STRONG CONVERGENCE THEOREMS FOR ϕ-BEST PROXIMITY POINTS OF NON-SELF NONEXTENSIVE MAPPINGS IN A BANACH SPACE

Priyanka Priyadarshini Behera[1], C. Nahak[2]

This study obtains strong convergence theorems for non-self nonextensive mappings in a Banach space, utilizing the new hybrid algorithmic techniques to address global minimization problems that pertain to common ϕ-best proximity points. Applying the generalized projection operator, we introduce the ϕ-proximal property and exhibit some existence results of ϕ-best proximity points. This serves as an imperative tool for establishing our strong convergence results.

Keywords: Generalized projection operator; ϕ-best proximity points; nonextensive mappings; ϕ_p-property; uniform convexity

1. Introduction

Let B be a real Banach space with norm $\|\cdot\|$ and B^* be its dual space. Let K be a nonempty, closed subset of B and $T : K \to K$ be a mapping. Let $F(T)$ symbolise the fixed point set of T, defined as $F(T) := \{x \in K : Tx = x\}$. We will use the notations $S_B = \{u \in B : \|u\| = 1\}$ for the unit sphere, $\langle \cdot, \cdot \rangle$ for the duality pair of B and B^*, and $\to$, $\xrightarrow{w}$ and $\xrightarrow{w^*}$ for strong, weak and weak* convergence, respectively.

The duality mapping J from B to B^* is defined by

$$Ju = \{\zeta \in B^* : \langle u, \zeta \rangle = \|u\|^2 = \|\zeta\|^2\}, \text{ for every } u \in B.$$

J is single-valued in a smooth Banach space B and uniformly norm-to-norm continuous on each bounded subset of a uniformly smooth Banach space B (see [12, 17, 20] for more details). A Banach space B is said to be strictly convex if for any two distinct elements $u, v \in S_B$, $\|\frac{u+v}{2}\| < 1$ and uniformly convex if for any two sequences $\{u_n\}$ and $\{v_n\}$ in S_B with $\|\frac{u_n+v_n}{2}\| \to 1$, we have $\|u_n - v_n\| \to 0$ as $n \to \infty$. B is said to have property (KK) if the norm and weak convergence coincide on the unit sphere S_B. It has been proved that every uniformly convex Banach space is strictly convex, reflexive and satisfies property (KK) (see [17, 8]). B is said to be smooth if for every $u, v \in S_B$ and $t \in \mathbb{R}$, the limit $\lim_{t\to 0} \frac{\|u+tv\|-\|u\|}{t}$, exists and it is called uniformly smooth if the limit is attained uniformly. We recall that [8] the duality mapping J from a smooth Banach space B into B^* is said to be weakly sequentially continuous if $u_n \xrightarrow{w} u$ implies $Ju_n \xrightarrow{w^*} Ju$.

Let B be a smooth Banach space. We define the functional [22] $\phi : B \times B \to \mathbb{R}$ by

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv \rangle + \|v\|^2, \ \forall\, u, v \in B. \tag{1}$$

Some properties of the functional ϕ are listed below (see [22, 24, 15, 16] for more details):

(ϕ1) $(\|u\| - \|v\|)^2 \le \phi(u, v) \le (\|u\| + \|v\|)^2$, $u, v \in B$.

(ϕ2) $\phi(u, v) = \phi(u, z) + \phi(z, v) + 2\langle u - z, Jz - Jv \rangle$, $u, v, z \in B$.

(ϕ3) [16, Remark 2.1] $\phi(u, v) = 0$ if and only if $u = v$, for u, v in a strictly convex and smooth Banach space B.

[1]Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721302, India, Email priyanka.pb@iitkgp.ac.in

[2]Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721302, India, Email cnahak@maths.iitkgp.ac.in

Let K be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth Banach space B and $u \in B$. Then there exists a unique element $u_0 \in K$ such that $\phi(u,u_0) = \inf_{v\in K} \phi(u,v)$. We collect such element u_0 in $\Pi_K u$ and refer to Π_K as the generalized projection onto K. To get more insight on generalized projections, one may refer to [7, 15, 16, 19]. Alber et al. [23] introduced the notion of nonextensive mappings in a Banach space applying the generalized projection operator. Inspired by this, we say that a mapping $T : M \to N$ is non-self nonextensive if

$$\phi(Tu,Tv) \leq \phi(u,v), \text{ for all } u,v \in M. \tag{2}$$

Certainly, in a Hilbert space H, $\phi(u,v) = \|u-v\|^2$, for $u,v \in H$ and in such a case, nonextensive mappings coincide with the nonexpansive mappings.

A mapping $T : B \to B$ is said to possess a fixed point if the equation $Tu = u$ has atleast one solution. When considering a generalized projection operator, which may not behave like a metric, we define a fixed point of T as $\phi(u,Tu) = 0$, for $u \in B$. This simplifies the fixed point equation in a strictly convex and smooth Banach space. In the case of a non-self mapping T, the equation $\phi(u,Tu) = 0$ may not have a solution, resulting in $\phi(u,Tu) > 0$. More precisely, let us assume two nonempty subsets M, N of a smooth Banach space B such that $T : M \to N$ and $\phi(u,Tu) > 0, u \in M$. In such a case, it is our objective to find an element $u_0 \in M$ such that $\phi(u_0,Tu_0)$ attains its minimum value $\text{dist}_\phi(M,N)$. This point u_0 is referred to as a ϕ-best proximity point of the mapping T in M.

Researchers have devised and investigated numerous iterative approaches to approximate a common fixed point of nonexpansive mappings. In this direction, Nakajo and Takahashi [9] developed an iterative scheme for approximating fixed points of nonexpansive mappings in a Hilbert space. Matsushita and Takahashi [16] obtained a hybrid algorithm for obtaining a strong convergence theorem for relatively nonexpansive mappings in a Banach space. By modifying the hybrid method introduced in [16], Inoue et al. [4] proposed a modified shrinking projection technique for relatively nonexpansive mappings in a Banach space. This approach is based on the modifying shrinking projection method introduced by Takahashi et al. [21] in a Hilbert space.

Strong convergence theorems by hybrid methods for the best proximity point of a nonself nonexpansive mapping in Hilbert spaces were recently proposed by Jacob et al. [5]. A shrinking projection approach was developed by Suparatulatorn et al. [14] to solve the iterative scheme in a Hilbert space, with the aim of making computation easier. Recently, Suparatulatorn et al. [13] introduced the general Mann algorithm for nonself nonexpansive mappings and proved the convergence result in a Hilbert space. They improved upon previous results of [14] by utilizing an approximate best proximity point sequence for a mapping T instead of the demi-closedness principle. Motivated by all the above research findings, our objective is to obtain strong convergence theorems for non-self nonextensive mappings in Banach spaces. We propose hybrid algorithms for strong convergence results of common ϕ-best proximity points in a Banach space.

This article organizes its contents into four parts. The fundamental concepts are concisely outlined in Section 2. Section 3 introduces the notion of ϕ_p-property and some of its basic properties. In Section 4, we investigate the ϕ-proximal property and its associated findings. In Section 5, we employ the ϕ-proximal property for achieving strong convergence results by shrinking projection method for common ϕ-best proximity points in Banach spaces.

2. **Preliminaries**

Let M and N be two nonempty, closed, convex subsets of a smooth Banach space B. For the rest of the paper, unless otherwise stated, we denote M_0 and N_0 as follows:

$$M_0 = \{u \in M : \phi(u,v) = \text{dist}_\phi(M,N), \text{ for some } v \in N\},$$
$$N_0 = \{v \in N : \phi(u,v) = \text{dist}_\phi(M,N), \text{ for some } u \in M\},$$

where $\text{dist}_\phi(M,N) = \inf\{\phi(u,v) : u \in M \text{ and } v \in N\}$. The pair (M_0,N_0) is referred to as the ϕ-best proximity pair associated with (M,N). This pair is nonempty whenever (M,N) is a nonempty, compact pair of subsets in a Frechet smooth Banach space.

Let us denote $\text{Best}^\phi_M(T)$ as the set of ϕ-best proximity points of T on M, where

$$\text{Best}^\phi_M(T) = \{u \in M : \phi(u,Tu) = \text{dist}_\phi(M,N)\}.$$

It is easy to see that the set $\text{Best}^\phi_M(T)$ is contained in M_0.

The following findings are essential for exhibiting the main theorems that will be presented in subsequent sections.

Lemma 2.1. *(Kamimura and Takahashi [15]) Let B be a uniformly convex and smooth Banach space and let $\{u_n\}$ and $\{v_n\}$ be two sequences in B. If $\phi(u_n,v_n) \to 0$ and either of $\{u_n\}$ or $\{v_n\}$ is bounded, then $u_n - v_n \to 0$ as $n \to \infty$.*

Lemma 2.2. *(Alber [22], Alber and Reich [24], Kamimura and Takahashi [15]) Let K be a nonempty, closed, convex subset of a smooth Banach space B and $u \in B$. Then $u_0 = \Pi_K u$ if and only if $\langle u_0 - v, Ju - Ju_0\rangle \geq 0$, for all $v \in K$.*

Lemma 2.3. *(Alber [22], Kamimura and Takahashi [15]) Let B be a reflexive, strictly convex and smooth Banach space and let K be a nonempty, closed, convex subset of B and $u \in B$. Then $\phi(v,\Pi_K u) + \phi(\Pi_K u,u) \leq \phi(v,u)$, $\forall v \in K$.*

Lemma 2.4. *(Xu [6], Zălinescu [3]) Let B be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $\|\eta u + (1-\eta)v\|^2 \leq \eta\|u\|^2 + (1-\eta)\|v\|^2 - \eta(1-\eta)\psi(\|u-v\|)$, for all $u,v \in B_r = \{z \in B : \|z\| \leq r\}$ and $\eta \in [0,1]$.*

3. ϕ_p-property

We now introduce a concept called the ϕ_p-property, which generalizes the previously defined P-property in [18]. While the P-property relies on a metric, the ϕ_p-property utilizes the functional ϕ in a smooth Banach space.

Definition 3.1. *[ϕ_p-property] Let (M,N) be a pair of non-empty subsets of a smooth Banach space B with M_0 is non-empty. Then (M,N) is said to have the ϕ_p-property if and only if*

$$\begin{cases} \phi(u_1,v_1) = dist_\phi(M,N) \\ \phi(u_2,v_2) = dist_\phi(M,N) \end{cases} \Rightarrow \phi(u_1,u_2) = \phi(v_1,v_2),$$

for $u_1,u_2 \in M_0$ and $v_1,v_2 \in N_0$.

Lemma 3.1. *Any pair (M,N) of non-empty, closed, convex subsets of a real Hilbert space H satisfies the ϕ_p-property.*

Proof. In a Hilbert space $\phi(u,v) = \|u-v\|^2$. Let us take $u_1,u_2 \in M_0$ and $v_1,v_2 \in N_0$ satisfying $\begin{cases} \phi(u_1,v_1) = \text{dist}_\phi(M,N), \\ \phi(u_2,v_2) = \text{dist}_\phi(M,N). \end{cases}$ This gives, $\|u_1 - v_1\|^2 = \text{dist}_\phi(M,N)$ and $\|u_2 - v_2\|^2 = \text{dist}_\phi(M,N)$. Then, by the uniqueness of generalized projection operator ϕ on H, we get $v_1 = \Pi_N(u_1)$ and $v_2 = \Pi_N(u_2)$. Also, $\|u_1 - v_2\| = \|u_2 - v_1\|$. Hence, $\|u_1 - v_2\|^2 = \|u_1 - u_2\|^2 + \text{dist}_\phi(M,N)$; and $\|u_2 - v_1\|^2 = \|v_1 - v_2\|^2 + \text{dist}_\phi(M,N)$. Therefore, $\|u_1 - u_2\|^2 = \|v_1 - v_2\|^2$. So, $\phi(u_1,u_2) = \phi(v_1,v_2)$. Hence H satisfies the ϕ_p-property. □

Lemma 3.2. *Let B be a strictly convex and smooth Banach space and (M,N) be a pair of non-empty subsets of B with M_0 being non-empty. Then, (M,N) satisfies the ϕ_p-property whenever $dist_\phi(M,N) = 0$.*

Proof. Let us consider $\phi(u_1,v_1) = 0 = \phi(u_2,v_2)$; for $u_1,u_2 \in M_0$ and $v_1,v_2 \in N_0$. Then, $u_1 = v_1$ and $u_2 = v_2$. Hence, $\phi(u_1,u_2) - \phi(v_1,v_2) = 0$. So, $\phi(u_1,u_2) = \phi(v_1,v_2)$. □

Remark 3.1. *It is noted that if B is a uniformly convex and smooth Banach space and M_0 is non-empty, then according to Lemma 3.2, the pair (M,N) satisfies the ϕ_p-property, when the ϕ-distance function $dist_\phi(M,N)$ equals 0. However, if $dist_\phi(M,N) \neq 0$, then a non-empty pair of subsets need not satisfy the ϕ_p-property, even if the space is uniformly smooth and uniformly convex. This is shown in the following example.*

Example 3.1. *Let us consider $(\mathbb{R}^3, \|\cdot\|_3)$, where the norm is defined by*

$$\|x\|_3 = (|x_1|^3 + |x_2|^3 + |x_3|^3)^{\frac{1}{3}},\ x = (x_1,x_2,x_3) \in \mathbb{R}^3.$$

Let us take

$$M = \{(x_1,x_2,x_3) : 0 \leq x_1 \leq 1, x_2^3 + x_3^3 = 1\} \text{ and } N = \{(0,0,0)\}.$$

Then both M and N are compact subsets of $\mathbb{R}^3$. Here $dist_\phi(M,N) = 1$ and $M_0 = \{(0,1,0),(0,0,1)\}$ and $N_0 = N$. Now, $\phi((0,1,0),(0,0,0)) = 1 = \phi((0,0,1),(0,0,0))$; but $\phi((0,1,0),(0,0,1)) \neq \phi((0,0,0))$. So, (M,N) fails to satisfy the ϕ_p-property.

Remark 3.2. *Any pair (M,N) of nonempty subsets of a smooth Banach space B with $dist_\phi(M,N) \neq 0$ may not satisfy the ϕ_p-property if one of the sets is singleton.*

Lemma 3.3. *Let B be a strictly convex and smooth Banach space and (M,N) be a pair of non-empty, closed, convex subsets of B with M_0 being non-empty and (M,N) satisfies the ϕ_p-property. Then, the generalized projection Π_{M_0} restricted to N_0 is an ϕ-isometry.*

Proof. For elements $v_1,v_2 \in N_0$, there exists $u_1,u_2 \in M_0$ such that $\phi(u_i,v_i) = \text{dist}_\phi(M,N)$, for $i = 1,2$. i.e., $\Pi_{M_0}(v_i) = u_i, i = 1,2$. By the ϕ_p-property, $\phi(\Pi_{M_0}(v_1),\Pi_{M_0}(v_2)) = \phi(v_1,v_2) = \phi(u_1,u_2)$. Thus, Π_{M_0} is an ϕ-isometry. □

The subsequent conclusions are essential for proving the main findings.

Lemma 3.4. *Let (M,N) be a pair of nonempty, closed, convex subsets of a smooth Banach space B. Then $\phi(u,\Pi_N u) = dist_\phi(M,N)$, $\forall u \in M_0$ and $\phi(\Pi_M v,v) = dist_\phi(M,N)$, $\forall v \in N_0$.*

Proof. Let $u \in M_0$, then we can find an element $z \in N$ such that $\phi(u,z) = \text{dist}_\phi(M,N)$. Using the definition of $\Pi_N u$, we ascertain that

$$\phi(u,\Pi_N u) = \inf_{\hat{u} \in N} \phi(u,\hat{u}) \leq \phi(u,z) = \text{dist}_\phi(M,N).$$

Thus, $\phi(u,\Pi_N u) = \text{dist}_\phi(M,N)$. The other claim follows likewise. □

Lemma 3.5. *Let (M,N) be a pair of nonempty subsets of a strictly convex and smooth Banach space B with M being closed and convex. Let $T : M \to N$ be a mapping satisfying $T(M_0) \subseteq N_0$ and that (M,N) has the ϕ_p-property. Then $F(\Pi_M \circ T|_{M_0}) = F(\Pi_M \circ T) \cap M_0 = Best_M^\phi(T)$.*

Proof. Let $u \in F(\Pi_M \circ T) \cap M_0$. Since $u \in F(\Pi_M \circ T)$, it implies that $\Pi_M \circ Tu = u$. Then,

$$\begin{aligned}\phi(u,Tu) &= \phi(u,\Pi_M \circ Tu) + \phi(\Pi_M \circ Tu,Tu) + 2\langle u - \Pi_M \circ Tu, J\Pi_M \circ Tu - JTu\rangle \\ &= \phi(\Pi_M \circ Tu,Tu) \\ &= \text{dist}_\phi(M,N). \text{ (by Lemma 3.4)}\end{aligned}$$

This shows that $u \in \text{Best}_M^\phi(T)$.

Conversely, let $u \in \text{Best}_M^\phi(T)$. Clearly, $u \in M_0$ and $\phi(u,Tu) = \text{dist}_\phi(M,N)$. Also, by Lemma 3.4, we deduce that $\phi(\Pi_M \circ Tu,Tu) = \text{dist}_\phi(M,N)$. Since (M,N) has the ϕ_p-property, we obtain that $\phi(u,\Pi_M \circ Tu) = 0$. i.e., $u \in F(\Pi_M \circ T)$ and as a result, $u \in F(\Pi_M \circ T) \cap M_0$. □

Lemma 3.6. *Let (M,N) be a pair of nonempty subsets of a smooth and strictly convex Banach space B with N being closed and convex. Let $T : M \to N$ be a mapping that satisfies $T(M_0) \subseteq N_0$. Then, $Tu = \Pi_N u$, $\forall u \in Best_M^{\phi}(T)$.*

Proof. Let $u \in \mathrm{Best}_M^{\phi}(T)$. i.e., $\phi(u,Tu) = \mathrm{dist}_{\phi}(M,N)$. Now, since $\mathrm{Best}_M^{\phi}(T) \subseteq M_0$ and $\phi(u,\Pi_N u) = \mathrm{dist}_{\phi}(M,N)$ (by Lemma 3.4); using ϕ_p-property, we deduce that

$$\phi(Tu,\Pi_N u) = 0.$$

Hence, by (ϕ3), $Tu = \Pi_N u$. □

4. ϕ-proximal property

In this section, we study the ϕ-proximal property, which is an encompassing term derived from [11] in the context of a generalized projection.

Definition 4.1. *[Aϕ-BPS] Let (M,N) be a nonempty pair of subsets of a smooth Banach space B and $T : M \to N$ be a non-self mapping. A sequence $\{u_n\}$ in M is said to be an approximate ϕ-best proximity point sequence (Aϕ-BPS) for T if and only if $\lim_{n\to\infty} \phi(u_n,Tu_n) = dist_{\phi}(M,N)$.*

It is noteworthy that in a Hilbert space, this notion is analogous to the idea of an approximate best-proximity point sequence as studied in [11]. The following lemma ensures the existence of an (Aϕ-BPS) for a non-self nonextensive mapping.

Lemma 4.1. *Let (M,N) be a pair of nonempty, convex subsets of a smooth and uniformly convex Banach space that satisfies the ϕ_p-property. Let $T : M \to N$ be a non-self nonextensive mapping satisfying $T(M_0) \subseteq N_0$, then there exists an (Aϕ-BPS) for T in M.*

Proof. It is evident that M_0 is a convex subset of M. Let $u_0 \in M_0$. Since $T(M_0) \subseteq N_0$, there exists an element $u_1 \in M_0$ such that $\phi(u_1,Tu_0) = \mathrm{dist}_{\phi}(M,N)$. Now, since $u_1 \in M_0$ and by using the fact that $T(M_0) \subseteq N_0$, it is guaranteed to find an element $u_2 \in M_0$ such that $\phi(u_2,Tu_1) = \mathrm{dist}_{\phi}(M,N)$. By proceeding in this way, we can identify a sequence $\{u_n\}$ in M_0 such that

$$\phi(u_{n+1},Tu_n) = \mathrm{dist}_{\phi}(M,N), \text{ for each } n \in \mathbb{N} \cup \{0\}. \tag{3}$$

Since (M,N) satisfies the ϕ_p-property and T is nonextensive, it follows that $\phi(u_n,u_{n+1}) = \phi(Tu_{n-1}, Tu_n) \leq \phi(u_{n-1},u_n)$. As a result, $\{\phi(u_n,u_{n+1})\}$ is a decreasing, bounded sequence and so it converges. Consequently, we find that as n approaches infinity, $\phi(u_n,u_{n+1}) \to 0$ and Proposition 2.1 gives

$$\lim_{n\to\infty} \|u_n - u_{n+1}\| = 0. \tag{4}$$

Therefore, we conclude that

$$\phi(u_n,Tu_n) = \phi(u_n,u_{n+1}) + \phi(u_{n+1},Tu_n) + 2\langle u_n - u_{n+1}, Ju_{n+1} - Ju_n\rangle. \tag{5}$$

Employing (3) and (4) in (5), we get $\lim_{n\to\infty} \phi(u_n,Tu_n) = \mathrm{dist}_{\phi}(M,N)$. This proves the claim. □

Motivated by [11] and [20], the subsequent definition is provided.

Definition 4.2. *[ϕ-proximal property] Let (M,N) be a pair of nonempty subsets of a smooth Banach space B. A non-self mapping $T : M \to N$ is said to satisfy the ϕ-proximal property if and only if for each sequence $\{u_n\}$ in M such that $u_n \xrightarrow{w} u_0 \in M$ and $\lim_{n\to\infty} \phi(u_n,Tu_n) = dist_{\phi}(M,N)$, we have $\phi(u_0,Tu_0) = dist_{\phi}(M,N)$.*

Note that if $\mathrm{dist}_{\phi}(M,N) = 0$ and B is smooth and strictly convex, then the ϕ-proximal property reduces to the demi-closedness principle of $I - T$ at 0, where I is the identity operator on M. Recall that the map $I - T : M \to B$ is demi-closed at 0 if whenever $\{u_n\}$ is a sequence in M such that $u_n \xrightarrow{w} u_0 \in M$ and $(I-T)u_n \to 0$ as $n \to \infty$; then $(I-T)u_0 = 0$.

The following theorem asserts the existence of ϕ-best proximity points for non-self nonextensive mappings in a uniformly convex Banach space.

Theorem 4.1. *Let (M,N) be a pair of nonempty, convex subsets of a smooth and uniformly convex Banach space B with M being weakly compact and that (M,N) satisfies the ϕ_p-property. Suppose $T: M \to N$ be a non-self nonextensive mapping satisfying $T(M_0) \subseteq N_0$ and M_0 is nonempty. Then T has a ϕ-best proximity point if one of the following condition hold.*
(1) J is weakly sequentially continuous and T is weakly continuous.
(2) T satisfies the ϕ-proximal property.

Proof. It is apparent from Lemma 4.1 that there exists a (Aϕ-BPS) sequence $\{u_n\}$ in M_0. i.e., $\lim_{n\to\infty} \phi(u_n, Tu_n) = \text{dist}_\phi(M,N)$. Now, since T is weakly compact, we may assume that $u_n \xrightarrow{w} u_0 \in M$.

(1) If T is weakly continuous; then $Tu_n \xrightarrow{w} Tu_0$, and since J is weakly sequentially continuous, $JTu_n \xrightarrow{w^*} JTu_0$. Therefore,

$$\begin{aligned}\phi(u_0, Tu_0) \leq \liminf_{n\to\infty} \phi(u_n, Tu_0) &= \lim_{n\to\infty} \phi(u_n, Tu_n) + \|Tu_0\|^2 - \|Tu_n\|^2 + 2\langle u_n, JTu_n - JTu_0\rangle \\ &= \text{dist}_\phi(M,N).\end{aligned}$$

(2) If T satisfies the ϕ-proximal property, we obtain from the Definition 4.2 that $\phi(u_0, Tu_0) = \text{dist}_\phi(M,N)$.

□

Motivated by the Property (UC), studied in [1], we introduce the notion of property (ϕ-UC) in a Banach space.

Definition 4.3. *[Property (ϕ-UC)] A pair (M,N) of nonempty subsets of a smooth Banach space B is said to have the property (ϕ-UC) if for any sequences $\{u_n\}, \{v_n\}$ in M and $\{t_n\}$ in N,*

$$\left.\begin{aligned}\lim_{n\to\infty} \phi(u_n, t_n) = dist_\phi(M,N) \\ \lim_{n\to\infty} \phi(v_n, t_n) = dist_\phi(M,N)\end{aligned}\right\} \Rightarrow \lim_{n\to\infty} \phi(u_n, v_n) = 0.$$

We now state another version of Theorem 4.1 for a uniformly convex Banach space.

Theorem 4.2. *Let (M,N) be a pair of nonempty, convex subsets of a uniformly convex and Frechet smooth Banach space B with N being compact and M is closed and bounded. Assume that $T: M \to N$ is a non-self nonextensive mapping with $T(M_0) \subseteq N_0$ and that (M,N) satisfies the property (ϕ-UC). Then T has a ϕ-best proximity point in M.*

Proof. Lemma 4.1 gives a sequence $\{u_n\}$ in M_0 satisfying $\lim_{n\to\infty} \phi(u_n, Tu_n) = \text{dist}_\phi(M,N)$. Since M is bounded and N is compact, it follows that $u_n \xrightarrow{w} u_0 \in M_0$ and $Tu_n \to v_0 \in N$. Therefore,

$$\begin{aligned}\phi(u_0, v_0) \leq \liminf_{n\to\infty} \phi(u_n, v_0) &= \lim_{n\to\infty} \phi(u_n, Tu_n) + \phi(Tu_n, v_0) + 2\langle u_n - Tu_n, JTu_n - Jv_0\rangle \\ &= \text{dist}_\phi(M,N), \end{aligned} \tag{6}$$

which follows using the fact that $JTu_n - Jv_n \to 0$ as $n \to \infty$ in a Frechet smooth Banach space. On the other hand, for each $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \phi(u_n, v_0) = \text{dist}_\phi(M,N)$. Since (M,N) satisfies the property (ϕ-UC); it follows that $\lim_{n\to\infty} \phi(u_n, u_0) = 0$. Consequently, by Proposition 2.1, $\lim_{n\to\infty} \|u_n - u_0\| = 0$. As a result, $\phi(v_0, Tu_0) = 0$; which implies that $v_0 = Tu_0$. Thus, $\phi(u_0, Tu_0) = \text{dist}_\phi(M,N)$. □

5. Algorithms and strong convergence results

This section commences some strong convergence results of the iterate sequence to the ϕ-best proximity point of non-self nonextensive mappings using hybrid algorithms. We start with proving the strong convergence theorem using the shrinking projection method in a Banach space.

Theorem 5.1. *Let (M,N) be a pair of nonempty, closed, convex subsets of a uniformly convex and uniformly smooth Banach space B that satisfies the ϕ_p-property. Let $T : M \to N$ be a non-self nonextensive mapping satisfying $T(M_0) \subseteq N_0$ and that T has the ϕ-proximal property. Let us consider the sequence $\{u_n\}$ generated by*

$$\begin{cases} u_1 = \Pi_{M_0}u, u \in B \text{ be arbitrary}, \\ H_1 = M_0, \\ v_n = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)J\Pi_M Tu_n), n \in \mathbb{N}, \\ H_{n+1} = \{z \in H_n : \phi(z,v_n) \le \phi(z,u_n)\}, \\ u_{n+1} = \Pi_{H_{n+1}}u, \end{cases} \tag{7}$$

for each $n \in \mathbb{N}$, where $\alpha_n \in [0,a]$, for some $a \in [0,1)$. If $Best^{\phi}_M(T)$ is nonempty, then the sequence $\{u_n\}$ strongly converges to $u^* = \Pi_{Best^{\phi}_M(T)}u$.

Proof. From the definition of H_n, it is obvious that H_n is closed. The convexity of H_n is inferred from the inequality $\phi(z,v_n) \le \phi(z,u_n)$, which is equivalent to $2\langle z, Ju_n - Jv_n\rangle + \|v_n\|^2 - \|u_n\|^2 \le 0$. Next, we show by induction that the set $\text{Best}^{\phi}_M(T)$ is contained in H_n, for each $n \in \mathbb{N}$. For $n = 1$, $\text{Best}^{\phi}_M(T) \subset H_1 = M_0$. Let us assume that $\text{Best}^{\phi}_M(T) \subset H_k$, for some $k \in \mathbb{N}$ and $p \in \text{Best}^{\phi}_M(T) \subset H_k$. Then

$$\begin{aligned} \phi(p,v_k) &= \phi(p, J^{-1}(\alpha_k Ju_k + (1-\alpha_k)J\Pi_M Tu_k)) \\ &\le \|p\|^2 - 2\alpha_k\langle p, Ju_k\rangle - 2(1-\alpha_k)\langle p, J\Pi_M Tu_k\rangle + \alpha_k\|u_k\|^2 + (1-\alpha_k)\|\Pi_M Tu_k\|^2 \\ &= \alpha_k\phi(p,u_k) + (1-\alpha_k)\phi(p,\Pi_M Tu_k). \end{aligned} \tag{8}$$

By Lemma 3.4, it follows that $\phi(\Pi_M Tu_k, Tu_k) = \text{dist}_\phi(M,N)$ and $\phi(p,Tp) = \text{dist}_\phi(M,N)$. By using the ϕ_p-property and the fact that T is nonextensive mapping, we deduce that

$$\phi(p,\Pi_M Tu_k) = \phi(Tp, Tu_k) \le \phi(p,u_k). \tag{9}$$

So, (8) reduces to, $\phi(p,v_k) \le \phi(p,u_k)$; indicating that $p \in H_{k+1}$. Therefore, it follows that $\text{Best}^{\phi}_M(T) \subset H_n$, for all $n \in \mathbb{N}$. This also proves that $\{u_n\}$ is well-defined.
On the other hand, from the definition H_n, $u_n = \Pi_{H_n}u$. Applying Proposition 2.3, we get

$$\phi(u_n,u) \le \phi(p,u) - \phi(p,u_n) \le \phi(p,u), \text{ for each } n \in \mathbb{N}. \tag{10}$$

This shows that $\{\phi(u_n,u)\}$ is bounded and hence by the inequality $(\|u_n\| - \|u\|)^2 \le \phi(u_n,u)$; it follows that $\{u_n\}$ is bounded. Next, since $u_{n+1} = \Pi_{H_{n+1}}u \in H_n$, using Proposition 2.3, we obtain that

$$\phi(u_n,u) \le \phi(u_{n+1},u), \text{ for each } n \in \mathbb{N}. \tag{11}$$

Thus, $\{\phi(u_n,u)\}$ is nondecreasing and so, it converges to a limit. Further, we have

$$\phi(u_{n+1},u_n) = \phi(u_{n+1},\Pi_{H_n}u) \le \phi(u_{n+1},u) - \phi(\Pi_{H_n}u,u) = \phi(u_{n+1},u) - \phi(u_n,u),$$

for each $n \in \mathbb{N}$. This implies that $\phi(u_{n+1},u_n) = 0$ and hence by Proposition 2.1, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{12}$$

Besides, we can see that $\phi(u_{n+1},v_n) \le \phi(u_{n+1},u_n)$, for each $n \in \mathbb{N}$, which follows from the fact that $u_{n+1} \in H_{n+1}$. Thus, we can conclude that

$$\lim_{n\to\infty} \phi(u_{n+1},v_n) = 0 \text{ and so } \lim_{n\to\infty} \|u_{n+1} - v_n\| = 0. \tag{13}$$

Now,

$$\begin{aligned} \phi(u_n,v_n) &= \phi(u_n, J^{-1}(\alpha_n Ju_n + (1-\alpha_n)J\Pi_M Tu_n)) \\ &\le \alpha_n\phi(u_n,u_n) + (1-\alpha_n)\phi(u_n,\Pi_M Tu_n) \\ &= (1-\alpha_n)\phi(u_n,\Pi_M Tu_n). \end{aligned} \tag{14}$$

From (12) and (13), we have $\|u_n - v_n\| \to 0$ and since $\{u_n\}$ is bounded, we have $\phi(u_n, v_n) \to 0$ as $n \to \infty$. So, (14) together with the fact that α_n does not converge to 1 gives

$$\lim_{n\to\infty} \phi(u_n, \Pi_M T u_n) = 0. \tag{15}$$

Thus,

$$\begin{aligned}\lim_{n\to\infty} \phi(u_n, Tu_n) &= \lim_{n\to\infty} \phi(u_n, \Pi_M Tu_n) + \phi(\Pi_M Tu_n, Tu_n) + 2\langle u_n - \Pi_M Tu_n, J\Pi_M Tu_n - JTu_n\rangle \\ &= \lim_{n\to\infty} \phi(\Pi_M Tu_n, Tu_n) \text{ (by (15) and Frechet smoothness of } B) \\ &= \operatorname{dist}_\phi(M,N). \end{aligned}\tag{16}$$

As a result, $\{u_n\}$ is an (Aϕ-BPS) for the mapping T. Our aim now is to show that the set of weak accumulation points of the sequence $\{u_n\}$ is contained in $\text{Best}_M^\phi(T)$. To show this, let q be the weak limit point of the sequence $\{u_n\}$. i.e., we can find a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \xrightarrow{w} q$. From (16) and using the fact that T satisfies the ϕ-proximal property, we obtain that $\phi(q, Tq) = \operatorname{dist}_\phi(M,N)$. i.e., $q \in \text{Best}_M^\phi(T)$.
Let $u^* = \Pi_{\text{Best}_M^\phi(T)} u$. By (10), we get $\phi(u_n, u) \le \phi(u^*, u)$, for all $n \in \mathbb{N}$. Then

$$\phi(q,u) \le \liminf_{k\to\infty} \phi(u_{n_k}, u) \le \limsup_{k\to\infty} \phi(u_{n_k}, u) \le \phi(u^*, u).$$

On the other hand, using the fact that $u_{n+1} = \Pi_{H_{n+1}} u$ and $u^* \in \text{Best}_M^\phi(T) \subset H_n$, we get $\phi(u_{n+1}, u) \le \phi(u^*, u)$. From the definition of $\Pi_{\text{Best}_M^\phi(T)} u$, we obtain that $q = u^*$ and so, $\lim_{k\to\infty} \phi(u_{n_k}, u) = \phi(u^*, u)$ which further gives $\|u_{n_k}\| \to \|u^*\|$. Therefore, by the property (KK), we conclude that $\{u_{n_k}\}$ strongly converges to $u^* = \Pi_{\text{Best}_M^\phi(T)} u$ and since $\{u_{n_k}\}$ is arbitrary, the assertion follows. ☐

Theorem 5.2. *Let (M,N) be a pair of nonempty, closed, convex subsets of a uniformly convex and uniformly smooth Banach space B that satisfies the ϕ_p-property. Let $T : M \to N$ be a non-self nonextensive mapping satisfying $T(M_0) \subseteq N_0$ and that T has the ϕ-proximal property. Let us consider the sequence $\{u_n\}$ generated by*

$$\begin{cases} u_1 = u \in M_0 \text{ be arbitrary}, \\ v_n = J^{-1}(\alpha_n J u_n + (1-\alpha_n) J\Pi_M T u_n), \\ H_n = \{z \in M_0 : \phi(z, v_n) \le \phi(z, u_n)\}, \\ W_n = \{z \in M_0 : \langle u_n - z, Ju_n - Ju\rangle \le 0\}, \\ u_{n+1} = \Pi_{H_n \cap W_n} u, \end{cases} \tag{17}$$

for each $n \in \mathbb{N}$, where $\alpha_n \in [0,a]$, for some $a \in [0,1)$. If $Best_M^\phi(T)$ is nonempty, then the sequence $\{u_n\}$ strongly converges to $u^* = \Pi_{Best_M^\phi(T)} u$.

Proof. Firstly, we show that $\text{Best}_M^\phi(T)$ is contained in $H_n \cap W_n$. By Theorem 5.1, it is assured that H_n is closed and convex and $\text{Best}_M^\phi(T)$ is contained in H_n. It can be easily seen that W_n is closed and convex. So, it remains to only prove that $\text{Best}_M^\phi(T) \subset W_n$, for each $n \in \mathbb{N}$. For $n = 1$, $\text{Best}_M^\phi(T) \subset M_0 = W_1$ and assume that $\text{Best}_M^\phi(T) \subset W_k$, for some $k \in \mathbb{N}$. Since, $u_{k+1} = \Pi_{H_k \cap W_k} u$, we obtain that

$$\langle u_{k+1} - z, Ju - Ju_{k+1}\rangle \ge 0, \ \forall z \in H_k \cap W_k. \tag{18}$$

Since, $\text{Best}_M^\phi(T) \subset H_k \cap W_k$; (18) holds for all $z \in \text{Best}_M^\phi(T)$. Hence, $\text{Best}_M^\phi(T) \subset W_{k+1}$. Thus, $\text{Best}_M^\phi(T) \subset H_n \cap W_n$. It follows from the definition of W_n and Proposition 2.3 that, $u_n = \Pi_{W_n} u$ and so, $\phi(u_n, u) \le \phi(p, u) - \phi(p, u_n) \le \phi(p, u)$, for each $p \in \text{Best}_M^\phi(T)$. Therefore, $\{\phi(u_n, u)\}$ is bounded. Moreover, $\{u_n\}$ is bounded. Since, $u_{n+1} = \Pi_{H_n \cap W_n} u \in W_n$; by Proposition 2.3, $\phi(u_n,u) \leq \phi(u_{n+1},u)$, for each $n \in \mathbb{N}$. Therefore, $\{\phi(u_n,u)\}$ is nondecreasing and so, it converges. Now,

$$\phi(u_{n+1},u_n) = \phi(u_{n+1},\Pi_{W_n}u) \leq \phi(u_{n+1},u) - \phi(u_n,u), \text{ for each } n \in \mathbb{N}.$$

Consequently,

$$\lim_{n\to\infty} \phi(u_{n+1},u_n) = 0 \text{ and } \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{19}$$

Since, $u_{n+1} = \Pi_{H_n \cap W_n} u \in H_n$, from the definition of H_n, we have $\phi(u_{n+1},v_n) \leq \phi(u_{n+1},u_n), \forall n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty} \phi(u_{n+1},v_n) = 0 \text{ and } \lim_{n\to\infty} \|u_{n+1} - v_n\| = 0. \tag{20}$$

By broadly applying the proof of Theorem 5.1, one can show that $\{u_n\}$ converges strongly to $u^* = \Pi_{\text{Best}^{\phi}_M(T)} u$. □

We now state the strong convergence result, which a modified shrinking projection method defined in [2].

Theorem 5.3. *Let (M,N) be a pair of nonempty, closed, convex subsets of a uniformly smooth and uniformly convex Banach space B that satisfies the ϕ_p-property. Let $T : M \to N$ be a nonself nonextensive mapping satisfying $T(M_0) \subseteq N_0$ and that T has the ϕ-proximal property. Let us consider the sequence $\{u_n\}$ generated by*

$$\begin{cases} u_0 = u \in M_0 \text{ be arbitrary}, \\ H_1 = M_0,\ u_1 = \Pi_{M_0} u, \\ v_n = \Pi_M J^{-1}(\alpha_n J \Pi_N u_n + (1-\alpha_n) J T u_n), \\ H_{n+1} = \{z \in H_n : \phi(z,v_n) \leq \phi(z,u_n)\}, \\ u_{n+1} = \Pi_{H_{n+1}} u, \end{cases} \tag{21}$$

for each $n \in \mathbb{N}$, where $\alpha_n \in (0,1)$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. If $Best^{\phi}_M(T)$ is nonempty, then the sequence $\{u_n\}$ strongly converges to $u^* = \Pi_{Best^{\phi}_M(T)} u$.

Proof. Clearly, H_n is closed and convex, for each $n \in \mathbb{N}$. Firstly, we claim that $\text{Best}^{\phi}_M(T) \subset H_n$, for each $n \in \mathbb{N}$. For $n = 1$, $\text{Best}^{\phi}_M(T) \subset H_1 = M_0$ is obvious. Assume that $\text{Best}^{\phi}_M(T) \subset H_k$, for some $k \in \mathbb{N}$ and $p \in \text{Best}^{\phi}_M(T)$. By Lemma 3.5, we have $p = \Pi_M \circ Tp$. Then,

$$\begin{aligned} \phi(p,v_k) &= \phi(\Pi_M \circ Tp, \Pi_M J^{-1}(\alpha_k J \Pi_N u_k + (1-\alpha_k) J T u_k)) \\ &\leq \phi(Tp, J^{-1}(\alpha_k J \Pi_N u_k + (1-\alpha_k) J T u_k)) \\ &\leq \alpha_n \phi(Tp, \Pi_N u_k) + (1-\alpha_k) \phi(Tp, Tu_k). \end{aligned} \tag{22}$$

Since, $\phi(u_k,\Pi_N u_k) = \text{dist}_{\phi}(M,N)$ and $\phi(p,Tp) = \text{dist}_{\phi}(M,N)$; by ϕ_p-property, we have

$$\phi(Tp,\Pi_N u_k) = \phi(p,u_k). \tag{23}$$

Using (23) and the fact that T is nonextensive, (22) reduces to $\phi(p,v_k) \leq \phi(p,u_k)$, for some $k \in \mathbb{N}$. This shows that $p \in H_{k+1}$. Thus, by induction, it is proved that $\text{Best}^{\phi}_M(T) \subset H_n$, for all $n \in \mathbb{N}$. This also shows that $\{u_n\}$ is a well-defined sequence.

Besides this, it is also observed that $\text{Best}^{\phi}_M(T)$ is closed and convex. This follows from Lemma 3.5, which yields $F(\Pi_M \circ T|_{M_0}) = \text{Best}^{\phi}_M(T)$. So, if we consider $\hat{u} \in M_0$ and $q \in F(\Pi_M \circ T|_{M_0})$; then $\phi(\Pi_M \circ T|_{M_0} q, \Pi_M \circ T|_{M_0} \hat{u}) \leq \phi(T|_{M_0} q, T|_{M_0} \hat{u}) \leq \phi(q,\hat{u})$. Consequently, using the arguments from

[16, Proposition 2.4], we can show that $\text{Best}^{\phi}_{M}(T)$ is closed and convex. Next, since $u_{n+1} = \Pi_{H_{n+1}} u$ and $\text{Best}^{\phi}_{M}(T) \subset H_n$, for all $n \in \mathbb{N}$; it follows by Proposition 2.3 that

$$\phi(u_{n+1}, u) \le \phi(p, u), \text{ for } p \in \text{Best}^{\phi}_{M}(T). \tag{24}$$

Thus, $\{\phi(u_n, u)\}$ is bounded and so, by the inequality $(\|u_n\| - \|u\|)^2 \le \phi(u_n, u)$; $\{u_n\}$ is bounded. Again, since $u_n = \Pi_{H_n} u$, we obtain that,

$$\phi(u_n, u) \le \phi(u_{n+1}, u). \tag{25}$$

Therefore, $\{\phi(u_n, u)\}$ is nondecreasing and so, it has a limit. From Proposition 2.3, it also follows that

$$\phi(u_{n+1}, u_n) \le \phi(u_{n+1}, u) - \phi(u_n, u), \ \forall n \in \mathbb{N}. \tag{26}$$

Thus,

$$\lim_{n\to\infty} \phi(u_{n+1}, u_n) = 0 \Rightarrow \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \text{ (by Proposition 2.1)} \tag{27}$$

From the definition of H_n, we also have, $\phi(u_{n+1}, v_n) \le \phi(u_{n+1}, u_n)$, $\forall n \in \mathbb{N}$; which results

$$\lim_{n\to\infty} \phi(u_{n+1}, v_n) = 0 \Rightarrow \lim_{n\to\infty} \|u_{n+1} - v_n\| = 0. \text{ (by Proposition 2.1)} \tag{28}$$

From (27) and (28), it follows that

$$\lim_{n\to\infty} \|u_n - v_n\| = 0. \tag{29}$$

Since, J is norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Ju_n - Jv_n\| = 0. \tag{30}$$

Let us take $r = \sup_{n\in\mathbb{N}}\{\|\Pi_N u_n\|, \|Tu_n\|\}$. We know that B^* is uniformly convex, since B is uniformly smooth; thus by Lemma 2.4, we can find a continuous, strictly increasing and convex function ψ with $\psi(0) = 0$ such that $\|\eta f + (1-\eta)g\|^2 \le \eta\|f\|^2 + (1-\eta)\|g\|^2 - \eta(1-\eta)\psi(\|f-g\|)$, for $f, g \in B^*_r$ and $\eta \in [0,1]$. Therefore, for $p \in \text{Best}^{\phi}_{M}(T)$, one has

$$\begin{aligned}
\phi(p, v_n) &= \phi(\Pi_M \circ Tp, \Pi_M J^{-1}(\alpha_n J\Pi_N u_n + (1-\alpha_n)JTu_n)) \\
&\le \phi(Tp, J^{-1}(\alpha_n J\Pi_N u_n + (1-\alpha_n)JTu_n)) \\
&\le \|Tp\|^2 - 2\alpha_n\langle Tp, J\Pi_N u_n\rangle - 2(1-\alpha_n)\langle Tp, JTu_n\rangle \\
&\quad + \alpha_n\|\Pi_N u_n\|^2 + (1-\alpha_n)\|Tu_n\|^2 - \alpha_n(1-\alpha_n)\psi(\|J\Pi_N u_n - JTu_n\|) \\
&\le \alpha_n\phi(Tp, \Pi_N u_n) + (1-\alpha_n)\phi(Tp, Tu_n) - \alpha_n(1-\alpha_n)\psi(\|J\Pi_N u_n - JTu_n\|) \\
&\le \phi(p, u_n) - \alpha_n(1-\alpha_n)\psi(\|J\Pi_N u_n - JTu_n\|). \text{ (by (23) and the nonextensiveness of } T)
\end{aligned}$$

So,

$$\begin{aligned}
\alpha_n(1-\alpha_n)\psi(\|J\Pi_N u_n - JTu_n\|) &\le \phi(p, u_n) - \phi(p, v_n) \\
&= \|u_n\|^2 - \|v_n\|^2 - 2\langle p, Ju_n - Jv_n\rangle \\
&\le \|u_n - v_n\|(\|u_n\| + \|v_n\|) + 2\|p\|\|Ju_n - Jv_n\|.
\end{aligned} \tag{31}$$

Substituting (29) and (30) in (31), we have $\alpha_n(1-\alpha_n)\psi(\|J\Pi_N u_n - JTu_n\|) \to 0$ as $n \to \infty$. Since, $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, it follows that $\lim_{n\to\infty} \psi(\|J\Pi_N u_n - JTu_n\|) = 0$. The properties of ψ yield that

$$\lim_{n\to\infty} \|J\Pi_N u_n - JTu_n\| = 0. \tag{32}$$

Since B is uniformly smooth, J^{-1} is uniformly norm-to-norm continuous on bounded sets and so we get

$$\lim_{n\to\infty} \|\Pi_N u_n - Tu_n\| = \lim_{n\to\infty} \|J^{-1}(J\Pi_N u_n - JTu_n)\| = 0. \tag{33}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\phi(u_n,Tu_n) &= \lim_{n\to\infty}\phi(u_n,\Pi_N u_n)+\phi(\Pi_N u_n,Tu_n)+2\langle u_n-\Pi_N u_n, J\Pi_N u_n-JTu_n\rangle \\ &= \lim_{n\to\infty}\phi(u_n,\Pi_N u_n) \text{ (by (32) and (33))} \\ &= \mathrm{dist}_\phi(M,N).\end{aligned}$$

This shows that $\{u_n\}$ is an (Aϕ-BPS). The strong convergence can be now obtained by the same arguments followed in Theorem 5.1. □

Remark 5.1. *Theorem 5.1 can be used to solve the strong convergence problem concerning a nonself nonexpansive mapping in a Hilbert space, which is equivalent to* [14, Theorem 3.2]. *In a Hilbert space, Theorem 5.2 is equivalent to the convergence result determined in* [5].

6. Conclusion

In conclusion, we employ the shrinking projection approach to identify the ϕ-best proximity points of a non-self nonextensive mapping in a uniformly convex and uniformly smooth Banach space. We have proved the strong convergence of the generated sequence by the proposed algorithm under the assumption that the nonextensive mapping has the ϕ-proximal property. New iterative techniques for two or more non-self nonextensive mappings in Banach spaces may be developed from this work, guiding the authors' future work.

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REFERENCES

[1] *A. Anthony Eldred and P. Veeramani*, Existence and convergence of best proximity points. *J. Math. Anal. Appl.* **323**(2006), 1001–1006.

[2] *C. Klin-eam, S. Suantai and W. Takahashi*, Strong convergence theorems by monotone hybrid method for a family of generalized nonexpansive mappings in Banach spaces. *Taiwanese J. Math.* **16**(2012), No. 6, 1971–1989.

[3] *C. Zălinescu*, On uniformly convex functions. *J. Math. Anal. Appl.* **95**(1983), No. 2, 344–374.

[4] *G. Inoue, W. Takahashi and K. Zembayashi*, Strong convergence theorems by hybrid methods for maximal monotone operator and relatively nonexpansive mappings in Banach spaces. *J. of Convex Anal.* **16**(2009), No. 3-4, 791–806.

[5] *G. K. Jacob, M. Postolache, M. Marudai and V. Raja*, Norm convergence iterations for best proximity points of non-self non-expansive mappings. *UPB Sci. Bull. Ser. A Appl. Math. Phys.* **79**(2017), 49–56.

[6] *H. K. Xu*, Inequalities in Banach spaces with applications. *Nonlinear Anal.* **16**(1991), No. 12, 1127–1138.

[7] *Jinlu Li*, The generalized projection operator on reflexive Banach spaces and its applications. *J. Math. Anal. Appl.* **306**(2005), 55–71.

[8] *J. Diestel*, Geometry of Banach space-selected topics Lect. Notes in Math., **485**, 1975.

[9] *K. Nakajo and W. Takahashi*, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. *J. Math. Anal. Appl.* **279**(2003), No. 2, 372–379.

[10] *M. A. Al-Thagafi and N. Shahzad*, Convergence and existence results for best proximity points. *Nonlinear Anal.* **70**(2009), No. 10, 3665–3671.

[11] *M. Gabeleh*, Best proximity point theorems via proximal non-self mappings. *J. Optim. Theory Appl.* **164**(2015), No. 2, 565–576.

[12] *Ravi P. Agarwal, Donal O'Regan and D. R. Sahu*, Fixed point theory for Lipschitzian-type mappings with applications. Vol. 6. New York: Springer, 2009.

[13] *R. Suparatulatorn, W. Cholamjiak and S. Suantai*, Existence and convergence theorems for global minimization of best proximity points in Hilbert spaces. *Acta Appl. Math.* **165**(2020), 81–90.

[14] *R. Suparatulatorn and S. Suantai*, A new hybrid algorithm for global minimization of best proximity points in Hilbert spaces. *Carpathian J. Math.***35**(2019), No. 1, 95–102.

[15] *S. Kamimura and W. Takahashi*, Strong convergence of a proximal-type algorithm in a Banach space. *SIAM J. Optim.* **13**(2002), 938–945.

[16] *S. Matsushita and W. Takahashi*, A strong convergence theorem for relatively nonexpansive mappings in a Banach space. *J. Approx. Theory.* **134**(2005), 257–266.

[17] *V. Barbu and Th. Pewcupanu*, Convexity and Optimization in Banach spaces. Romania International Publishers, Bucuresti, 1978.

[18] *V. Sankar Raj*, A best proximity point theorem for weakly contractive non-self-mappings, *Nonlinear Anal.* **74**(2011), 4804–4808.

[19] *W. B. Guan and W. Song*, W-approximative compactness and continuity of the generalized projection operator in Banach spaces. *J. Approx. Theory.* **62**(2010), 64–71.

[20] *W. Takahashi*, Nonlinear Functional Analysis, Yokohama Publishers 2000.

[21] *W. Takahashi, Y. Takeuchi and R. Kubota*, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* **341**(2008), No. 1, 276–286.

[22] *Ya. Alber,* Metric and generalized projection operators in Banach spaces: Properties and applications, in: A. Kartsatos (Ed.), *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, Marcel Dekker, Inc. 15–50, 1996.

[23] *Ya. Alber and S. Guerre-Delabriere*, On the projection methods for fixed point problems, *Analysis.* **21** (2001), 17–39.

[24] *Ya. Alber and S. Reich*, An iterative method for solving a class of nonlinear operator equations in Banach spaces. *Panam. Math. J.* **4** (1994), 39–54.