NUMERICAL SOLUTION OF STOCHASTIC VOLTERRA-FREDHOLM INTEGRAL EQUATIONS USING HAAR WAVELETS

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In this paper, we present a computational method for solving stochastic Voltera- Fredholm integral equations which is based on the Haar wavelets and their stochastic operational matrix. Convergence and error analysis of the proposed method are worked out. Numerical results are compared with the block pulse functions method for some non-trivial examples. The obtained results reveal efficiency and reliability of the proposed method.

Keywords: Itô integral, Stochastic Volterra-Fredholm integral equations, Haar wavelets, Stochastic operational matrix, Error analysis.

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1. **Introduction**

Random or stochastic integral equations are very important in the study of many phenomena in physics, mechanics, medical, finance, sociology, biology, etc. The study of problems in such fields are often dependent on a noise source, on a Gaussian white noise, governed by certain probability laws. So, modeling such phenomena naturally requires the use of various stochastic differential equations, stochastic integral equations or stochastic integro-differential equations. In many cases it is difficult to derive an explicit form of the solution for stochastic differential and integral equations. So, numerical approximation becomes a practical way to face this difficulty. Many papers have been appeared on the problem of approximate the solution of stochastic integral and differential equations [1–12].

Recently, different orthogonal basis functions, such as block pulse functions, Walsh functions, Fourier series, orthogonal polynomials and wavelets, were used to estimate solutions of functional equations. As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis, and many other areas. Wavelets permit the accurate representation of a variety of functions and operators [13, 14]. Haar wavelets have been widely applied in system analysis, system identification, optimal control and numerical solution of integral and differential equations [15–17].

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In this paper we consider the following stochastic Voltera-Fredholm integral equation

$$X(t) = f(t) + \int_{\alpha}^{\beta} X(s)k_1(s,t)ds + \int_{0}^{t} X(s)k_2(s,t)ds$$
$$+ \int_{0}^{t} X(s)k_3(s,t)dB(s), \ \ s,t \in [0,T), \tag{1}$$

where $X(t)$, $f(t)$ and $k_i(s,t), i = 1,2,3$ are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_3(s,t)X(s)dB(s)$ is the Itô integral [2,18]. We first describe Haar wavelets and their properties. Then a new stochastic operational matrix for Haar is introduced. After that a computational method is proposed for approximate solution of this stochastic Voltera-Fredholm integral equation.

This paper is organized as follows: In section 2 some basic properties of the Haar wavelets are described. In section 3 stochastic operational matrix for Haar wavelets and a general procedure for deriving this matrix are introduced. In section 4 a new computational method based on stochastic operational matrix for Haar wavelets are proposed for solving Voltera-Fredholm integral equations. Section 5 presents the convergence and error analysis of the proposed method. Numerical examples are presented in section 6. Finally, a conclusion is given in section 7.

2. Haar wavelets and Block pulse functions

In this section we describe some basic properties of the Haar wavelets. For this purpose we first introduce the block pulse functions (BPFs), function approximation by BPFs and their operational matrices. Then the relations between Haar wavelets and BPFs are investigated. Finally, we derive some important formulas for Haar wavelets that are useful for the next sections.

2.1. Block pulse functions

BPFs have been studied by many authors and applied for solving different problems. In this section we recall definition and some properties of the block pulse functions [3,8,19].

The m-set of BPFs are defined as

$$b_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih \\ 0 & otherwise \end{cases} \tag{2}$$

in which $t \in [0,T)$, $i = 1,2,...,m$ and $h = \frac{T}{m}$. The set of BPFs are disjointed with each other in the interval $[0,T)$ and

$$b_i(t)b_j(t) = \delta_{ij}b_i(t), i,j = 1,2,...,m, \tag{3}$$

where δ_{ij} is the Kronecker delta. The set of BPFs defined in the interval $[0, T)$ are orthogonal with each other, that is

$$\int_0^T b_i(t) b_j(t) = h\delta_{ij}, \ i, j = 1, 2, ..., m. \tag{4}$$

If $m \to \infty$ the set of BPFs is a complete basis for $L^2[0, T)$, so an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0, T)$, can be expanded into a block pulse series as

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t), \tag{5}$$

where

$$f_i = \frac{1}{h} \int_0^T b_i(t) f(t), \ i = 1, 2, ..., m. \tag{6}$$

Rewritting Eq. (37) in the vector form we have

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t) = F^T \Phi(t) = \Phi^T(t) F, \tag{7}$$

in which

$$\Phi(t) = [b_1(t), b_2(t), ..., b_m(t)]^T,$$
$$F = [f_1, f_2, ..., f_m]^T. \tag{8}$$

Morever, any two dimensional function $k(s, t) \in L^2([0, T_1] \times [0, T_2])$ can be expanded with respect to BPFs such as

$$k(s, t) = \Phi^T(t) K \Phi(s), \tag{9}$$

where $\Phi(t)$ is the m-dimensional BPFs vectors and K is the $m \times m$ BPFs coefficient matrix with (i, j)-th element

$$k_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} k(s, t) b_i(t) b_j(s) dt ds, \ i, j = 1, 2, ..., m, \tag{10}$$

and $h_1 = \frac{T_1}{m}$ and $h_2 = \frac{T_2}{m}$. Let $\Phi(t)$ be the BPFs vector, then we have

$$\Phi^T(t)\Phi(t) = 1, \tag{11}$$

and

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} b_1(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_m(t) \end{pmatrix}_{m \times m}. \tag{12}$$

For an m-vector F we have

$$\Phi(t)\Phi^T(t) F = \tilde{F} \Phi(t), \tag{13}$$

where $\tilde{F}$ is an $m \times m$ diagonal matrix with the elements of the vector F on the main diagonal. Also, it is easy to show that for an $m \times m$ matrix A

$$\Phi^T(t)A\Phi(t) = \hat{A}^T\Phi(t), \tag{14}$$

where $\hat{A} = diag(A)$ is an m-vector.

2.2. Haar wavelets

The orthogonal set of Haar wavelets $h_n(t)$ constitute a set of square waves defined as follows [13, 15, 16]

$$h_n(t) = 2^{\frac{j}{2}}\psi\left(2^j t - k\right),\ j \geq 0,\ 0 \leq k < 2^j,\ n = 2^j + k,\ n, j, k \in \mathbb{N}, \tag{15}$$

where

$$h_0(t) = 1,\ 0 \leq t < 1,\ \psi(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t < 1 \end{cases} \tag{16}$$

Each Haar wavelet $h_n(t)$ has the support $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$, so that it is zero elsewhere in the interval $[0,1)$. The Haar wavelets $h_n(t)$ are pairwise orthonormal in the interval $[0,1)$ and

$$\int_0^1 h_i(t)h_j(t)dt = \delta_{ij}, \tag{17}$$

where δ_{ij} is the Kronecker delta. Any square integrable function $f(t)$ in the interval $[0,1)$ can be expanded in terms of Haar wavelets as

$$f(t) = c_0h_0(t) + \sum_{i=1}^{\infty} c_ih_i(t), \tag{18}$$

where c_i is given by

$$c_i = \int_0^1 f(t)h_i(t)dt, \tag{19}$$

The infinite series in Eq. (18) can be truncated after $m = 2^J$ terms (J is level of wavelet resolution), that is

$$f(t) \simeq c_0h_0(t) + \sum_{i=1}^{m-1} c_ih_i(t),\ i = 2^j + k,\ j = 0, 1, ..., J-1,\ 0 \leq k < 2^j, \tag{20}$$

rewritting this equation in the vector form we have,

$$f(t) \simeq C^TH(t) = H(t)^TC, \tag{21}$$

in which C and $H(t)$ are Haar coefficients and wavelets vectors as

$$C = [c_0, c_1, ..., c_{m-1}]^T, \tag{22}$$

$$H(t) = [h_0(t), h_1(t), ..., h_{m-1}(t)]^T. \quad (23)$$

Any two dimensional function $k(s,t) \in L^2[0,1) \times L^2[0,1)$ can be expanded with respect to Haar wavelets as

$$k(s,t) = H^T(t)KH(t), \quad (24)$$

where $H(t)$ is the Haar wavelets vector and K is the $m \times m$ Haar wavelets coefficients matrix with (i,l)-th element can be obtained as

$$k_{il} = \int_0^1 \int_0^1 k(s,t)H_i(t)H_l(s)dtds, \; i,l = 1,2,...,m. \quad (25)$$

2.3. Relation between the BPFs and Haar wavelets

In this section we will derive the relation between the BPFs and Haar wavelets. It is worth mention that in this section we set $T = 1$ in definition of BPFs.

Theorem 2.1. *Let $H(x)$ and $\Phi(x)$ be the m-dimensional Haar wavelets and BPFs vector respectively, the vector $H(x)$ can be expanded by BPFs vector $\Phi(x)$ as*

$$H(t) = Q\Phi(t), \; m = 2^J, \quad (26)$$

where Q is an $m \times m$ matrix and

$$Q_{il} = h_{i-1}\left(\frac{2l-1}{2m}\right), \; i,l = 1,2,...m, \; i-1 = 2^j + k, \; 0 \le k < 2^j, \quad (27)$$

Proof. Let $H_i(t), i = 1,2,...,m$ be the i-th element of Haar wavelets vector. Expanding $H_i(t)$ into an m-term vector of BPFs, we have

$$H_i(t) = \sum_{l=1}^{m} Q_{il}b_l(t) = Q_i^T B(t), \; i = 1,2,...,m, \quad (28)$$

where Q_i is the i-th column and Q_{il} is the (i,l)-th element of matrix Q. By using the orthogonality of BPFs we have

$$Q_{il} = \frac{1}{h}\int_0^1 H_i(t)b_l(t)dt = \frac{1}{h}\int_{\frac{l-1}{m}}^{\frac{l}{m}} H_i(t)dt = m\int_{\frac{l-1}{m}}^{\frac{l}{m}} h_{i-1}(t)dt, \quad (29)$$

by using mean value theorem for integrals in the last equation we can write

$$Q_{il} = m\left(\frac{l}{m} - \frac{l-1}{m}\right)h_{i-1}(\eta_l) = h_{i-1}(\eta_l), \; \eta_l \in \left(\frac{l-1}{m}, \frac{l}{m}\right), \quad (30)$$

As $h_{i-1}(t)$ is constant on the interval $\left(\frac{l-1}{m}, \frac{l}{m}\right)$ we can choose $\eta_l = \frac{2l-1}{2m}$ so we have

$$Q_{il} = h_{i-1}\left(\frac{2l-1}{2m}\right), \; i,l = 1,2,...m. \quad (31)$$

and this proves the desired result. □

For an example the matrix $Q_{8\times 8}$ has the following form

$$Q_{8\times 8}=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2}\\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}. \tag{32}$$

Remark 2.1. *According to the definition of matrix Q in (26) it is easy to see that*

$$Q^{-1}=\frac{1}{m}Q^{T}. \tag{33}$$

The following Remark is the consequence of relations (13), (14) and Theorem 2.1.

Remark 2.2. *For an m-vector F we have*

$$H(t)H^{T}(t)F=\tilde{F}H(t), \tag{34}$$

in which $\tilde{F}$ is an $m\times m$ matrix as

$$\tilde{F}=Q\bar{F}Q^{-1}, \tag{35}$$

where $\bar{F}=diag\,(Q^{-1}F)$. Moreover, it can be easy to show that for an $m\times m$ matrix A

$$H^{T}(t)AH(t)=\hat{A}^{T}H(t), \tag{36}$$

where $\hat{A}^{T}=UQ^{-1}$ and $U=diag(Q^{T}AQ)$ is a m-vector.

3. Stochastic integration operational matrix of Haar wavelets

In this section we obtain the stochastic integration operational matrix for Haar wavelets. For this purpose we remind some useful results for BPFs [3,8].

Lemma 3.1. *[3] Let $\Phi(t)$ be the BPFs vector defined in (8), then integration of this vector can be derived as*

$$\int_0^t \Phi(s)ds\simeq P\Phi(t), \tag{37}$$

where $P_{m\times m}$ is called the operational matrix of integration for BPFs and is given by

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{m\times m}. \qquad (38)$$

Lemma 3.2. *[3] Let $\Phi(t)$ be the BPFs vector defined in (8), the Itô integral of this vector can be derived as*

$$\int_0^t \Phi(s)dB(s) \simeq P_s\Phi(t), \qquad (39)$$

where P_s is called the stochastic operational matrix of integration for BPFs and is given by

$$P_s = \begin{bmatrix} B\left(\frac{h}{2}\right) & B\left(h\right) & B\left(h\right) & \dots & B\left(h\right) \\ 0 & B\left(\frac{3h}{2}\right) - B\left(h\right) & B\left(2h\right) - B(h) & \dots & B\left(2h\right) - B(h) \\ 0 & 0 & B\left(\frac{5h}{2}\right) - B\left(2h\right) & \dots & B\left(3h\right) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B\left(\frac{(2m-1)h}{2}\right) - B\left((m-1)h\right) \end{bmatrix}$$

Now we are ready to derive a new operational matrix of stochastic integration for the Haar wavelets basis. For this end we use BPFs and the matrix Q introduced in (26).

Theorem 3.1. *Suppose $H(t)$ be the Haar wavelets vector defined in (23), the integral of this vector can be derived as*

$$\int_0^t H(s)ds \simeq \frac{1}{m}QPQ^T H(t) = \Lambda H(t), \qquad (40)$$

where Λ is called the operational matrix for BPFs, Q is introduced in (26) and P is the operational matrix of integration for BPFs derived in (38).

Proof. Let $H(t)$ be the Haar wavelets vector, by using Theorem 2.1 and Lemma 3.1 we have

$$\int_0^t H(s)ds \simeq \int_0^t Q\Phi(s)ds = Q\int_0^t \Phi(s)ds = QP\Phi(t), \qquad (41)$$

now, Theorem 2.1 and Remark 2.1 give

$$\int_0^t H(s)ds \simeq QP\Phi(t) = \frac{1}{m}QPQ^T H(t) = \Lambda H(t), \qquad (42)$$

and this complete the proof. □

Theorem 3.2. *Suppose $H(t)$ be the Haar wavelets vector defined in (23), the Itô integral of this vector can be derived as*

$$\int_0^t H(s)dB(s) \simeq \frac{1}{m}QP_sQ^T H(t) = \Lambda_s H(t), \tag{43}$$

where Λ_s is called stochastic operational matrix for Haar wavelets, Q is introduced in (26) and P_s is the stochastic operational matrix of integration for BPFs derived in (39).

Proof. Let $H(t)$ be the Haar wavelets vector, by using Theorem 2.1 and Lemma 3.2 we have

$$\int_0^t H(s)dB(s) \simeq \int_0^t Q\Phi(s)dB(s) = Q\int_0^t \Phi(s)dB(s) = QP_s\Phi(t), \tag{44}$$

now, by Theorem 2.1 and Remark 2.1 we have

$$\int_0^t H(s)dB(s) = QP_s\Phi(t) = \frac{1}{m}QP_sQ^T H(t) = \Lambda_s H(t), \tag{45}$$

and this complete the proof. □

4. Solving stochastic Voltera-Fredholm integral equations

In this section, we apply the stochastic operational matrix of Haar wavelets for solving stochastic Voltera-Fredholm integral equation. Consider the following stochastic Voltera-Fredholm integral equation

$$X(t) = f(t) + \int_\alpha^\beta X(s)k_1(s,t)ds + \int_0^t X(s)k_2(s,t)ds$$
$$+ \int_0^t X(s)k_3(s,t)dB(s), \ t \in [0,T), \tag{46}$$

where $X(t)$, $f(t)$ and $k_i(s,t), i = 1,2,3$ are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_3(s,t)X(s)dB(s)$ is the Itô integral [2,18]. For sake of simplicity and without loss of generality we set $(\alpha,\beta) = (0,1)$. Now, by using the stochastic operational matrix of Haar wavelets, we approximate $X(t)$, $f(t)$ and $k_i(s,t), i = 1,2,3$ in terms of Haar wavelets as follows

$$f(t) = F^T H(t) = H^T(t)F, \tag{47}$$

$$X(t) = X^T H(t) = H^T(t)X, \tag{48}$$

$$k_i(s,t) = H^T(s)K_iH(t) = H^T(t)K_i^T H(s), i = 1,2,3, \tag{49}$$

where X and F are Haar wavelets coefficients vector, and $K_i, i = 1,2,3$ are Haar wavelets coefficient matrices defined in Eqs. (23) and (24). Substituting above approximations in Eq. (46), we have

$$X^T H(t) = F^T H(t) + X^T \left(\int_0^1 H(s) H^T(s) \right) K_1 H(t)$$

$$+ H^T(t) K_2^T \left(\int_0^t H(s) H^T(s) X ds \right) + H^T(t) K_3^T \left(\int_0^t H(s) H^T(s) X dB(s) \right),$$

using relation $\int_0^1 H(s) H^T(s) ds = I_{m\times m}$ and Remark 2.2 we get

$$X^T H(t) = F^T H(t) + X^T K_1 H(t) + H^T(t) K_2^T \left(\int_0^t \tilde{X} H(s) ds \right)$$

$$+ H^T(t) K_3^T \left(\int_0^t \tilde{X} H(s) dB_i(s) \right),$$

where $\tilde{X}$ is an $m \times m$ matrix. Now applying the operational matrices Λ and Λ_s for Haar wavelets derived in Eqs. (40) and (43) we have

$$X^T H(t) = F^T H(t) + X^T K_1 H(t) + H^T(t) K_2^T \tilde{X} \Lambda H(t) + H^T(t) K_3^T \tilde{X} \Lambda_s H(t) \quad (50)$$

by setting $Y_2 = K_2^T \tilde{X} \Lambda,\ Y_3 = K_3^T \tilde{X} \Lambda_s$ and using Remark 2.2 we derive

$$X^T H(t) - X^T K_1 H(t) - \hat{Y}_2^T H(t) - \hat{Y}_3^T H(t) = F^T H(t), \quad (51)$$

in which $\hat{Y}_2$ and $\hat{Y}_3$ are $m \times m$ matrices and they are linear functions of vector X. This equation is hold for all $t \in [0, 1)$, so we can write

$$X^T - X^T K_1 - \hat{Y}_2^T - \hat{Y}_3^T = F^T, \quad (52)$$

Since $\hat{Y}_2$ and $\hat{Y}_3$ are linear function of X, Eq. (52) is a linear system of equations for unknown vector X. By solving this linear system and determining X, we can approximate solution of stochastic Voltera-Fredholm integral equation (46) by substituting obtained vector X in Eq. (48).

5. **Error Analysis**

In this section, we investigate the convergence and error analysis of the presented method for solving stochastic Voltera-Fredholm integral equations.

Theorem 5.1. *Suppose that $f(t) \in L^2[0, 1)$ is an arbitrary function with bounded first derivative, $|f'(t)| \le M$, and $e_m(t) = f(t) - \sum_{i=0}^{m-1} f_i h_i(t)$, then*

$$\|e_m(t)\|_2 \le \frac{M}{\sqrt{3}m}, \quad (53)$$

that means the Haar wavelets series will be convergent.

Proof. By definition of the error $e_m(t)$ we have

$$\|e_m(t)\|_2^2 = \int_0^1 \left(\sum_{i=m}^{\infty} f_i h_i(t)\right)^2 dt = \sum_{i=m}^{\infty} f_i^2, \tag{54}$$

where $i = 2^j + k,\ m = 2^J,\ J > 0$ and

$$f_i = \int_0^1 h_i(t) f(t) dt = 2^{\frac{j}{2}} \left(\int_{k2^{-j}}^{\left(k+\frac{1}{2}\right)2^{-j}} f(t)dt - \int_{\left(k+\frac{1}{2}\right)2^{-j}}^{(k+1)2^{-j}} f(t)dt \right),$$

by the mean value theorem for integrals there are $\eta_{j1} \in \left(k2^{-j}, \left(k+\frac{1}{2}\right)2^{-j}\right)$ and $\eta_{j2} \in \left(\left(k+\frac{1}{2}\right)2^{-j}, (k+1)\,2^{-j}\right)$ such that

$$f_i = \int_0^1 h_i(t) f(t) dt = 2^{\frac{j}{2}} \left(f(\eta_{j1}) \int_{k2^{-j}}^{\left(k+\frac{1}{2}\right)2^{-j}} dt - f(\eta_{j2}) \int_{\left(k+\frac{1}{2}\right)2^{-j}}^{(k+1)2^{-j}} dt \right)$$

$$2^{\frac{j}{2}} \left(f(\eta_{j1}) \left[\left(k + \frac{1}{2}\right) 2^{-j} - k2^{-j} \right] - f(\eta_{j2}) \left[(k+1)\,2^{-j} - \left(k + \frac{1}{2}\right) 2^{-j} \right] \right)$$

$$= 2^{-\frac{j}{2}-1} \left(f(\eta_{j1}) - f(\eta_{j2})\right) = 2^{-\frac{j}{2}-1} \left(\eta_{j1} - \eta_{j2}\right) f^{'}(\eta_j),\ \eta_{j1} < \eta_j < \eta_{j2}, \tag{55}$$

this results

$$\|e_m(t)\|_2^2 = \sum_{i=m}^{\infty} f_i^2 = \sum_{i=m}^{\infty} 2^{-j-2} \left(\eta_{j1} - \eta_{j2}\right)^2 \left(f^{'}(\eta_j)\right)^2 \leq \sum_{i=m}^{\infty} 2^{-j-2} 2^{-2j} M^2$$

$$= \frac{M^2}{4} \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} 2^{-3j} = \frac{M^2}{4} \sum_{j=J}^{\infty} 2^{-3j} \left(\sum_{k=0}^{2^j-1} 1\right) = \frac{M^2}{4} \sum_{j=J}^{\infty} 2^{-2j} = \frac{M^2}{3} 2^{-2J}, \tag{56}$$

since $m = 2^J$, we have

$$\|e_m(t)\|_2 \leq \frac{M}{\sqrt{3}m}. \tag{57}$$

□

Theorem 5.2. *Suppose that $f(s,t) \in L^2\left([0,1) \times [0,1)\right)$ is a function with bounded partial derivative, $\left|\frac{\partial^2 f}{\partial s \partial t}\right| \leq M$, and $e_m(s,t) = f(s,t) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} f_{ij} h_i(s) h_j(t)$ then*

$$\|e_m(s,t)\|_2 \leq \frac{M}{3m^2}. \tag{58}$$

Proof. By definition of error $e_m(s,t)$ we have

$$\|e_m(s,t)\|_2^2 = \int_0^1 \left(\sum_{i=m}^{\infty} \sum_{l=m}^{\infty} f_{il} h_i(s) h_l(t)\right)^2 dt = \sum_{i=m}^{\infty} \sum_{l=m}^{\infty} f_{il}^2,$$

where $i = 2^j + k,\ l = 2^{j'} + k,\ m = 2^J, J > 0$ and

$$f_{ij} = \int_0^1 \int_0^1 h_i(s)h_l(t)f(s,t)dsdt,$$

from Theorem 5.1 there are $\eta_j, \eta_{j1},\ \eta_{j2}, \eta_{j'},\ \eta_{j'_1}$ and $\eta_{j'_2}$ such that

$$f_{ij} = \int_0^1 h_i(s)\left(\int_0^1 h_l(t)f(s,t)dt\right)ds = \int_0^1 h_i(s)\left[2^{-\frac{j'}{2}-1}\left(\eta_{j'_1} - \eta_{j'_2}\right)\frac{\delta f(s,\eta_{j'})}{\delta t}\right]ds,$$

$$2^{-\frac{j'}{2}-1}\left(\eta_{j'_1} - \eta_{j'_2}\right)\int_0^1 \frac{\delta f(s,\eta_{j'})}{\delta t}h_i(s) = 2^{-\frac{j}{2}-\frac{j'}{2}-2}\left(\eta_{j'_1} - \eta_{j'_2}\right)\left(\eta_{j_1} - \eta_{j_2}\right)\frac{\partial^2 f(\eta_j,\eta_{j'})}{\partial t\partial s},$$

this means

$$\|e_m(s,t)\|_2^2 = \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} f_{il}^2 = \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} 2^{-j-j'-4}\left(\eta_{j'_1} - \eta_{j'_2}\right)^2\left(\eta_{j_1} - \eta_{j_2}\right)^2\left|\frac{\partial^2 f(\eta_j,\eta_{j'})}{\partial t\partial s}\right|^2$$

$$\leq \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} M^2 2^{-3j-3j'-4},$$

by using Eq.(56) we can derive

$$\|e_m(s,t)\|_2^2 \leq M^2\sum_{i=m}^{\infty} 2^{-3j-2}\sum_{l=m}^{\infty} 2^{-3j'-2} = \frac{M^2}{(3m^2)^2},$$

in other words

$$\|e_m(s,t)\|_2 \leq \frac{M}{3m^2}. \tag{59}$$

□

Theorem 5.3. *Suppose $X(t)$ is the exact solution of (1) and $X_m(t)$ is its Haar wavelets approximate solution whose coeffcients are obtained by (19). Also assume that*

$$a)\ \|X(t)\| \leq \rho,\ t \in [0,1], \tag{60}$$

$$b)\ \|k_i(s,t)\| \leq M_i,\ s,t \in [0,1]\times[0,1], i = 1,2,3, \tag{61}$$

$$c)\ (\beta-\alpha)\left(M_1 + \Gamma_{1m}\right) + \left(M_2 + \Gamma_{2m}\right) + \|B(t)\|\left(M_3 + \Gamma_{3m}\right) < 1, \tag{62}$$

then

$$\|X(t) - X_m(t)\| \leq \frac{\Upsilon_m + \rho\left((\beta-\alpha)\Gamma_{1m} + \Gamma_{2m} + \|B(t)\|\Gamma_{3m}\right)}{1 - \left[(\beta-\alpha)\left(M_1 + \Gamma_{1m}\right) + \left(M_2 + \Gamma_{2m}\right) + \|B(t)\|\left(M_3 + \Gamma_{3m}\right)\right]},$$

where

$$\Upsilon_m = \sup_{t\in[0,1]}\frac{\left|f'(t)\right|}{\sqrt{3}\,m},\quad \Gamma_{im} = \frac{1}{3m^2}\sup_{s,t\in[0,1]}\left|\frac{\partial^2 k_i(s,t)}{\partial s\partial t}\right|,\ i = 1,2,3. \tag{63}$$

Proof. From (1) we have

$$X(t) - X_m(t) = f(t) - f_m(t) + \int_\alpha^\beta (k_1(s,t)X(s) - k_{1m}(s,t)X_m(s))\, ds$$

$$\int_0^t (k_2(s,t)X(s) - k_{2m}(s,t)X_m(s))\, ds + \int_0^t (k_3(s,t)X(s) - k_{3m}(s,t)X_m(s))\, dB(s),$$

so, by the mean value theorem, we can write

$$\|X(t) - X_m(t)\| \leq \|f(t) - f_m(t)\| + (\beta - \alpha) \|(k_1(s,t)X(s) - k_{1m}(s,t)X_m(s))\| \quad (64)$$

$$+ t \|(k_2(s,t)X(s) - k_{2m}(s,t)X_m(s))\| + B(t) \|(k_3(s,t)X(s) - k_{3m}(s,t)X_m(s))\|,$$

now by using Theorems 5.1 and 5.2 we have

$$\|(k_i(s,t)X(s) - k_{im}(s,t)X_m(s))\| \leq \|k_i(s,t)\| \|X(t) - X_m(t)\|$$

$$+ \|(k_i(s,t) - k_{im}(s,t))\| \|X(t)\| + \|(k_i(s,t) - k_{im}(s,t))\| \|X(t) - X_m(t)\|$$

$$\leq (M_i + \Gamma_{im}) \|X(t) - X_m(t)\| + \rho\Gamma_{im}, \; i = 1, 2, 3, \quad (65)$$

substituting (65) in (64), we get

$$\|X(t) - X_m(t)\| \leq \Upsilon_m + (\beta - \alpha) [(M_1 + \Gamma_{1m}) \|X(t) - X_m(t)\| + \rho\Gamma_{1m}]$$

$$+ t [(M_2 + \Gamma_{2m}) \|X(t) - X_m(t)\| + \rho\Gamma_{2m}]$$

$$+ B(t) [(M_3 + \Gamma_{3m}) \|X(t) - X_m(t)\| + \rho\Gamma_{3m}], \quad (66)$$

as assumption (c) holds we get the inequality

$$\|X(t) - X_m(t)\| \leq \frac{\Upsilon_m + \rho ((\beta - \alpha) \Gamma_{1m} + \Gamma_{2m} + \|B(t)\| \Gamma_{3m})}{1 - [(\beta - \alpha)(M_1 + \Gamma_{1m}) + (M_2 + \Gamma_{2m}) + \|B(t)\| (M_3 + \Gamma_{3m})]},$$

and this proves the desired result. □

6. **Numerical examples**

In this section, we consider some nontrivial numerical examples to illustrate the efficiency and reliability of the Haar wavelets operational matrices in solving stochastic Volterra-Fredholm integral equation.

Example 6.1. *Consider the following stochastic Volterra-Fredholm integral equation [8]*

$$X(t) = f(t) + \int_0^1 \cos(s+t)X(s)ds + \int_0^t (s+t)X(s)ds$$

$$+ \int_0^t e^{-3(s+t)}X(s)dB(s), \;\; s, t \in [0, 1],$$

in which $f(t) = t^2 + \sin(1+t) - 2\cos(1+t) - 2\sin(t) - \frac{7t^4}{12} + \frac{1}{40}B(t)$, $X(t)$ *is an unknown stochastic process defined on the probability space* $(\Omega, \mathcal{F}, P)$ *and* $B(t)$ *is a Brownian motion process. This stochastic Volterra-Fredholm integral equation is solved by using the Haar wavelets stochastic operational matrix and the proposed method in section 4 for different values of* $m = 2^J$. *In Fig. 6.1 the approximate solution for* $m = 2^7$ *is presented. A comparison between the numetrical solutions given by the Haar wavelets method (HWM) and the BPFs method proposed in [8] is shown in Table 2.*

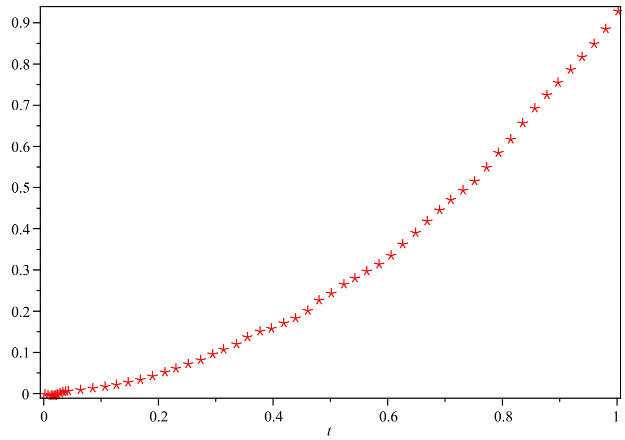

FIGURE 1. The approximate solution and exact solution for $m = 2^7$.

	$m = 2^5$		$m = 2^6$	
t	*HWM*	*BPFs [8]*	*HWM*	*BPFs [8]*
0.1	0.01894037	0.01991100	0.018461086	0.01551376
0.3	0.10263681	0.11746767	0.10332699	0.05832510
0.5	0.24699817	0.27412074	0.24627347	0.27753509
0.7	0.46248377	0.51447080	0.46447317	0.48867600
0.9	0.76428458	0.76857228	0.76405099	0.82223316

TABLE 1. Numerical results of Example 1 for different values of m.

Example 6.2. *Consider the following stochastic Volterra-Fredholm integral equation [8]*

$$X(t) = f(t) + \int_0^1 (s+t)X(s)ds + \int_0^t (s-t)X(s)ds$$

$$+ \frac{1}{125}\int_0^t \sin(s+t)X(s)dB(s), \;\; s,t \in [0,1],$$

where $f(t) = 2 - \cos(1) - (1 + t)\sin(1) + \frac{1}{250}\sin(B(t))$, $X(t)$ *is an unknown stochastic process defined on the probability space* $(\Omega, \mathcal{F}, P)$ *and* $B(t)$ *is a Brownian motion process. The proposed method in section 4 is used for approximate solution of this stochastic Volterra-Fredholm integral equation for different values of* $m = 2^J$*. In Fig. 6.2 the approximate solution for* $m = 2^7$ *is presented. A comparison between the numetrical solutions given by the Haar wavelets method (HWM) and the BPFs method proposed in [8] are shown in Table* **??**.

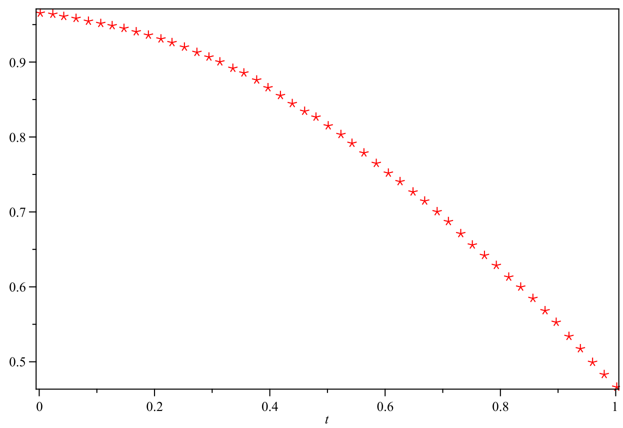

FIGURE 2. The approximate solution and exact solution for $m = 2^7$.

	$m = 2^5$		$m = 2^6$	
t	*HWM*	*BPFs [8]*	*HWM*	*BPFs [8]*
0.1	0.95261751	0.99832325	0.95351151	0.99586771
0.3	0.90442995	0.94271558	0.90583308	0.96183409
0.5	0.81494616	0.89309254	0.81603608	0.85038394
0.7	0.69226490	0.76959231	0.69438254	0.75669689
0.9	0.54802651	0.69244110	0.54967133	0.61203566

TABLE 2. Numerical results of Example 2 for different values of m.

7. Conclusion

A numerical method based on Haar wavelets and their stochastic operational matrix are proposed for solving stochastic Volterra-Fredholm integral equations. The main characteristic of this method is that it reduces these stochastic integral equations to those of solving a linear system of algebraic equations, thus greatly simplifying the problem and speeds up the computation. The convergence and error analysis of this method are investigated. Non-trivial examples demonstrate the efficiency and accuracy of the proposed method.

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