Maxwell-like equations

Let $H^n = (E^*, H)$ be a time-dependent Hamilton space, endowed with its canonical nonlinear connection (5), which produces the adapted vertical distinguished 1-forms $\delta p_i^1 = dp_i^1 + \underset{1}{N}{}_{(i)1}^{(1)} dt + \underset{2}{N}{}_{(i)j}^{(1)} dx^j$. Let $C\Gamma(N)$ be the Cartan canonical linear connection of the space H^n, locally defined by (6). Let us also consider the canonical Liouville-Hamilton $d-$tensor field of momenta $\mathbb{C}^* = p_i^1(\partial/\partial p_i^1)$, together with the fundamental vertical metrical $d-$tensor (2). All these geometrical objects allow us to define the *metrical deflection $d-$tensors*

$$\Delta_{(1)1}^{(i)} = p_{(1)/1}^{(i)}, \quad \Delta_{(1)j}^{(i)} = p_{(1)|j}^{(i)}, \quad \vartheta_{(1)(1)}^{(i)(j)} = p_{(1)}^{(i)}|_{(1)}^{(j)},$$

where $p_{(1)}^{(i)} = G_{(1)(1)}^{(i)(k)} p_k^1$ and "$_{/1}$", "$_{|j}$" and "$|_{(1)}^{(j)}$" are the local covariant derivatives induced by the Cartan connection $C\Gamma(N)$. Taking into account the form of the local covariant derivatives of the Cartan canonical connection $C\Gamma(N)$, by direct computations, we get

Proposition 6.1. *The metrical deflection $d-$tensors of the time-dependent Hamilton space H^n are given by*

$$\Delta_{(1)1}^{(i)} = -h_{11} g^{ik} A_{k1}^r p_r^1, \quad \Delta_{(1)j}^{(i)} = h_{11} g^{ik} \left[-\underset{2}{N}{}_{(k)j}^{(1)} - H_{kj}^r p_r^1 \right],$$

$$\vartheta_{(1)(1)}^{(i)(j)} = h_{11} g^{ij} - h_{11} g^{ik} C_{k(1)}^{r(j)} p_r^1.$$

In order to construct our time-dependent Hamiltonian theory of electromagnetism, we introduce the following geometric-physical notion:

Definition 6.1. *The distinguished* 2*-form on the* 1*-jet space* E^*, *defined by*

$$\mathbb{F} = F^{(i)}_{(1)j}\delta p^1_i \wedge dx^j + f^{(i)(j)}_{(1)(1)}\delta p^1_i \wedge \delta p^1_j, \tag{7}$$

where

$$F^{(i)}_{(1)j} = \frac{1}{2}\left[\Delta^{(i)}_{(1)j} - \Delta^{(j)}_{(1)i}\right], \qquad f^{(i)(j)}_{(1)(1)} = \frac{1}{2}\left[\vartheta^{(i)(j)}_{(1)(1)} - \vartheta^{(j)(i)}_{(1)(1)}\right], \tag{8}$$

is called the electromagnetic field of the time-dependent Hamilton space H^n *or* momentum electromagnetic field.

By a straightforward calculation, we infer the following

Proposition 6.2. *The local components* $F^{(i)}_{(1)j}$ *and* $f^{(i)(j)}_{(1)(1)}$ *of the electromagnetic field* $\mathbb{F}$, *associated with the Hamilton space* H^n, *have the following expressions:*

$$F^{(i)}_{(1)j} = \frac{h^{11}}{2}\left[g^{jk}\underset{2}{N}{}^{(1)}_{(k)i} - g^{ik}\underset{2}{N}{}^{(1)}_{(k)j} + \left(g^{jk}H^r_{ki} - g^{ik}H^r_{kj}\right)p^1_r\right], \quad f^{(i)(j)}_{(1)(1)} = 0.$$

The main result of our abstract geometrical Hamilton time-dependent electromagnetism of momenta is

Theorem 6.1. *The electromagnetic components* $F^{(i)}_{(1)j}$ *of the space* H^n *are governed by the following* geometrical Maxwell-like equations*:*

$$\begin{cases} F^{(i)}_{(1)k/1} = \dfrac{1}{2}\mathcal{A}_{\{i,k\}}\left\{\Delta^{(i)}_{(1)1|k} + \Delta^{(i)}_{(1)r}T^r_{1k} + \vartheta^{(i)(r)}_{(1)(1)}R^{(1)}_{(r)1k} + R^i_{r1k}p^{(r)}_{(1)}\right\} \\ \sum_{\{i,j,k\}} F^{(i)}_{(1)j|k} = -\dfrac{1}{2}\sum_{\{i,j,k\}}\left\{\vartheta^{(i)(r)}_{(1)(1)}R^{(1)}_{(r)jk} + R^i_{rjk}p^{(r)}_{(1)}\right\} \\ F^{(i)}_{(1)j}|^{(k)}_{(1)} = \dfrac{1}{2}\mathcal{A}_{\{i,j\}}\left\{\vartheta^{(i)(k)}_{(1)(1)|j} - P^{i\ \ (k)}_{rj(1)}p^{(r)}_{(1)} - \Delta^{(i)}_{(1)r}C^{r(k)}_{j(1)} - \vartheta^{(i)(r)}_{(1)(1)}P^{(1)\ \ (k)}_{(r)j(1)}\right\}, \end{cases}$$

where $\mathcal{A}_{\{i,j\}}$ *means an alternate sum and* $\sum_{\{i,j,k\}}$ *means a cyclic summation over these indices.*

Proof. The general Ricci identities applied to the metric g^{ij} yield the equalities (see [3]):

$$\begin{gathered} g^{ir}R^j_{r1k} + g^{jr}R^i_{r1k} = 0, \qquad g^{ir}R^j_{rkl} + g^{jr}R^i_{rkl} = 0, \\ g^{ir}P^{j\ \ (l)}_{rk(1)} + g^{jr}P^{i\ \ (l)}_{rk(1)} = 0. \end{gathered} \tag{9}$$

Let us consider now the following non-metrical deflection $d-$tensor identities (see [4]):

(d_1) $\Delta^{(1)}_{(p)1|k} - \Delta^{(1)}_{(p)k/1} = p^1_r R^r_{p1k} - \Delta^{(1)}_{(p)r}T^r_{1k} - \vartheta^{(1)(r)}_{(p)(1)}R^{(1)}_{(r)1k}$,

(d_2) $\Delta^{(1)}_{(p)j|k} - \Delta^{(1)}_{(p)k|j} = p^1_r R^r_{pjk} - \vartheta^{(1)(r)}_{(p)(1)}R^{(1)}_{(r)jk}$,

(d_3) $\Delta^{(1)}_{(p)j}|^{(k)}_{(1)} - \vartheta^{(1)(k)}_{(p)(1)|j} = p^1_r P^{r\ \ (k)}_{pj(1)} - \Delta^{(1)}_{(p)r}C^{r(k)}_{j(1)} - \vartheta^{(1)(r)}_{(p)(1)}P^{(1)\ \ (k)}_{(r)j(1)}$,

where $\Delta^{(1)}_{(i)1} = p^1_{i/1}$, $\Delta^{(1)}_{(i)j} = p^1_{i|j}$, $\vartheta^{(1)(j)}_{(i)(1)} = p^1_i|^{(j)}_{(1)}$.

By contracting the above deflection $d-$tensor identities with the fundamental vertical metrical $d-$tensor $G^{(i)(p)}_{(1)(1)}$, and using the equalities (9), we obtain the following *metrical deflection d−tensor identities:*

$$(d_1')\quad \Delta^{(i)}_{(1)1|k} - \Delta^{(i)}_{(1)k/1} = -p^{(r)}_{(1)} R^i_{r1k} - \Delta^{(i)}_{(1)r} T^r_{1k} - \vartheta^{(i)(r)}_{(1)(1)} R^{(1)}_{(r)1k},$$

$$(d_2')\quad \Delta^{(i)}_{(1)j|k} - \Delta^{(i)}_{(1)k|j} = -p^{(r)}_{(1)} R^i_{rjk} - \vartheta^{(i)(r)}_{(1)(1)} R^{(1)}_{(r)jk},$$

$$(d_3')\quad \Delta^{(i)}_{(1)j}|^{(k)}_{(1)} - \vartheta^{(i)(k)}_{(1)(1)|j} = -p^{(r)}_{(1)} P^{i\ \ (k)}_{rj(1)} - \Delta^{(i)}_{(1)r} C^{r(k)}_{j(1)} - \vartheta^{(i)(r)}_{(1)(1)} P^{(1)\ \ (k)}_{(r)j(1)}.$$

To obtain the first (respectively, the third) geometrical Maxwell-like equation, we permute the indices i and k in the identity (d_1') (respectively, the indices i and j in the identity (d_3')), and we subtract this new identity from the initial one. Moreover, by doing a cyclic sum by indices $\{i, j, k\}$ in the identity (d_2'), it follows the second geometrical Maxwell-like equation. □

6.2. Geometrical momentum Einstein-like equations

On a time-dependent Hamilton space $H^n = (E^*, H)$, via its fundamental vertical metrical $d-$tensor given by (2) and its canonical nonlinear connection (5), we construct a corresponding *momentum time-dependent gravitational h-potential*, by taking

$$\mathbb{G} = h_{11} dt \otimes dt + g_{ij} dx^i \otimes dx^j + h_{11} g^{ij} \delta p^1_i \otimes \delta p^1_j.$$

At the same time, let us consider that $C\Gamma(N)$, which is given by (6), is the Cartan canonical connection of the time-dependent Hamilton space H^n. We postulate that the geometrical momentum Einstein-like equations, which govern the time-dependent gravitational h-potential G of the Hamilton space H^n, are the abstract geometrical Einstein equations associated with the Cartan canonical connection $C\Gamma(N)$ and to the adapted metric G on E^*, namely

$$\text{Ric}(C\Gamma(N)) - \frac{\text{Sc}(C\Gamma(N))}{2}\mathbb{G} = \mathcal{K}\mathbb{T}, \tag{10}$$

where Ric$(C\Gamma(N))$ represents the distinguished *Ricci tensor* of the Cartan connection, Sc$(C\Gamma(N))$ is the *scalar curvature*, $\mathcal{K}$ is the *Einstein constant* and $\mathbb{T}$ is an intrinsic $d-$tensor of matter, which is called the *momentum stress-energy d−tensor*.

In the adapted basis of vector fields $(X_A) = (\delta/\delta t, \delta/\delta x^i, \partial/\partial p^1_i)$, produced by the canonical nonlinear connection (5), the curvature tensor $\mathbf{R}$ of the Cartan canonical connection $C\Gamma(N)$ is locally expressed by $\mathbf{R}(X_C, X_B)X_A = \mathbf{R}^D_{ABC} X_D$. It follows that we have $R_{AB} =$Ric$(X_A, X_B) = \mathbf{R}^D_{ABD}$, and Sc$(C\Gamma) =$

$G^{AB}R_{AB}$, where

$$G^{AB}=\begin{cases} h^{11}, & \text{for } A=1,\ B=1\\ g^{ij}, & \text{for } A=i,\ B=j\\ h^{11}g_{ij}, & \text{for } A={}^{(1)}_{(i)},\ B={}^{(1)}_{(j)}\\ 0, & \text{otherwise.}\end{cases} \tag{11}$$

Taking into account, on one hand, the form of the inverse metrical $d-$tensor $\mathbf{G}^*=(G_{AB})$ of the time-dependent Hamilton space H^n, and, on the other hand, the expressions of the local curvature $d-$tensors attached to the Cartan canonical connection $C\Gamma(N)$, by direct computations, we get

Proposition 6.3. *The Ricci $d-$tensor* $\mathrm{Ric}(C\Gamma(N))$ *of the Cartan canonical connection $C\Gamma(N)$ of the time-dependent Hamilton space H^n is determined by the following adapted components:*

$$\begin{array}{ll} R_{11}:=H_{11}=0, & R_{1i}=R^1_{1i1}=0,\\ R_{1(1)}^{\ \ (i)}=-P^{1(i)}_{1(1)1}=0, & R_{i1}=R^r_{i1r},\qquad R_{ij}=R^r_{ijr},\\ R^{(i)}_{(1)1}:=-P^{(i)}_{(1)1}=-P^{i\ \ (r)}_{r1(1)}, & R_{i(1)}^{\ (j)}:=-P_{i(1)}^{\ (j)}=-P^{r\ (j)}_{ir(1)},\\ R^{(i)(j)}_{(1)(1)}:=-S^{(i)(j)}_{(1)(1)}=-S^{i(j)(r)}_{r(1)(1)}, & R^{(i)}_{(1)j}:=-P^{(i)}_{(1)j}=-P^{i\ \ (r)}_{rj(1)}. \end{array}$$

By using the notations $R=g^{ij}R_{ij}$ and $S=h^{11}g_{ij}S^{(i)(j)}_{(1)(1)}$, we find

Corollary 6.1. *The scalar curvature* $\mathrm{Sc}(C\Gamma(N))$ *of the Cartan canonical connection $C\Gamma(N)$ of the space H^n is* $\mathrm{Sc}(C\Gamma(N))=R-S$.

In this context, the main result of the Hamilton geometrical momentum gravitational theory is

Theorem 6.2. *The* geometrical Einstein-like equations, *which govern the gravitational h-potential $\mathbb{G}$ of the time-dependent Hamilton space H^n, have the following adapted local form:*

$$\begin{cases} -\dfrac{R-S}{2}h_{11}=\mathcal{K}\mathbb{T}_{11}\\ R_{ij}-\dfrac{R-S}{2}g_{ij}=\mathcal{K}\mathbb{T}_{ij}\\ -S^{(i)(j)}_{(1)(1)}-\dfrac{R-S}{2}h_{11}g^{ij}=\mathcal{K}\mathbb{T}^{(i)(j)}_{(1)(1)}\\ 0=\mathbb{T}_{1i},\qquad R_{i1}=\mathcal{K}\mathbb{T}_{i1},\qquad -P^{(i)}_{(1)1}=\mathcal{K}\mathbb{T}^{(i)}_{(1)1},\\ 0=\mathbb{T}_{1(1)}^{\ \ (i)},\qquad -P_{i(1)}^{\ (j)}=\mathcal{K}\mathbb{T}_{i(1)}^{\ (j)},\qquad -P^{(i)}_{(1)j}=\mathcal{K}\mathbb{T}^{(i)}_{(1)j}, \end{cases} \tag{12}$$

where $\mathbb{T}_{AB},\ A,B\in\left\{1,\ i,\ {}^{(i)}_{(1)}\right\}$ *represent the adapted components of the momentum stress-energy $d-$tensor of matter* $\mathbb{T}$.

From a theoretical-physics point of view, it is well known that in the classical Riemannian theory of gravity, the stress-energy $d-$tensor of matter must verify some conservation laws. By a natural extension of the Riemannian conservation laws, in our geometrical Hamiltonian context, we postulate the following *momentum conservation laws* of the stress-energy $d-$tensor $\mathbb{T}$:

$$\mathbb{T}^B_{A|B} = 0, \qquad \forall\, A \in \left\{1,\ i,\ {}^{(1)}_{(i)}\right\},$$

where $\mathbb{T}^B_A = G^{BD}\mathbb{T}_{DA}$. Consequently, by direct computations, we find

Theorem 6.3. *The* momentum conservation laws *of the time-dependent Hamilton space H^n are given by the following equations:*

$$\begin{cases} \left[\dfrac{R-S}{2}\right]_{/1} = R^r_{1|r} - P^{(1)}_{(r)1}|^{(r)}_{(1)} \\ \left[R^r_j - \dfrac{R-S}{2}\delta^r_j\right]_{|r} = P^{(1)}_{(r)j}|^{(r)}_{(1)} \\ \left[S^{(1)(j)}_{(r)(1)} + \dfrac{R-S}{2}\delta^j_r\right]\Big|^{(r)}_{(1)} = -P^{r(j)}_{\ \ (1)|r}, \end{cases} \tag{13}$$

where

$$R^i_1 = g^{iq}R_{q1}, \qquad P^{(1)}_{(i)1} = h^{11}g_{iq}P^{(q)}_{(1)1}, \quad R^i_j = g^{iq}R_{qj},$$
$$P^{(1)}_{(i)j} = h^{11}g_{iq}P^{(q)}_{(1)j}, \quad P^{i(j)}_{\ \ (1)} = g^{iq}P^{\ \ (j)}_{q(1)} \qquad S^{(1)(j)}_{(i)(1)} = h^{11}g_{iq}S^{(q)(j)}_{(1)(1)}.$$

7. Geometrization of the time-dependent Hamiltonian of the least squares variational method

7.1. Hamiltonian $d-$torsions and $d-$curvatures of a dynamical system

Let us consider a non-autonomous dynamical system, given by

$$\frac{dx^i}{dt} = X^{(i)}_{(1)}(t, x^k(t)), \tag{14}$$

where $X^{(i)}_{(1)}(t,x)$ is a $d-$tensor on $\mathbb{R}\times M$, whose solutions are the global minimum points of the *least squares Lagrangian function* (see Udrişte [14] and Neagu-Udrişte [11])

$$\begin{aligned} L &= h^{11}(t)\varphi_{ij}(x)\left(y^i_1 - X^{(i)}_{(1)}\right)\left(y^j_1 - X^{(j)}_{(1)}\right) = \\ &= h^{11}\varphi_{ij}y^i_1y^j_1 - 2h^{11}\varphi_{ij}X^{(i)}_{(1)}y^j_1 + h^{11}\varphi_{ij}X^{(i)}_{(1)}X^{(j)}_{(1)}, \end{aligned} \tag{15}$$

where $y^i_1 = dx^i/dt$ and $\varphi_{ij}(x)$ is a Riemannian metric on the spatial manifold M, whose Christoffel symbols are $\gamma^i_{jk}(x)$. The Hamiltonian associated with the Lagrangian (15) is given by

$$H = \frac{h_{11}\varphi^{ij}}{4}p_i^1 p_j^1 + X_{(1)}^{(k)} p_k^1, \tag{16}$$

where $p_k^1 = \partial L / dy_1^i$ and $H = p_k^1 y_1^k - L$. This is called the *least squares Hamiltonian* associated with the dynamical system (14).

But, the differential geometry oh such time-dependent Hamiltonians was developed in the preceding sections. Consequently, we can construct the distinguished geometry of the Hamiltonian (16), in the sense of canonical nonlinear connections, Cartan N-linear connections, $d-$torsions and $d-$curvatures or momentum electromagnetic-like 2-form. For instance, by direct computations, the canonical nonlinear connection N of the time-dependent Hamiltonian function (16) has the components (see also the formulas (5))

$$\underset{1}{\mathbf{N}}_{(i)1}^{(1)} = H_{11}^1 p_i^1, \quad \underset{2}{\mathbf{N}}_{(i)j}^{(1)} = -\gamma_{ij}^k p_k^1 + h^{11}\left(X_{i1\bullet j} + X_{j1\bullet i}\right), \tag{17}$$

where $X_{i1} = \varphi_{ik} X_{(1)}^{(k)}$, and

$$X_{k1\bullet r} = \frac{\partial X_{k1}}{\partial x^r} - X_{s1}\gamma_{kr}^s.$$

Moreover, the coefficients of the generalized Cartan canonical connection $C\Gamma(N)$ of the least squares Hamiltonian function (16) reduce to

$$A_{11}^1 = H_{11}^1, \quad A_{j1}^i = 0, \quad H_{jk}^i = \gamma_{jk}^i, \quad C_{j(1)}^{i(k)} = 0. \tag{18}$$

Remark 7.1. *If we have $h_{11} = 1$ and $\varphi_{ij} = \delta_{ij}$, we find the coefficients of the canonical nonlinear connection produced by the least squares Hamiltonian function (16) as being the following:*

$$\underset{1}{\mathbf{N}}_{(i)1}^{(1)} = 0, \quad \underset{2}{\mathbf{N}}_{(i)j}^{(1)} = \frac{\partial X_{(1)}^{(i)}}{\partial x^j} + \frac{\partial X_{(1)}^{(j)}}{\partial x^i}.$$

Moreover, all coefficients of the Cartan canonical connection $C\Gamma(N)$ of the least squares Hamiltonian function (16), are zero.

By applying the formulas that determine the local $d-$torsions and $d-$curvatures of the generalized Cartan canonical connection $C\Gamma(N)$, we obtain the following important geometrical results.

Theorem 7.1. *The torsion tensor $\mathbf{T}$ of the generalized Cartan canonical connection $C\Gamma(N)$ associated with the least squares Hamiltonian (16) is determined by the local $d-$components*

$$\mathbf{R}_{(r)1j}^{(1)} = -\frac{\partial \underset{2}{\mathbf{N}}_{(r)j}^{(1)}}{\partial t} - H_{11}^1 T_{(r)j}^{(1)}, \quad \mathbf{R}_{(r)ij}^{(1)} = -\mathfrak{R}_{rij}^k p_k^1 + \left[T_{(r)i|j}^{(1)} - T_{(r)j|i}^{(1)}\right],$$

where

$$\mathfrak{R}^r_{kij} = \frac{\partial \gamma^r_{ki}}{\partial x^j} - \frac{\partial \gamma^r_{kj}}{\partial x^i} + \gamma^p_{ki}\gamma^r_{pj} - \gamma^p_{kj}\gamma^r_{pi}, \quad T^{(1)}_{(i)j} = h^{11}\left(X_{i1\bullet j} + X_{j1\bullet i}\right).$$

Moreover, all the curvature $d-$tensors of the Cartan canonical connection $C\Gamma(N)$ of the least squares Hamiltonian (16) are zero, except

$$\mathbf{R}^{(1)(l)}_{(i)(1)jk} = -R^{(1)(l)}_{(i)(1)jk} = -R^l_{ijk} := -\mathfrak{R}^l_{ijk}.$$

Remark 7.2. *If we have $h_{11} = 1$ and $\varphi_{ij} = \delta_{ij}$, we find the torsion components produced by the least squares Hamiltonian function (16) as being the following:*

$$\mathbf{R}^{(1)}_{(r)1j} = -\left(\frac{\partial^2 X^{(r)}_{(1)}}{\partial t \partial x^j} + \frac{\partial^2 X^{(j)}_{(1)}}{\partial t \partial x^r}\right), \quad \mathbf{R}^{(1)}_{(r)ij} = \frac{\partial^2 X^{(i)}_{(1)}}{\partial x^r \partial x^j} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial x^r \partial x^i}.$$

Moreover, all the curvature $d-$tensors produced by the least squares Hamiltonian (16) are zero.

The local components $F^{(i)}_{(1)j}$ and $f^{(i)(j)}_{(1)(1)}$ of the momentum electromagnetic-like field $\mathbb{F}$, which are attached to the least squares Hamiltonian function (16), are given by

$$F^{(i)}_{(1)j} = \frac{1}{8}\left[\varphi^{jk}X_{k1\bullet i} - \varphi^{ik}X_{k1\bullet j} + \varphi^{jk}X_{i1\bullet k} - \varphi^{ik}X_{j1\bullet k}\right], \quad f^{(i)(j)}_{(a)(b)} = 0.$$

Remark 7.3. *If we have $h_{11} = 1$ and $\varphi_{ij} = \delta_{ij}$, we find that $F^{(i)}_{(1)j} = 0$, that is the momentum electromagnetic-like field in this case is trivial, i.e. $\mathbb{F} = 0$.*

7.2. Geometrization of an Economy dynamical system

We study now the dynamical of competition between two economical sectors governed by the first order differential system (see [15] and [9])

$$\begin{cases} \dfrac{dE_1}{dt} = g_1 E_1 \left(1 - \dfrac{E_1}{K_1} - \beta_1 \dfrac{E_2}{K_1}\right) \\ \dfrac{dE_2}{dt} = g_2 E_2 \left(1 - \dfrac{E_2}{K_2} - \beta_2 \dfrac{E_1}{K_2}\right), \end{cases} \tag{19}$$

where:

- E_1 and E_2 are two populations of new firms born in the above economical sectors;
- g_1 and g_2 are strictly positive constants representing the growth rates of the two economical sectors;
- K_1 and K_2 are strictly positive constants representing the investments of capitals;
- β_1 and β_2 are strictly positive constants representing the competitive interaction coefficients.

The differential system (19) can be reconsidered on the 1-jet space $J^1(\mathbb{R}, M)$, whose coordinates are $(t,\ x^1 = E_1,\ x^2 = E_2,\ y_1^1 = dE_1/dt,\ y_1^2 = dE_2/dt)$.

In this context, the solutions of class C^2 of the system (19) are the global minimum points of the least square Lagrangian (see [9])

$$\begin{aligned} L &= \left(y_1^1 - X_{(1)}^{(1)}(t, E_1, E_2)\right)^2 + \left(y_1^2 - X_{(1)}^{(2)}(t, E_1, E_2)\right)^2 = \\ &= \sum_{i,j=1}^{2} 1 \cdot \delta_{ij} \left(y_1^i - X_{(1)}^{(i)}(t, E_1, E_2)\right) \left(y_1^j - X_{(1)}^{(j)}(t, E_1, E_2)\right). \end{aligned}$$

where

$$\begin{cases} X_{(1)}^{(1)}(t, E_1, E_2) = g_1 E_1 \left(1 - \dfrac{E_1}{K_1} - \beta_1 \dfrac{E_2}{K_1}\right) \\ X_{(1)}^{(2)}(t, E_1, E_2) = g_2 E_2 \left(1 - \dfrac{E_2}{K_2} - \beta_2 \dfrac{E_1}{K_2}\right), \end{cases}$$

whose corresponding least squares Hamiltonian is given by

$$H = \frac{\delta^{ij}}{4} p_i^1 p_j^1 + X_{(1)}^{(k)} p_k^1 = \frac{1}{4}\left[\left(p_1^1\right)^2 + \left(p_2^1\right)^2\right] + X_{(1)}^{(1)} p_1^1 + X_{(1)}^{(2)} p_2^1.$$

By applying the preceding geometrical theory, it follows that, using the Jacobian notation $J(X) = \left(\dfrac{\partial X_{(1)}^{(i)}}{\partial E_j}\right)_{i,j=\overline{1,2}} =$

$$= \begin{pmatrix} g_1 - 2g_1 \dfrac{E_1}{K_1} - g_1\beta_1 \dfrac{E_2}{K_1} & -g_1\beta_1 \dfrac{E_1}{K_1} \\ -g_2\beta_2 \dfrac{E_2}{K_2} & g_2 - 2g_2 \dfrac{E_2}{K_2} - g_2\beta_2 \dfrac{E_1}{K_2} \end{pmatrix},$$

we find the following geometrical objects associated with the dynamical system (19) (here we have $i, j \in \{1, 2\}$):

(1) The coefficients of the *canonical nonlinear connection* produced by the dynamical system (19) are given by the temporal components $\underset{1}{\mathbf{N}}{}_{(i)1}^{(1)} = 0$, and the spatial components are the entries of the symmetric matrix

$$\begin{aligned} \underset{2}{\mathbf{N}} &= \left(\underset{2}{\mathbf{N}}{}_{(i)j}^{(1)}\right) = \left(\frac{\partial X_{(1)}^{(i)}}{\partial E_j} + \frac{\partial X_{(1)}^{(j)}}{\partial E_i}\right) = J(X) + J(X)^T = \\ &= \begin{pmatrix} 2\left(g_1 - 2g_1 \dfrac{E_1}{K_1} - g_1\beta_1 \dfrac{E_2}{K_1}\right) & -g_1\beta_1 \dfrac{E_1}{K_1} - g_2\beta_2 \dfrac{E_2}{K_2} \\ -g_1\beta_1 \dfrac{E_1}{K_1} - g_2\beta_2 \dfrac{E_2}{K_2} & 2\left(g_2 - 2g_2 \dfrac{E_2}{K_2} - g_2\beta_2 \dfrac{E_1}{K_2}\right) \end{pmatrix}. \end{aligned}$$

Moreover, all the coefficients of the *Cartan canonical connection* $C\Gamma(N)$ of the least squares Hamiltonian function (16) are zero.

(2) The nonzero *torsion components* produced by the dynamical system (19) are the entries of the matrices:

$$\mathbf{R}_{(1)} = \left(\mathbf{R}^{(1)}_{(1)ij}\right) = \left(\frac{\partial^2 X^{(i)}_{(1)}}{\partial E_1 \partial E_j} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial E_1 \partial E_i}\right) =$$

$$= \frac{d}{dE_1}\left[J(X) - J(X)^T\right] = \begin{pmatrix} 0 & -\dfrac{g_1\beta_1}{K_1} \\ \dfrac{g_1\beta_1}{K_1} & 0 \end{pmatrix};$$

$$\mathbf{R}_{(2)} = \left(\mathbf{R}^{(1)}_{(2)ij}\right) = \left(\frac{\partial^2 X^{(i)}_{(1)}}{\partial E_2 \partial E_j} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial E_2 \partial E_i}\right) =$$

$$= \frac{d}{dE_2}\left[J(X) - J(X)^T\right] = \begin{pmatrix} 0 & \dfrac{g_2\beta_2}{K_2} \\ -\dfrac{g_2\beta_2}{K_2} & 0 \end{pmatrix}.$$

Moreover, all the *curvature d−tensors* produced by the dynamical system (19) are zero.

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