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# POINTWISE $L_p^t$−FUNCTIONS ON LOCALLY COMPACT GROUPS

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*Let $G$ be a locally compact group and $1 \leq p < \infty$. This paper is mainly concerned with introducing the concept of pointwise $L_p^t$− functions and studying the structure of the space $L_p^{tp}(G)$, consisting of all these functions. Furthermore, some algebraic and topological properties of $L_p^{tp}(G)$ as a Banach algebra under pointwise multiplication are investigated.*

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## 1. Introduction and Preliminaries

Let $G$ be a locally compact group and $\lambda$ be a fixed left Haar measure on $G$. For $1 \leq p \leq \infty$, the Lebesgue space $L^p(G)$ with respect to $\lambda$ is as defined in [6]. The usual pointwise multiplication of functions will be denoted by "$\cdot$". We introduce the concept of *pointwise $L_p^t$−function* as the following;

**Definition 1.1.** A function $f \in L^p(G)$ is called a *pointwise $L_p^t$−function* if $g.f \in L^p(G)$, for each $g \in L^p(G)$ and

$$\|f\|_p^{tp} = \sup\{\|g.f\|_p,\ g \in L^p(G),\ \|g\|_p \leq 1\} < \infty. \tag{1.1}$$

The set of all pointwise $L_p^t$−functions will be denoted by $L_p^{tp}(G)$. Using some easy calculations, one can show that $L_p^{tp}(G)$ is a vector subspace of $L^p(G)$, and the function $\|.\|_p^{tp}$, defined in (1.1), is a norm on $L_p^{tp}(G)$ under which it is a normed algebra, i.e. for all $f, g \in L_p^{tp}(G)$,

$$\|g{\cdot}f\|_p^{tp} \leq \|g\|_p^{tp}\|f\|_p^{tp}.$$

It should be noted that these functions are known for the case where $p = 1$; see [7, Theorem 20.15]. Our main motivation for presenting Definition 1.1, stemmed from the general definition of $L_p^t$−functions, whenever $G$ is abelian, $1 \leq p \leq 2$ and $L^p(G)$ is considered under the convolution multiplication "$*$", defined in [6]. Indeed, a function $f \in L^p(G)$ is said to be a $L_p^t$−function if $g * f$ exists and belongs to $L^p(G)$, for all $g \in L^p(G)$ and

$$\|f\|_p^t = \sup\{\|g * f\|_p,\ g \in L^p(G),\ \|g\|_p \leq 1\} < \infty; \tag{1.2}$$

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see [4] and [5] for more details. Note that in [4], these functions have been named $p-$tempered functions. The set of all $L_p^t-$functions will be denoted by $L_p^t(G)$. The function $\|.\|_p^t$, defined in (1.2) is a norm on $L_p^t(G)$ and for all $f, g \in L_p^t(G)$,

$$\|g * f\|_p^t \leq \|g\|_p^t \|f\|_p^t.$$

The structure of $L_p^t(G)$ has been characterized in a partial case. Indeed, it has been proved that the set $L_p^t(G)^+$, consisting of all nonnegative-valued functions of $L_p^t(G)$, is just the set $L^1(G) \cap L^p(G)^+$, whenever $G$ is a unimodular and amenable locally compact group; see [4, Lemma 2.3]. We just cite that some of the main results related to this subject can be found in [4], [5] and [10].

In the present work, we first obtain an accurate structure for $L_p^{tp}(G)$, as a normed space. In fact similar to the result [7, Theorem 20.15], we show that $L_p^{tp}(G)$ is just the space $L^p(G) \cap L^\infty(G)$ and $\|.\|_p^{tp}$ is in fact the norm $\|.\|_\infty$ restricted to $L_p^{tp}(G)$, as a subspace of $L^\infty(G)$. Also we prove that unless $G$ is compact, $L_p^{tp}(G)$ under $\|.\|_p^{tp}$ is never complete. Then an attempt will be made to supply this deficiency by the introduction of the second norm on $L_p^{tp}(G)$, defined as the following

$$|||f||| = \|f\|_p + \|f\|_\infty.$$

We shall be principally concerned with comparing three norms $\|.\|_p$, $\|.\|_p^{tp}$ and $|||.|||$ on $L_p^{tp}(G)$, to show the different treatment of $L_p^{tp}(G)$, with respect to the alternative norm. The last section is devoted to some algebraic properties of $L_p^{tp}(G)$. For example, we show that the existence of an identity in $L_p^{tp}(G)$ is equivalent to the compactness of $G$. Moreover, we give some results about the amenability of $L_p^{tp}(G)$ under both norms $\|.\|_p^{tp}$ and $|||.|||$. We also study $L_p^{tp}(G)$ as an abstract Segal algebra with respect to $L^p(G)$ and also $L^\infty(G)$, whenever $L_p^{tp}(G)$ is considered under all introduced norms.

## 2. The structure of $L_p^{tp}(G)$

In this section we characterize $L_p^{tp}(G)$ for $1 \leq p \leq \infty$, and show that $L_p^{tp}(G)$ is exactly the set $L^p(G) \cap L^\infty(G)$. Our proof is nothing more than a modification of the result [7, Theorem 20.15]. Then we show that $\|.\|_p^{tp}$ is just norm $\|.\|_\infty$, restricted to $L_p^{tp}(G)$. For this purpose, we make some preparations.

For each $f \in L^\infty(G)$, the map

$$\phi_f : L^p(G) \to L^p(G),$$

defined by $g \mapsto f.g$ is well defined by [7, Theorem 20.13], and belongs to $B(L^p(G))$, consisting of all bonded linear operators from $L^p(G)$ into $L^p(G)$. Moreover $\|\phi_f\| \leq \|f\|_\infty$. It is obvious that $\|\phi_f\| = \|f\|_\infty$, whenever $p = \infty$. In the next lemma, we prove this truth for each $1 \leq p < \infty$. Indeed, we use similar arguments as in [9, Example 2.5.1], to show that

$$\Phi : L^\infty(G) \to B(L^p(G))$$

defined by $f \mapsto \phi_f$ is an isometric linear map from $L^\infty(G)$ onto the subspace

$$\Phi(L^\infty(G)) = \{\phi_f : f \in L^\infty(G)\}$$

of $B(L^p(G))$.

**Lemma 2.1.** *Let $G$ be a locally compact group and $1 \leq p < \infty$. Then the map $\Phi$ is an isometric linear map from $L^\infty(G)$ onto $\Phi(L^\infty(G))$.*

*Proof.* It is sufficient to show that $\|f\|_\infty \leq \|\phi_f\|$, for each $f \in L^\infty(G)$. If this is false, then there exists a positive number $\varepsilon$ such that

$$\|f\|_\infty > \|\phi_f\| + \varepsilon.$$

Thus there is a compact subset $K$ of $G$ with positive measure such that

$$|f(x)| \geq \|\phi_f\| + \varepsilon,$$

for each $x \in K$. However,

$$\begin{aligned}
\|\phi_f\|^p \lambda(K) &\geq \|\phi_f(\chi_K)\|_p^p \\
&= \int_G |f(x)\chi_K(x)|^p d\lambda(x) \\
&\geq \int_G (\|\phi_f\| + \varepsilon)^p \, \chi_K(x) d\lambda(x) \\
&= (\|\phi_f\| + \varepsilon)^p \, \lambda(K),
\end{aligned}$$

where $\chi_K$ is the characteristic function of the set $K$. It follows that

$$\|\phi_f\| \geq \|\phi_f\| + \varepsilon,$$

a contradiction. This together with the explanations preceding the lemma imply that

$$\|\phi_f\| = \|f\|_\infty,$$

as claimed. □

In fact we have a rather stronger result. If $f$ is a complex valued measurable function on $G$ such that $\phi_f \in B(L^p(G))$, then $f \in L^\infty(G)$. Suppose on the contrary that $f \notin L^\infty(G)$. Thus for each positive constant $N$, there is a compact subset $K_N$ of $G$ with positive measure such that

$$|f(x)| \geq N,$$

for each $x \in K_N$. Thus

$$\begin{aligned}
\|\phi_f\|^p \lambda(K_N) &\geq \|\phi_f(\chi_{K_N})\|_p^p \\
&= \int_G |f(x)\chi_{K_N}(x)|^p d\lambda(x) \\
&\geq N^p \int_G \chi_{K_N}(x) d\lambda(x) \\
&= N^p \lambda(K_N),
\end{aligned}$$

which is a contradiction. It follows that $f \in L^\infty(G)$.

**Theorem 2.2.** *Let $G$ be a locally compact group and $1 \leq p < \infty$. Then $L_p^{tp}(G) = L^p(G) \cap L^\infty(G)$, as two sets. Moreover, for each $f \in L_p^{tp}(G)$,*

$$\|f\|_p^{tp} = \|f\|_\infty \leq |||f|||.$$

*Proof.* If $f \in L^p(G) \cap L^\infty(G)$, then Lemma 2.1 implies that

$$\sup\{\|g.f\|_p : \|g\|_p \leq 1\} = \|\phi_f\| = \|f\|_\infty < \infty. \tag{2.1}$$

It follows that $f \in L_p^{tp}(G)$ and so $L^p(G) \cap L^\infty(G) \subseteq L_p^{tp}(G)$. The reverse of the inclusion and also the inequality

$$\|f\|_p^{tp} = \|f\|_\infty \leq |||f|||$$

can be easily implied by (2.1) and the explanations preceding the theorem. □

*Remark* 2.3. It should be noted that our results are based on the definitions of $L^p-$spaces, given in [6]. Now we show that $(L_p^{tp}(G), |||.|||)$ is a Banach algebra, which will be used several times in this paper. By Theorem 2.2,

$$L_p^{tp}(G) = L^p(G) \cap L^\infty(G).$$

It is not hard to see that $L_p^{tp}(G)$ is always an algebra under pointwise multiplication. Suppose that $(f_n)_n$ is a Cauchy sequence in $(L_p^{tp}(G), |||.|||)$. Thus $(f_n)_n$ is also a Cauchy sequence in $L^p(G)$ and $L^\infty(G)$, and so $(f_n)_n$ is convergent to $f$ and $g$ in $L^p(G)$ and $L^\infty(G)$, respectively. To that end, we show that $f = g$, almost everywhere on $G$. By the proof of [6, Theorem 12.8], there is a subsequence $(f_{n_k})_k$ of $(f_n)_n$ such that it is almost everywhere convergent to $f$. For each $j, l, k \in \mathbb{N}$, let

$$E_{l,j} = \{x \in G : |f_{n_l}(x) - f_{n_j}(x)| \geq \|f_{n_l} - f_{n_j}\|_\infty\},$$

and

$$E_k = \{x \in G : |f_{n_k}(x)| \geq \|f_{n_k}\|_\infty\}$$

and set

$$E = \bigcup_{l,j,k} (E_{l,j} \cup E_k).$$

By the definition of $L^\infty(G)$ in [6], all the sets $E_{l,j}$ and $E_k$ are locally null; i.e. $\lambda(E_{l,j} \cap F) = \lambda(E_k \cap F) = 0$, for every compact subset $F$ of $G$. It follows that $E$ is also a locally null set. Since for each $n \in \mathbb{N}$, $f_n \in L^p(G)$, it follows that for all $l, j, k \in \mathbb{N}$,

$$\lambda(E_{l,j}) < \infty$$

and

$$\lambda(E_k) < \infty,$$

which implies that $E$ is a $\sigma-$finite set. Consequently $\lambda(E) = 0$, by [6, Theorem 11.32] and [6, Note 11.33]. It follows that $f_{n_k}$ converges to $g$, almost everywhere on $G$. Consequently $f = g$, almost everywhere on $G$. It follows that $(L_p^{tp}(G), |||.|||)$ is a Banach space. Moreover inequality

$$|||f.g||| \leq |||f|||\ |||g||| \qquad (f, g \in L_p^{tp}(G))$$

is immediately obtained. These observations show that $L_p^{tp}(G)$ is a Banach algebra under $|||.|||$.

It is known that $L_p^t(G) = L^p(G)$ if and only if $G$ is compact. This is completely related to an ancient conjecture, called '$L^p$−conjecture' which was proved finally by Saeki [11], in the general case. This conjecture asserts that $f * g$ exists and belongs to $L^p(G)$ for all $f, g \in L^p(G)$ if and only if $G$ is compact. In the next theorem some equivalent conditions to the equality $L^p(G) = L_p^{tp}(G)$ are given.

**Theorem 2.4.** *Let $G$ be a locally compact group and $1 \le p < \infty$. Then the following assertions are equivalent.*

(i) *$L^p(G)$ is a Banach algebra under pointwise multiplication.*
(ii) *$L_p^{tp}(G) = L^p(G)$, as two sets.*
(iii) *The norms $\|.\|_p$ and $|||.|||$ are equivalent on $L_p^{tp}(G)$.*
(iv) *There exists a positive constant $K$ such that $\|f\|_p^{tp} \le K\|f\|_p$, for each $f \in L_p^{tp}(G)$.*
(v) *There exists a positive constant $K$ such that $|||f||| \le K\|f\|_p$, for each $f \in L_p^{tp}(G)$.*
(vi) *$G$ is discrete.*

*Proof.* $(i) \Rightarrow (vi)$. Let $L^p(G)$ be a Banach algebra under pointwise multiplication. Toward a contradiction, suppose that $G$ is not discrete. Thus there exists a sequence $(O_n)$ of disjoint relatively compact open subsets of $G$ such that $\lambda(\overline{O_n}) < n^{-2p}$, for each $n \in \mathbb{N}$. Indeed, we first choose a relatively compact open set $O_1 \subseteq G$ with $\lambda(\overline{O_1}) < 1$. Then there is a relatively compact open set $O_2 \subseteq G \setminus \overline{O_1}$ with $\lambda(\overline{O_2}) < 2^{-2p}$. Inductively, we may find a sequence $(O_n)$ of relatively compact open subsets in $G$ such that

$$O_n \subseteq G \setminus (\overline{O_1} \cup \cdots \cup \overline{O_{n-1}})$$

and

$$\lambda(\overline{O_n}) < n^{-2p}.$$

Let $\alpha$ be a positive constant with $\alpha \le (2p)^{-1}$ and define the function $g$ on $G$ by

$$g(x) = \sum_{n=1}^{\infty} n^{-1-\alpha} \lambda(O_n)^{-1/p} \chi_{O_n}(x) \qquad (x \in G).$$

It is clear that $g \in L^p(G)$. But

$$\begin{aligned}\int_G |g(x)g(x)|^p d\lambda(x) &= \sum_{n=1}^{\infty} \int_{O_n} \frac{d\lambda(x)}{\lambda(O_n)^2 n^{2p+2p\alpha}} \\ &\ge \sum_{n=1}^{\infty} n^{-2p\alpha} \\ &= \infty.\end{aligned}$$

It follows that $g^2 = g.g \notin L^p(G)$, which contradicts the hypothesis.

$(vi) \Rightarrow (i)$. If $G$ is discrete then for all $f, g \in L^p(G)$ we clearly have

$$\begin{aligned} \|f.g\|_p &= \left(\sum_{x\in G} |f(x)|^p |g(x)|^p\right)^{1/p} \\ &\leq \|f\|_p \|g\|_p \\ &< \infty, \end{aligned}$$

and so $(i)$ is obtained. Therefore $(i)$ and $(vi)$ are equivalent. The implications $(vi) \Rightarrow (v)$, $(v) \Rightarrow (iv)$ and also $(iv) \Rightarrow (iii)$ are clear.

$(iii) \Rightarrow (ii)$. Let the norms $\|.\|_p$ and $|||.|||$ be equivalent on $L_p^{tp}(G)$. By Remark 2.3, $L_p^{tp}(G)$ is always a Banach algebra under $|||.|||$. It follows that $L_p^{tp}(G)$ is also complete under $\|.\|_p$. Now the result follows by the density of $L_p^{tp}(G)$ in $L_p(G)$.

$(ii) \Rightarrow (vi)$. Since $L_p^{tp}(G)$ is an algebra under pointwise multiplication, then $L^p(G)$ is also an algebra under pointwise multiplication. Now the proof of $(i) \Rightarrow (vi)$ implies the discreteness of $G$. □

In the following theorem, some equivalent conditions to the completeness of $L_p^{tp}(G)$ under $\|.\|_p^{tp}$ are provided.

**Theorem 2.5.** *Let $G$ be a locally compact group and $1 \leq p < \infty$. Then the following assertions are equivalent.*

(i) *$L_p^{tp}(G)$ is a Banach algebra under $\|.\|_p^{tp}$.*
(ii) *$L_p^{tp}(G) = L^\infty(G)$, as two sets.*
(iii) *The norms $\|.\|_p^{tp}$ and $|||.|||$ are equivalent on $L_p^{tp}(G)$.*
(iv) *There exists a positive constant $M$ such that $\|f\|_p \leq M\|f\|_p^{tp}$, for each $f \in L_p^{tp}(G)$.*
(v) *There exists a positive constant $K$ such that $|||f||| \leq K\|f\|_p^{tp}$, for each $f \in L_p^{tp}(G)$.*
(vi) *$G$ is compact.*

*Proof.* $(i) \Rightarrow (vi)$. Let $L_p^{tp}(G)$ be a Banach algebra under $\|.\|_p^{tp}$ and consider the identity map

$$\iota : (L_p^{tp}(G), |||.|||) \to (L_p^{tp}(G), \|.\|_p^{tp}).$$

By Remark 2.3, $L_p^{tp}(G)$ is a Banach algebra under $|||.|||$. Since $\iota$ is continuous, then open mapping theorem implies that $\|.\|_p^{tp}$ and $|||.|||$ are equivalent and hence there is a constant $K > 1$ such that

$$|||f||| \leq K\|f\|_p^{tp},$$

for each $f \in L_p^{tp}(G)$. By Theorem 2.2 we have

$$\|f\|_p \leq (K-1)\|f\|_p^{tp}. \tag{2.2}$$

Now we show that $C_0(G) \subseteq L^p(G)$. Let $g \in C_0(G)$. Thus there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $C_{00}(G)$ such that

$$\|g_n - g\|_\infty \to 0.$$

Since $g_n - g \in C_0(G)$, it follows that for each $n \in \mathbb{N}$, the set

$$A_n = \{x \in G : |g_n(x) - g(x)| > \|g_n - g\|_\infty\}$$

is open and thus it is a null set; i.e. $\lambda(A_n) = 0$. So $A = \cup_{n\in\mathbb{N}} A_n$ is a null set, as well. Consequently $(g_n)$ is almost every where convergent to $g$. Moreover inequality (2.2) together with Theorem 2.2 imply that $(g_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $L^p(G)$. Thus there is a function $h \in L^p(G)$ such that

$$\|g_n - h\|_p \to 0.$$

It follows by the proof of [6, Theorem 12.8] that there is a subsequence of $h$ which almost every where convergent to $h$. Consequently $g = h$ almost every where on $G$, which implies $g \in L^p(G)$. Moreover

$$\|g\|_p \le (K-1)\|g\|_p^{tp} = (K-1)\|g\|_\infty. \tag{2.3}$$

In the sequel, we show that the inclusion $C_0(G) \subseteq L^p(G)$ implies the compactness of $G$. For this purpose, consider the inclusion map

$$\iota : C_0(G) \to L^p(G).$$

By Theorem 2.2 and also (2.3), $\iota$ is continuous. If $G$ were non-compact then there is a sequence $(F_n)_{n\in\mathbb{N}}$ of the compact subsets of $G$ with positive measure such that $\lambda(F_n) \to \infty$. Also for each $n \in \mathbb{N}$, there is a relatively compact open subset $U_n$ of $G$ such that $F_n \subseteq U_n$. For each $n \in \mathbb{N}$, let $f_n$ be a continuous function on $G$ such that vanishes outside $U_n$ and

$$0 \le f_n(x) \le 1,$$

for each $x \in G$, and equals to the constant 1 on $F_n$. Set

$$g_n = \frac{f_n}{\lambda(F_n)^{1/p}}.$$

Thus $g_n \in C_0(G)$ and $\|g_n\|_\infty \to 0$, whereas

$$\lim_{n\to\infty} \|g_n\|_p \ge 1,$$

which is in contradiction with the inequality (2.3). Therefore $G$ should be compact.

$(vi) \Rightarrow (v)$, $(v) \Rightarrow (iv)$ and $(iv) \Rightarrow (iii)$ are clearly obtained from Theorem 2.2.

$(iii) \Rightarrow (ii)$. Let $\|.\|_p^{tp}$ and $|||.|||$ be equivalent on $L_p^{tp}(G)$. Since $L_p^{tp}(G)$ is a Banach algebra under the norm $|||.|||$, it follows that $L_p^{tp}(G)$ is complete under $\|.\|_p^{tp}$, as well. By Theorem 2.2, $C_{00}(G) \subseteq L_p^{tp}(G)$, which implies that

$$C_0(G) \subseteq L_p^{tp}(G) \subseteq L^p(G).$$

Now the proof of implication $(i) \Rightarrow (vi)$ provides that $G$ is compact. It follows that $L^\infty(G) \subseteq L^p(G)$ and again by Theorem 2.2, $L_p^{tp}(G) = L^\infty(G)$.

$(ii) \Rightarrow (i)$. It is immediately obtained from Theorem 2.2 and the fact that $L^\infty(G)$ is a Banach algebra under pointwise multiplication. $\square$

The following result is automatically fulfilled from Theorems 2.2 and 2.4.

**Corollary 2.6.** *Let $G$ be a locally compact group and $1 \leq p < \infty$. Then $\|.\|_p$ and $\|.\|_p^{tp}$ are equivalent on $L_p^{tp}(G)$ if and only if $G$ is finite.*

As an application of these results, we end this section with the examples, on some known locally compact groups.

*Examples.* Let $1 \leq p < \infty$.

(1) Let $\mathbb{Z}$ be the additive group of integers and $\mathbb{T}$ be the multiplicative circle group. Theorems 2.4 and 2.5 provide that $L^p(\mathbb{Z}) = L_p^{tp}(\mathbb{Z})$ and $L_p^{tp}(\mathbb{T}) = L^\infty(\mathbb{T})$, as two sets.
(2) Let $\mathbb{R}$ be the additive group of real numbers and take $G$ to be one of the locally compact groups $\mathbb{R}$, $\mathbb{R} \times \mathbb{T}$, $\mathbb{R} \times \mathbb{Z}$ or $\mathbb{T} \times \mathbb{Z}$. Since these groups are non discrete and noncompact, then $L_p^{tp}(G)$ is a proper subset of $L^p(G)$ and also $L^\infty(G)$.

## 3. Some aspects of $L_p^{tp}(G)$ as a Banach algebra

Let $G$ be a locally compact group and $1 \leq p < \infty$. As some applications of the previous results, we study amenability of $L_p^{tp}(G)$ under both introduced norms $\|.\|_p^{tp}$ and $|||.|||$. Furthermore, we investigate when $L_p^{tp}(G)$ is an abstract Segal algebra with respect to $L^p(G)$ or $L^\infty(G)$. We commence our discussion with a verification on the existence of an identity in $L_p^{tp}(G)$. Since $L^p(G)$ is not generally an algebra under pointwise multiplication, we define the concept of quasi identity for $L^p(G)$, which is directly related to the concept of identity element. An element $e \in L^p(G)$ is called a quasi identity for $L^p(G)$ if for each $f \in L^p(G)$, $e.f \in L^p(G)$ and $e.f = f$.

**Proposition 3.1.** *Let $G$ be a locally compact group and $1 \leq p < \infty$. Then $L^p(G)$ has a quasi identity if and only if $G$ is compact.*

*Proof.* Let $e \in L^p(G)$ be a quasi identity element for $L^p(G)$. Thus for each subset $F$ of $G$ with $\lambda(F) < \infty$, we have $e.\chi_F = \chi_F$. Hence for each compact subset $F$ of $G$,

$$\lambda(F) \leq \|e\|_p^p,$$

which implies the compactness of $G$, by the regularity of Haar measure $\lambda$ and also [6, Theorem 15.9]. The converse is trivial, because the constant function 1 is a quasi identity for $L^p(G)$. □

One can follow an argument similar to Proposition 3.1 to get the following result.

**Corollary 3.2.** *Let $G$ be a locally compact group and $1 \leq p < \infty$. Then $L_p^{tp}(G)$ has an identity if and only if $G$ is compact.*

Some more clarified results of our verification are simplified in the next remarks.

*Remark* 3.3. Let $G$ be a locally compact group and $1 \leq p \leq \infty$.

(i) If $p=\infty$, then $L^\infty(G)=L^{tp}_\infty(G)$; indeed, for each $f\in L^\infty(G)$, we have

$$\begin{aligned}\|f\|_\infty &= \|1.f\|_\infty \le \|f\|^{tp}_\infty\\ &= \sup_{\|g\|_\infty\le 1}\|g.f\|_\infty\\ &\le \|f\|_\infty.\end{aligned}$$

Thus $\|f\|_\infty=\|f\|^{tp}_\infty$ and consequently $L^\infty(G)$ and $L^{tp}_\infty(G)$ are isometrically isomorphic.

(ii) If $1\le p<\infty$, Corollary 3.2 implies that the existence of an identity for $L^{tp}_p(G)$ is equivalent to the compactness of $G$. It is also equivalent to being $L^{tp}_p(G)$ a Banach algebra under $\|.\|^{tp}_p$, by Theorem 2.5.

(iii) If $G$ is discrete and $1\le p<\infty$, Theorem 2.4 implies that $L^{tp}_p(G)=L^p(G)$. Consider the net $(\chi_F)_{F\in\mathcal{F}}$ of the characteristic functions on the finite subsets $F$ of $G$, directed by upward inclusion. We claim that $(\chi_F)_{F\in\mathcal{F}}$ is an approximate identity for $L^p(G)$. For the proof, let $f\in L^p(G)$. Thus $f$ equals zero, outside of a countable subset $S_f=\{x_n : n\in\mathbb{N}\}$ of $G$. For each $\varepsilon>0$, there is a positive integer $N$ such that

$$\sum_{k=N+1}^{\infty}|f(x_k)|^p<\varepsilon^p.$$

Now for each finite subset $F$ of $G$ containing the set $F_0=\{x_1,\cdots,x_N\}$, we have

$$\begin{aligned}\|f-\chi_F.f\|_p^p &= \sum_{n=1}^{\infty}|f(x_n)-\chi_F(x_n)f(x_n)|^p\\ &= \sum_{n=1}^{N}|f(x_n)-\chi_F(x_n)f(x_n)|^p\\ &\quad+\sum_{n=N+1}^{\infty}|f(x_n)-\chi_F(x_n)f(x_n)|^p\\ &\le \sum_{k=N+1}^{\infty}|f(x_k)|^p\\ &< \varepsilon^p.\end{aligned}$$

Thus the claim is proved. Theorem 2.4 implies that for each $f\in L^{tp}_p(G)$ the inequality

$$\|f-\chi_F.f\|^{tp}_p<K\|f-\chi_F.f\|_p<K\varepsilon$$

is also valid for some positive constant $K$. It follows that although in this case $L^{tp}_p(G)$ is not necessarily a Banach algebra under $\|.\|^{tp}_p$, but the net $(\chi_F)_{F\in\mathcal{F}}$ is an approximate identity for $(L^{tp}_p(G),\|.\|^{tp}_p)$.

### 3.1. Amenability of $L_p^{tp}(G)$

Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}-$bimodule: A derivation $D$ from $\mathcal{A}$ into $X$ is inner if there is $\xi \in X$ such that

$$D(a) = a\xi - \xi,$$

for each $a \in \mathcal{A}$. The Banach algebra $\mathcal{A}$ is amenable if every continuous derivation $D : A \to X^*$ is inner for all Banach $\mathcal{A}-$bimodules $X$.

Since $L^\infty(G)$ is a commutative unital $C^*-$algebra, thus it is isometrical isomorphic with the space $C(K)$, for some compact Hausdorff space $K$. It follows that $L^\infty(G)$ is an amenable Banach algebra [2, Theorem 5.6.2(i)]; see also [3] for full information about the structure of $C^*-$algebras. Now Theorem 2.5 implies that if $G$ is compact, then $L_p^{tp}(G)$ is always amenable under $\|.\|_{tp}$ and also $|||.|||$.

**Proposition 3.4.** *Let $G$ be a locally compact group and $1 \leq p < \infty$. Then $L_p^{tp}(G)$ is amenable under norm $|||.|||$ if and only if $G$ is compact.*

*Proof.* By the explanations given before the proposition, we proceed to prove the "only if" part. Let $L_p^{tp}(G)$ is amenable under norm $|||.|||$. Thus it admits an approximate identity such as $(e_\alpha)_{\alpha\in\Lambda}$ bounded by the positive constant $M$ [8, Proposition 1.6]. Hence for each compact subset $F$ of $G$ with positive measure we have

$$|||\chi_F.e_\alpha - \chi_F||| \to 0.$$

and so

$$\|\chi_F.e_\alpha - \chi_F\|_p \to 0.$$

It follows that there is $\alpha_0 \in \Lambda$ such that

$$\|\chi_F.e_{\alpha_0} - \chi_F\|_p \leq 1.$$

Consequently

$$\begin{aligned} \lambda(F)^{1/p} &= \|\chi_F\|_p \\ &\leq 1 + \|e_{\alpha_0}\|_p \\ &\leq 1 + |||e_{\alpha_0}||| \\ &\leq 1 + M. \end{aligned}$$

Now the result is obtained by the regularity of Haar measure $\lambda$ and also [6, Theorem 15.9]. □

### 3.2. $L_p^{tp}(G)$ as an abstract Segal algebra

For the sake of completeness, we first repeat and review the basic definitions of abstract Segal algebras; see [1] for more details.

Let $(\mathcal{A}, \|.\|_\mathcal{A})$ be a Banach algebra. Then $(\mathcal{B}, \|.\|_\mathcal{B})$ is an abstract Segal algebra with respect to $(\mathcal{A}, \|.\|_\mathcal{A})$ if:

(1) $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$ and $\mathcal{B}$ is a Banach algebra with respect to $\|.\|_\mathcal{B}$.
(2) There exists $M > 0$ such that $\|f\|_\mathcal{A} \leq M\|f\|_\mathcal{B}$, for each $f \in \mathcal{B}$.

(3) There exists $C > 0$ such that $\|fg\|_{\mathcal{B}} \le C\|f\|_{\mathcal{A}}\|g\|_{\mathcal{B}}$, for each $f, g \in \mathcal{B}$.

The following propositions can be readily verified by Theorems 2.4 and 2.5.

**Proposition 3.5.** *Let $G$ be a discrete group and $1 \le p < \infty$. Then $L_p^{tp}(G)$ endowed with $\|.\|_p^{tp}$ is an abstract Segal algebra with respect to $L^p(G)$ if and only if $G$ is finite.*

**Proposition 3.6.** *Let $G$ be a discrete group and $1 \le p < \infty$. Then $L_p^{tp}(G)$ endowed with $|||.|||$ is always an abstract Segal algebra with respect to $L^p(G)$.*

**Proposition 3.7.** *Let $G$ be a locally compact and $\sigma-$compact group and $1 \le p < \infty$. Then the following assertions are equivalent.*

(i) *$L_p^{tp}(G)$ endowed with $\|.\|_p^{tp}$ is as an abstract Segal algebra with respect to $L^\infty(G)$.*

(ii) *$L_p^{tp}(G)$ endowed with $|||.|||$ is as an abstract Segal algebra with respect to $L^\infty(G)$.*

(iii) *$G$ is compact.*

*Proof.* It is clear from Theorem 2.5 that $(i)$ is equivalent to $(iii)$ and $(iii)$ implies $(ii)$. It suffices to show that $(ii) \Rightarrow (iii)$. Let $L_p^{tp}(G)$ endowed with $|||.|||$ be an abstract Segal algebra with respect to $L^\infty(G)$. It follows that $L_p^{tp}(G)$ is dense in $L^\infty(G)$ and thus for the constant function 1 in $L^\infty(G)$, there is $f \in L_p^{tp}(G)$ such that

$$\|1 - f\|_\infty < 1/2.$$

Consequently there is a subset $A$ of $G$ such that

$$|f(x)| > 1/2,$$

for each $x \in A$ and also $G \setminus A$ is locally null. Since $f \in L^p(G)$ it follows that $\lambda(A) < \infty$. Moreover by [7, Theorem 20.12] we have

$$\lambda(G \setminus A) = 0.$$

Consequently $\lambda(G) < \infty$, which implies the compactness of $G$ [6, Theorem 15.9]. □

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