SOME APPLICATIONS OF MATHEMATICAL MODELING OF RISK ASSESSMENT

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Mathematical modeling has become a useful tool for study the different life problems that appear in social sciences. The paper proposes to precise mathematical models to analyze both the rate of dropout and graduation rates very important at all levels of the educational ecosystem. We analyze some characteristics of behavior of two populations that interact in a dynamical system. For the application of the model we used educational high school data. Thus, are specified demographic models with two populations (dropout/not dropout, promoted/not promoted). The logistic model of growth as a dynamic system with parameters is used for the dominant population to forecast the basic parameters (e.g., drop out rate, graduation rate). The solution is determined and analyzed by various methods: statistical, wavelet analysis. The main results obtained are analyzed, and the specific conclusions are mentioned.

Keywords: mathematical models, statistical analysis methods, dynamic systems, differential equation models, dyadic wavelet analysis.

MSC2010: 93A 30,97M 10, 62L 10, 34A 34, 37N 40, 42C 40 .

1. Introduction

Education contributes directly to the construction and consolidation of the social capital of individuals. Hence the imperative for a constant, sustained and coherent participation in education, which at the same time responds to the needs and interests of the individual and society. The education system must find the means necessary to be fair in front of all the actors it acts on, especially where the social and economic conditions are deficient. Characteristics of temporal changes of two types of populations can be studied with mathematical models, as predator -prey models and logistic growth model, [9, 10, 11].

In Mathematical Statistics the estimation of functions in the context of regression by standard methods such as the kernel approximation, the orthogonal

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series method, the spline smoothing function method, requires additional assumptions related to the smoothness of the function that will be estimated. Thus, wavelet multiresolution analysis [13, 17, 16] considerably reduce these requirements, because wavelets have the property of spatial adaptability, which allows a scientific estimation of discontinuous functions, those with discontinuities of derivatives or those with rapid or sudden variations. Moreover, in some differential problems that are corresponding to certain statistical processes two main directions have been pursued with regard to the applicability of the wavelets (see [13, 7, 18, 17, 16]): determining the solution using wavelets with compact support and fulfillment of the conditions of the problem with some restrictions. Thus, it appeared a dilemma, namely, which wavelet family from a variety is the best choice to satisfy the axioms of multiresolution analysis (see [13, 19, 6, 7, 3, 4, 5]) - a major disadvantage of the wavelet theory being the arbitrary character of their choice. For example, the Daubechies wavelets (see [13]) generate a large number of calculations, even for the one simpler differential problem. In this respect, the dyadic wavelets package (defined analytic functions, infinitely derivable and band-limited) is the most appropriate tool for studying processes located in a Fourier domain. Solving such problems the wavelet solution become accessible. Hence, taking into account the main properties of the wavelets such as good localization and resolution control have been introduced the following concepts, namely the notion of orthonormal dyadic wavelet series (see [19, 7, 6]) and representation of a function which belongs to $L^2(\mathbb{R})$ (Hilbert space) by such an expression. In this regard, the method based on the dyadic wavelet (e.g., harmonic and Shannon wavelets [19, 7, 6, 3, 4, 5]) approximation represents an effective tool used to obtain numerical solutions for some complex differential problems.

2. Modeling the regional differences of the results at the national exam

Dynamical equations modeling ecological interaction of two complementary populations are used to model typical indicators for educational problems such as graduate or school leavers. Lotka-Voltera ecological interaction model generalized by [9] leads to a logistic equation for predator population.
We shall use usual notations

- $x(t)$ =number of preys at time t, $y(t)$ = number of predators at time t, x, $y \in L^2(\mathbb{R})$ functions.
- $S(t) = x(t) + y(t)$, $z(t) = \gamma \cdot \dfrac{y(t)}{S(t)}$, $\gamma = constant.$

Chen model (2009) is expressed in the system of equations (1)

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = a\,x(t) + b\,y(t) - \Phi\,x(t)\,y(t) \tag{1a}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = c\,y(t) + d\,x(t) + \phi\,x(t)\,y(t) \tag{1b}$$

with a, b, c, d, Φ, ϕ parameters of the phenomena supposed to be constants at first step, with $0 < a, \Phi, \phi < 1$,$0 \leq b, c, d \leq 1$ and $\Phi = \phi = \phi^*$ for a closed system, understanding by this the fact that once one part of the population increases the other decreases at the same rate.

From equations (1a) and (1b) the level of the dominant population expressed by $z(t)$ verify the equation

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = (\phi^* - a) \cdot z(t) \cdot \left(1 - \frac{z(t)}{\gamma}\right), \tag{2}$$

On standard logistic growth model (LGM), [10, 11] with three parameters $\gamma \in (0, 1]$, $\alpha > 0$, $\beta > 0$, the level of the dominant population is the solution of differential equation depending on the three parameters:

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \beta \cdot z(t) \cdot \left(1 - \frac{z(t)}{\gamma}\right), \tag{3}$$

The solution $z(t)$ of (3) has the form

$$z(t) = \frac{\gamma}{1 + \alpha \mathrm{e}^{-\beta t}}, \tag{4}$$

in which the initial value $z(t_0)$, that express the uniqueness of the solution, leads to a relation between the three constants of the model. The parameter α has the significance of initial ratio of the two populations $\frac{x(t_0)}{y(t_0)}$, a location parameter, β relative growth rate and γ the upper limit of $z(t)$. The model (1) is generalized by Mulligan (2013) and Hsieh (see [15])

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = a\, x(t) - \Phi \cdot \frac{x(t)\, y(t)}{x(t) + y(t)} \tag{5a}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -d\, y(t) + \phi \cdot \frac{x(t)\, y(t)}{x(t) + y(t)}, \tag{5b}$$

with a, b, c, d constants. From (5), for a closed system with $\Phi = \phi$ and $\gamma = 1$ the equation for $z(t)$ becomes

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = (\phi - a - d) \cdot z(t) \cdot (1 - z(t)). \tag{6}$$

For any γ and not a closed closed system, (5a) and (5b) equations leads to

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = (\phi - a - d) \cdot z(t) + [(\Phi - \phi) - (\phi - a - d)] \cdot \frac{z^2(t)}{\gamma} + (\Phi - \phi) \cdot \frac{z^3(t)}{\gamma^2}. \tag{7}$$

3. Mathematical models

3.1. Logistical model

Looking for $z(t)$ as solution of type (4) for (3) , one observe that $z(t)$ verify (4) and we can estimate the parameters α, β as follows. From (4) results

$$\log\left(\frac{\gamma}{z(t)} - 1\right) = \log \alpha - \beta\, t. \tag{8}$$

For the first approximation when $\Phi = \phi$, we conclude that $\beta = \phi^* - a - d$. If considered $\frac{x}{x+y} = 1 - \frac{z(t)}{\gamma}$ then $\frac{1}{1 + \frac{y}{x}} = 1 - \frac{z(t)}{\gamma}$ or

$$\frac{y(t)}{x(t)} = \frac{\frac{z(t)}{\gamma}}{1 - \frac{z(t)}{\gamma}}, \tag{9}$$

meaning that $\dfrac{y}{x} = \dfrac{\mathrm{e}^{\beta\, t}}{\alpha}$ or

$$\log(y) - \log(x) = \beta t - \log(\alpha)\,, \tag{10}$$

α and β could be estimated from (10) by linear regression model.

With logistic solution (3) the system (5) becomes

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \left(a - \frac{\phi}{1+\alpha \mathrm{e}^{-\beta t}}\right) \cdot x(t) \tag{11a}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -d\, y(t) + \frac{\phi\, x(t)}{1+\alpha \mathrm{e}^{-\beta t}}\,, \tag{11b}$$

with solution for (11a)

$$x(t) = x(t_0) \cdot \left(\frac{\alpha + \mathrm{e}^{\beta t_0}}{\alpha + \mathrm{e}^{\beta t}}\right)^{\frac{\phi}{\beta}} \cdot \mathrm{e}^{a\,(t-t_0)},\ t > t_0\,, \tag{12}$$

consequently

$$\log\left(x(t)\right) = \log\left(x(t_0)\right) + a\,(t - t_0) + \frac{\phi}{\beta} \cdot \log\left(\frac{\alpha + \mathrm{e}^{\beta t_0}}{\alpha + \mathrm{e}^{\beta t}}\right)\,. \tag{13}$$

Using the estimations obtained above for α and β we continue to look forward for an estimation of parameters a and ϕ that will be obtained from a regression model namely: A, a and ϕ are parameters that express the function $\log\left(x(t)\right)$ as (14) with less errors for the known values $x(t_i) = x_i, y(t_i) = y_i$ and $r(t_i) = x_i/y_i$, in the sense of least squares method (MLS) The function

$$f(t \mid A, a, \phi) = A + (a - \phi)\, t - \frac{\phi}{\beta} \cdot \log\left(1 + r_y(t)\right)\,, \tag{14}$$

estimate $\cong \log\left(x(t)\right)$ after minimization the errors for the associated function

$$\mathcal{F}(t, A, a, \Phi) = \sum_{i=1}^{n} \left(A + (a - \phi)\, t - \frac{\phi}{\beta} \cdot \log\left(1 + r_y(t)\right) - f_i\right)^2\,, \tag{15}$$

that use the real data for $i \in \overline{i, n}$ and $f_i = f(t_i)$, t_i being the time values for which we have the data $x_i = x(t_i), y_i = y(t_i), r_i = r_y(t_i)$. Solving the three equations $\dfrac{\partial \mathcal{F}}{\partial A} = 0$, $\dfrac{\partial \mathcal{F}}{\partial a} = 0$ and $\dfrac{\partial \mathcal{F}}{\partial \Phi} = 0$ we obtain the system of equations for A, a and Φ

$$A \cdot A_{11} + (a - \phi) \cdot A_{12} + \phi \cdot A_{13} = B_1\,,\ B_1 = \sum_{i=1}^{n} \frac{f_i}{n} \tag{16a}$$

$$A \cdot A_{21} + (a - \phi) \cdot A_{22} + \phi A_{23} = B_2,\ B_2 = \sum_{i=1}^{n} f_i \cdot t_i \tag{16b}$$

$$A \cdot A_{31} + (a - \phi) \cdot A_{32} + \phi A_{33} = B_3\,,\ B_3 = \sum_{i=1}^{n} f_i \cdot \left(1 + \frac{1}{\beta} \cdot \log\left(1 + r_i\right)\right)\,, \tag{16c}$$

respectively

$$A_{11} = 1\,,\ A_{12} = \sum_{i=1}^{n} \frac{t_i}{n}\,,\ A_{13} = \frac{1}{\beta \cdot n} \sum_{i=1}^{n} \log\left(1 + r_i\right) \tag{17a}$$

$$A_{21} = \sum_{i=1}^{n} t_i \, , \, A_{22} = \sum_{i=1}^{n} t_i^2 \, , \, A_{23} = \frac{1}{\beta \cdot n} \sum_{i=1}^{n} t_i \cdot \log\left(1 + r_i\right) \tag{17b}$$

$$A_{31} = \sum_{i=1}^{n} \left(1 + \frac{1}{\beta} \cdot \log\left(1 + r_i\right)\right) \, , \, A_{32} = \sum_{i=1}^{n} t_i \cdot \left(1 + \frac{1}{\beta} \cdot \log\left(1 + r_i\right)\right) \, , \tag{17c}$$

$$A_{33} = \sum_{i=1}^{n} \log\left(1 + r_i\right) \cdot \left(1 + \frac{1}{\beta} \cdot \log\left(1 + r_i\right)\right) \, .$$

Also, for one type of data, the Chen model, [8], is simplified as Keyfitz model

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -a\, x(t) \, , \, a > 0 \tag{18a}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = m\, x(t) + c\, y(t) \, , \tag{18b}$$

with type of solutions

$$x(t) = K_1 \, \mathrm{e}^{-at}, \, y(t) = K_{21} \, \mathrm{e}^{-at} + K_{22} \, \mathrm{e}^{ct} \, , K_{21} = -mK_1/(a+c), c \neq -a. \tag{19a}$$

$$x(t) = K_1 \, \mathrm{e}^{-at}, \, y(t) = \left(K_{21} + K_{22} t\right) \, \mathrm{e}^{-at} \, , \quad K_{22} = mK_1, c = -a. \tag{19b}$$

3.2. Net ratio of migration estimation

The ratio $\frac{y(t)}{x(t)}$ estimate the relative growth of one kind of population upon the complementary one. Also, $r_y(t) = \frac{x(t)}{y(t)} \equiv \frac{\gamma}{z(t)} - 1$ describe the net ratio of migration of non-dominant population to the growing one. In first case when $\Delta = \Phi - \phi$ is zero with $\beta = \phi - a - d > 0$ the equation of r_y function is

$$\frac{\mathrm{d}r_y}{\mathrm{d}t} = -\beta\, r_y \, , \tag{20}$$

with solution $r_y(t) = \alpha\, \mathrm{e}^{-\beta\, t}$.

In second case, $\Delta \neq 0$, but for $\beta = 0$ then equations (7) and (20) lead to

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \frac{\Delta}{\gamma^2} \cdot z^2 \cdot (\gamma + z) \, , \, \frac{\mathrm{d}r_y}{\mathrm{d}t} = -\Delta \cdot \left(1 + \frac{1}{1 + r_y}\right) \, , \tag{21}$$

and one obtain an implicit solution

$$\log\left(1 + r_y\right) - r_y = \Delta\, t + K, K = \log\left(1 + r_y(t_0)\right) - r_y(t_0) - \Delta\, t_0 \, . \tag{22}$$

In general case, $\Delta \neq 0$, $\beta \neq 0$ one obtain the equation

$$\frac{\mathrm{d}r_y}{\mathrm{d}t} = -\beta\, r_y - (\Delta - \beta) - \Delta \cdot \frac{1}{r_y} \, . \tag{23}$$

3.3. Shannon wavelet type solution

If denoted by $Z(t) = \dfrac{z(t)}{\varphi - a - d}$ and $\delta = \varphi - a - d$, then the equation (6) can be rewritten as

$$\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = \delta Z(t) - \delta^2 Z^2(t)\,. \tag{24}$$

In the following is presented the method of obtaining the Shannon wavelet type solution of the equation (24) by using the Frobenius method (the power series form of the solution) and connection coefficients. Thus, the differential problem considered will turn into a finite-dimensional algebraic system that can solve by fixing a finite approximation scale.

The orthonormal dyadic wavelet is a function $\psi \in L^2(\mathbb{R})$ which provides the details necessary to increase the resolution of a signal approximation [16], such that the family $(D^n T_k[\psi])_{n,\,k\in\mathbb{Z}}$ constitute an orthonormal basis for $L^2(\mathbb{R})$, where D and T represent the expansion (dilatation) and translation unitary operators on $L^2(\mathbb{R})$ expressed as

$$D[\psi(t)] = \sqrt{2}\,\psi(2t)\ ,\ T[\psi(t)] = \psi(t-1)\,. \tag{25}$$

Based on the multiresolution analysis [13, 16, 17], the next result is immediate.

Lemma 3.1 (Translation invariance)**.** *For* $(\forall)\, n,\, s \in \mathbb{Z}$ *(dyadic values) and* $\psi \in L^2(\mathbb{R})$*, then*

$$D^n T_k[\psi(t)] = T_{\frac{k}{2^n}} D^n[\psi(t)] = \psi_{n\,k}(t)\,, \tag{26}$$

where $D^n[\psi(t)] = 2^{\frac{n}{2}}\psi\left(2^n t\right)$*,* $T_k[\psi(t)] = \psi\left(t-k\right)$ *and the wavelet function* $\psi_{n\,k}(t) = 2^n\psi\left(2^n t - k\right)$ *which depends on the pair* $(2^{-n}, 2^{-n}k)$ *(scaling and translation parameters).*

Proof. Taking into account (25), it immediately follows that

$$\begin{aligned}\mathrm{D}^n\mathrm{T}_k\left[\psi(t)\right] &= 2^{\frac{n}{2}}\,\mathrm{T}_k\left[\psi(t)\right] = 2^{\frac{n}{2}}\psi\left[2^n\left(t-\frac{k}{2^n}\right)\right]\\ &= \mathrm{D}^n\left[\psi\left(t-\frac{k}{2^n}\right)\right] = \mathrm{T}_{\frac{k}{2^n}}\mathrm{D}^n\left[\psi(t)\right]\,.\end{aligned}$$

□

The Shannon scaling and wavelet functions $\varphi^{\texttt{SW}}(t)$, $\psi^{\texttt{SW}}(t)$ are both real functions with bonded spectrum defined by the following relations (see [6, 7])

$$\begin{aligned}\varphi^{\texttt{SW}},\psi^{\texttt{SW}}:\mathbb{R}\to\mathbb{R}\quad,\quad &\varphi^{\texttt{SW}}(t) = \mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}\\ &\psi^{\texttt{SW}}(t) = 2\varphi^{\texttt{SW}}(2t-1) - \varphi^{\texttt{SW}}\left(t-\frac{1}{2}\right)\,.\end{aligned} \tag{27}$$

In according to (26) is obtained a family of dilated and translated instances of the scaling, respectively wavelet Shannon functions

$$\begin{aligned}\varphi^{\text{SW}}_{n\,k}(t) &= \sqrt[n]{2}\varphi^{\text{SW}}\left(2^n t-k\right)\\ \psi^{\text{SW}}_{n\,s}(t) &= \sqrt[n]{2}\psi^{\text{SW}}(2^n t-k)\\ &= \sqrt[n]{2}\left[2\varphi^{\text{SW}}_{n\,k}\left(2^{n+1}t-2k-1\right)-\varphi^{\text{SW}}_{n\,k}\left(2^n-k-\frac{1}{2}\right)\right] \quad, \ \forall n,\, k\in\mathbb{Z}\,,\end{aligned} \tag{28}$$

and for $n=k=0$ results that $\varphi^{\text{SW}}_{0\,0}(t)=\varphi^{\text{SW}}(t)$, respectively $\psi^{\text{SW}}_{0\,0}(t)=\psi^{\text{SW}}(t)$. Also, it should be noted that the Shannon wavelets [6] are a particular case of harmonic wavelets (see [19, 7]), because they derived from them. The main characteristic of the Shannon wavelet (the **sinc** function) is that its Fourier transform is approximately constant over an interval around the origin and null at the rest, i.e.,

$$\begin{aligned}\widehat{\varphi}^{\text{SW}}(\omega) &= \aleph\left(\omega+3\pi\right)\\ \widehat{\psi}^{\text{SW}}(\omega) &= \mathrm{e}^{-\frac{i\omega}{2}}\left[\aleph\left(\frac{\omega}{2}+3\pi\right)-\aleph(\omega+3\pi)\right]\,.\end{aligned} \tag{29}$$

where the indicator or characteristic function $\aleph(\omega)$ is defined as

$$\aleph\left(\omega\right)=\begin{cases}1, & \text{if} \quad 2\pi\le\omega\le 4\pi\\ 0, & \quad\text{otherwise}\,.\end{cases} \tag{30}$$

In this respect, it shows that the Shannon wavelet fulfills the necessary condition to be a wavelet, namely, the admissibility condition with the Calderon's admissibility constant $\mathbf{C}_{\psi^{\text{SW}}}=\ln(4)$ and its energy $\mathbf{E}_{\psi^{\text{SW}}}=1$, $\widehat{\psi}^{\text{SW}}(0)=0$. Also, according to multiresolution analysis it is easily to prove that the followings conditions are satisfied $\int_{\mathbb{R}}\psi^{\text{SW}}(t)\mathrm{d}t=0$ and $\int_{\mathbb{R}}\varphi^{\text{SW}}(t)\mathrm{d}t=1$, where the Cauchy principal value of the the doubly infinite improper integral of a function $f(t)$ is defined by $PV\left(\int_{\mathbb{R}}f(t)\mathrm{d}t\right)=\lim_{R\to\infty}\left(\int_{-R}^{R}f(t)\mathrm{d}t\right)$.

Taking into account the Plâncherel-Parceval theorem for $f,g\in L^2(\mathbb{R})$

$$\langle f,\,g\rangle\,(t)=\int_{\mathbb{R}}f(t)\cdot\overline{g(t)}\mathrm{d}t=\frac{1}{2\pi}\left\langle\widehat{f},\,\widehat{g}\right\rangle(\omega)\,, \tag{31}$$

the Shannon scaling and wavelet functions satisfy

$$\begin{cases}\left\langle\varphi^{\text{SW}},\varphi^{\text{SW}}\right\rangle=\left\langle\psi^{\text{SW}},\psi^{\text{SW}}\right\rangle=1\\ \left\langle\varphi^{\text{SW}},\psi^{\text{SW}}\right\rangle=\left\langle\psi^{\text{SW}},\varphi^{\text{SW}}\right\rangle=0\,.\end{cases} \tag{32}$$

In the Fourier domain, it takes place

$$\mathcal{F}\left[f^{(s)}(t)\right]=(i\omega)^s\,\widehat{f}(\omega)\,. \tag{33}$$

In addition, in order to obtain some results, it is necessary to use frequency convolution

$$\widehat{f(t)\cdot g(t)}=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\omega-\tau)\cdot\widehat{g}(\tau)\mathrm{d}\tau\,. \tag{34}$$

Let us consider the simplest case of the Shannon wavelet type solution for the equation (24), namely

$$Z^{\text{SW}}(t) = \alpha^{\text{SW}}\varphi^{\text{SW}}(t) + \beta^{\text{SW}}\psi^{\text{SW}}(t)\,, \tag{35}$$

and it represents the wavelet approximation of level zero (lower form) of the solution, where the constants α^{SW} and β^{SW} are the Shannon scaling, respectively wavelet coefficients. In this respect, equation (24) becomes

$$\begin{aligned}\alpha^{\text{SW}}\frac{\mathrm{d}\varphi^{\text{SW}}(t)}{\mathrm{d}t} + \beta^{\text{SW}}\frac{\mathrm{d}\psi^{\text{SW}}(t)}{\mathrm{d}t} &= \delta\left(\alpha^{\text{SW}}\varphi^{\text{SW}}(t) + \beta^{\text{SW}}\psi^{\text{SW}}(t)\right) - \\ &- \delta^2\left(\alpha^{\text{SW}\,2}\varphi^{\text{SW}\,2}(t) + 2\alpha^{\text{SW}}\beta^{\text{SW}}\varphi^{\text{SW}}(t)\psi^{\text{SW}}(t) + \beta^{\text{SW}\,2}\psi^{\text{SW}\,2}(t)\right)\end{aligned} \tag{36}$$

By inner product with functions $\varphi^{\text{SW}}(t)$ and $\psi^{\text{SW}}(t)$ and according to relations (29) - (34), the Shannon coefficients α^{SW} and β^{SW} are determined by solving the finite-dimensional algebraic system of nonlinear equations (see [3, 4])

$$\begin{cases} \dfrac{3}{4}\alpha^{\text{SW}\,2} + \dfrac{4-2\pi}{\pi^2}\alpha^{\text{SW}}\beta^{\text{SW}} + \dfrac{4-2\pi}{\pi^2}\beta^{\text{SW}\,2} - \dfrac{1}{\delta}\alpha^{\text{SW}} = 0 \\[2ex] \dfrac{2-\pi}{\pi^2}\alpha^{\text{SW}\,2} + \dfrac{8}{\pi^2}\alpha^{\text{SW}}\beta^{\text{SW}} - \dfrac{1}{2}\beta^{\text{SW}\,2} - \dfrac{1}{\delta}\beta^{\text{SW}} = 0\,. \end{cases} \tag{37}$$

Finally, Shannon coefficients determined from (37), respectively the Shannon wavelet solution of (6)

$$z^{\text{SW}}(t) = \delta \cdot Z^{\text{SW}}(t) \tag{38}$$

are depend by the values of the parameter δ; for e.g., if $\delta = 1$ the nonzero solution of the system (37) is $\alpha^{\text{SW}} = 11.6226849440416$, $\beta^{\text{SW}} = 14.7188362529956$ and the solution (38) has the graph

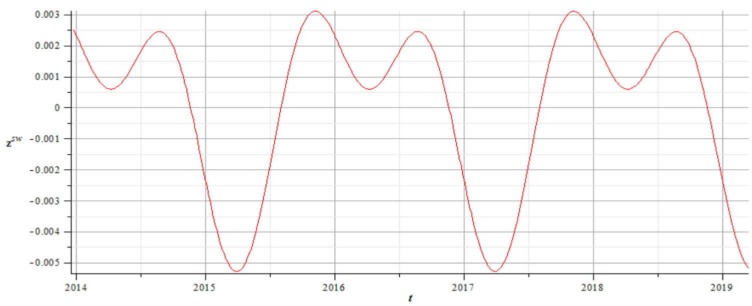


FIGURE 1. Shannon wavelet solution $z^{\text{SW}}(t)$ for $\delta = 1$.

4. Parameters estimations and analysis

We shall use mathematical models precised in order to make an overview of the risk of school drop out and of the decreasing process of the percentage of the rejected candidates of the baccalaureate exam. Public data from Ministry of National Education are computed and primary values are precised in `Table 1` and `Table 2`.

No.	Type of candidates	2014	2015	2016	2017	2018	2019
1.	Present Candidates	147837	158934	126222	124516	128521	107927
2.	Passed Candidates	90433	106817	86964	91357	88356	81159
3.	Rejected Candidates, average mark 5-5.49	2879	2396	2113	1541	1984	1055
4.	Rejected Candidates, average mark 5,5-5.98	6748	6346	5342	4254	5207	3169
5.	Candidates with average mark 6-6,49	12938	12836	10159	9323	10824	6236
6.	Candidates with average mark 6,5-6,99	12202	12978	10545	9375	10373	8118
7.	Candidates with average mark 7-7,49	12474	13835	10903	10763	11118	9934
8.	Candidates with average mark 7,5-7,99	12314	14577	11450	12016	11435	10848
9.	Candidates with average mark 8-8,49	12737	15185	12291	13569	12577	12201
10.	Candidates with average mark 8,5-8,99	12116	15677	12548	14229	12635	13138
11.	Candidates with average mark 9-9,49	10479	14684	12537	14257	12708	13109
12.	Candidates with average mark 9,5-9,99	5065	6964	6454	7699	6556	7386
13.	Candidates with average mark 10	108	81	77	126	130	189

TABLE 1. Urban unit of high school education structure on the baccalaureate exam over 2014-2019 years

No	Type of candidates	2014	2015	2016	2017	2018	2019
1.	Present Candidates	9289	9629	6820	6216	6239	4468
2.	Passed Candidates	3001	3741	2728	2876	2553	2040
3.	Rejected Candidates, average mark 5-5.49	252	237	150	113	122	73
4.	Rejected Candidates, average mark 5,5-5.98	470	473	352	260	312	163
5.	Candidates with average mark 6-6,49	745	781	570	560	573	311
6.	Candidates with average mark 6,5-6,99	559	715	502	517	456	303
7.	Candidates with average mark 7-7,49	482	631	436	458	443	386
8.	Candidates with average mark 7,5-7,99	438	531	373	387	336	321
9.	Candidates with average mark 8-8,49	329	426	308	377	319	287
10.	Candidates with average mark 8,5-8,99	260	365	263	316	208	225
11.	Candidates with average mark 9-9,49	146	215	198	200	170	151
12.	Candidates with average mark 9,5-9,99	42	77	77	61	46	56
13.	Candidates with average mark 10	0	0	1	0	2	0

TABLE 2. Rural unit of high school education structure on the baccalaureate exam over 2014-2019 years

We aim to observe also the regional differences of development of education and find levers to reduce these inequalities.

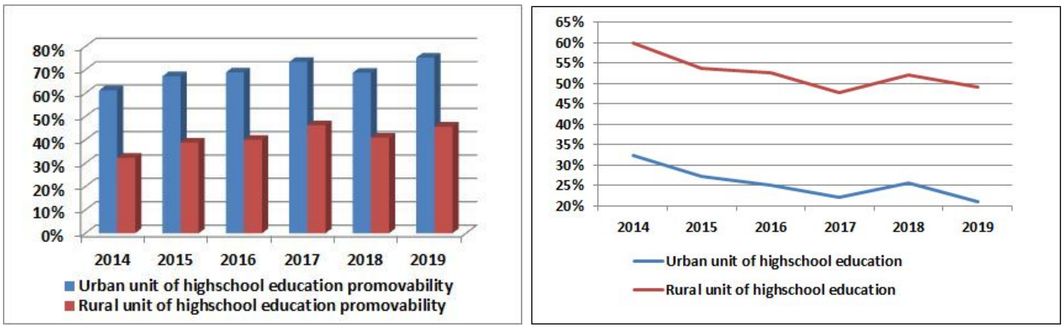
If we denote with $x(t)$ values from Rural unit of high school education table and $y(t)$ values from Urban unit of high school education table for the same characteristic. We compute some public data from Ministry of National Education and primary values are depicted from `Figure` 2 to `Figure` 5.

In `Figure` 6 is represented the function $r_y(t) = \dfrac{x(t)}{y(t)}$ for some of the variables, for those for which that the model is representative. For these variables, the parameters α and β are estimated and precised in `Table 3`.

No	Type of candidates	α	β	R^2
1.	Present Candidates	0.0695	0.081	0.9639
2.	Passed Candidates	0.037	0.056	0.8071
3.	Rejected Candidates with average mark 5,5-5.98	0.0793	0.064	0.85
4.	Candidates with average mark 7,5-7,99	0.0381	0.045	0.8828

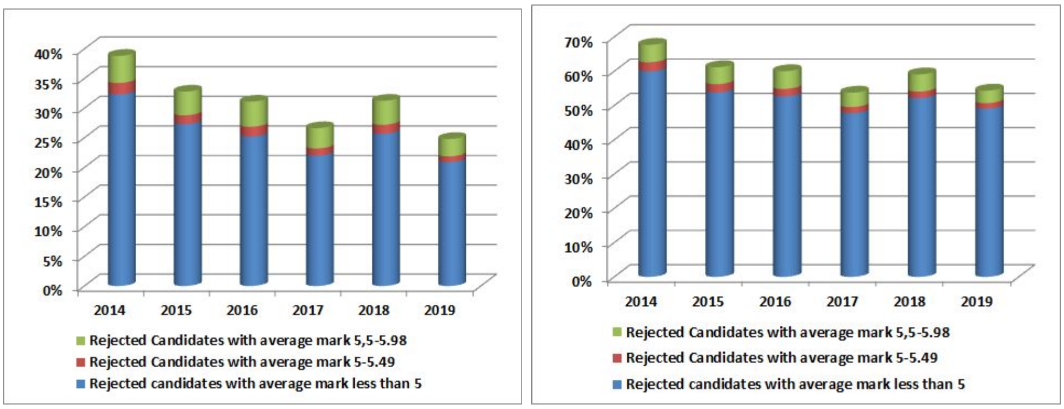
TABLE 3. Estimated Parameters α and β for $r_y(t)$ function over 2014-2019 years

Further, using estimated parameters α and β and relations (16)-(17) one obtain the parameters a, Φ and d as is presented in `Table 4`. A carefully study of



(A) graduation candidates (B) rejectes candidates

FIGURE 2. (**A**) Graduation. (**B**) Rejected candidates with average mark less than 5.



(A) urban unit (B) rural unit

FIGURE 3. Structure of rejected candidates in (**a**) Urban unit of highschool education. (**b**) Rural unit of high school education.

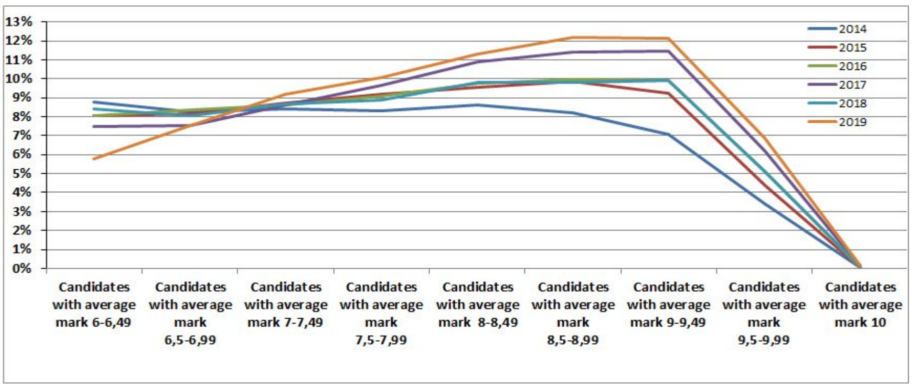


FIGURE 4. Structure of graduated candidates in Urban unit of high school education.

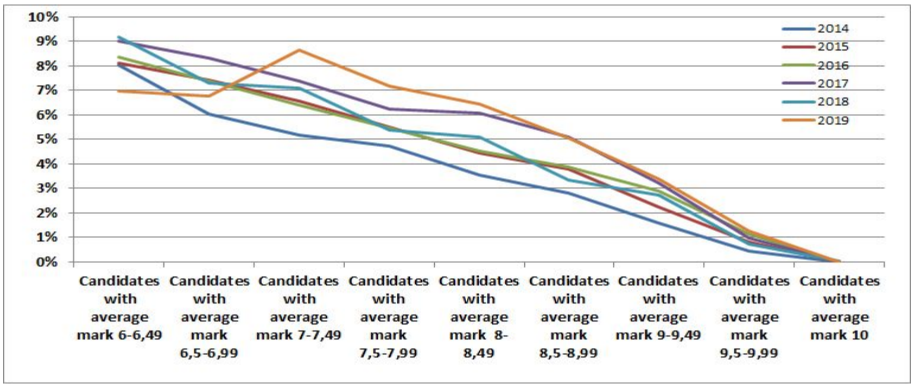


FIGURE 5. Structure of graduated candidates in Rural unit of high school education

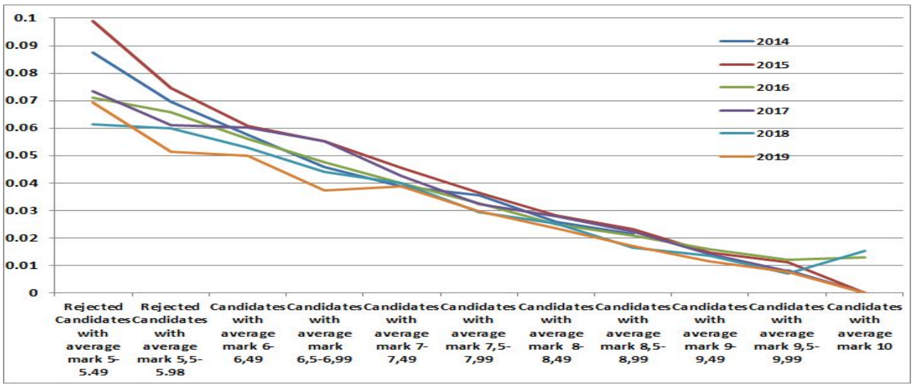


FIGURE 6. Function $r_y(t) = x(t)/y(t)$ over 2014-2019 years

No	Type of candidates	a	ϕ	d	A
1.	Present Candidates	-5.7514	-5.8983	–0.2279	-291.1932
2.	Passed Candidates	-4.30797	-4.34745	-0.0956	-74.3445
3.	Rejected Candidates, average mark 5,5-5.98	-2.7307	-2.6931	-0.0264	78.9086
4.	Candidates with average mark 7,5-7,99	-4.1096	-4.1584	-0.0938	-95.4469

TABLE 4. Estimated Parameters a, $\phi^* = \Phi = \phi$, d and A for data values over 2014-2019 years

Rural/Urban high school education number of candidates over the range data leads, also, that *Rejected candidates with average mark less than 5* could be applied the Keyfitz model. One obtain $K_1 = 6728.5$ and $a = 0.178$ values, with accuracy $R^2 = 0.9048$.

Also for prognosis for 2020 year according with parameters determined in `Table 4`, we obtain for graduation values 45.47% for rural unit candidates and 71.7063% as can be computed for the `Table 5`. The Shannon wavelet solution (38) represents a method to approximation the solution of equation (6) and interpretation of it. Based on the estimate of the parameter δ (calculated for the ratio r_y rural/urban) for 2014-2019 are determined the corresponding values of the Shannon

No	Type of candidates	Rural candidates	Urban candidates
1.	Present Candidates	4513	114472
2.	Passed Candidates	2052	82083
3.	Rejected Candidates with average mark 5,5-5.98	166	3272
4.	Candidates with average mark 7,5-7,99	-302	10853

TABLE 5. Prognosis for 2020 year

coefficients (see `Table 6` and the graph of the Shannon wavelet solution for each case/instance (see `Figure 7`). It should be noted that the analysis was achieved for some type of candidates, the other values being unrepresentative. Also, from `Figure 1` and `Figure 7` it observes a trend smoothing of the graph for certain values of the δ. In `Figure 8` is illustrated the Shannon ratio function $r_y^{SW}(t)$ calculated by formula $r_y^{SW}(t) = \dfrac{\gamma}{z^{SW}(t)} - 1$, for $\gamma = 1$ and different instances of the Shannon wavelet solution (38) depending on the Shannon coefficients values (37). Moreover, based on the study related to presence/graduation, by comparison, it can be concluded that the net ratio of migration (rural/ urban) has significant values for PrC (Present Candidates), PaC (Passed Candidates), RC (Rejected Candidates with average mark 5.5-5.98) in the period 2014-2017 and close or similar values in the period 2018-2019, the trend being to minimize the RC value by assimilation in one of the categories C_{6-7} (Candidates with average mark 6.5-6.99) or C_{7-8} (Candidates with average mark 7.5-7.99), at least, which confirms the results obtained by using the statistical model.

No	Type of candidates	Parameter δ	Value of α^{SW}	Value of β^{SW}
1.	Present Candidates	0.081	143.4899389	181.7140286
2.	Passed Candidates	0.056	207.5479474	262.8363629
3.	Rejected Candidates with average mark 5.5-5.98	0.064	181.6044539	229.9818175
4.	Candidates with average mark 6.5-6.99	0.044	264.1519330	334.5190074
5.	Candidates with average mark 7.5-7.99	0.045	258.2818900	327.0852516

TABLE 6. Values of Shannon coefficients based on the estimation of parameter δ

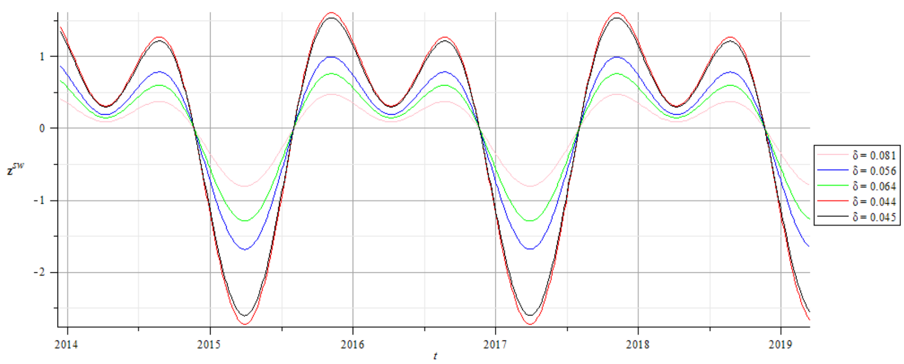
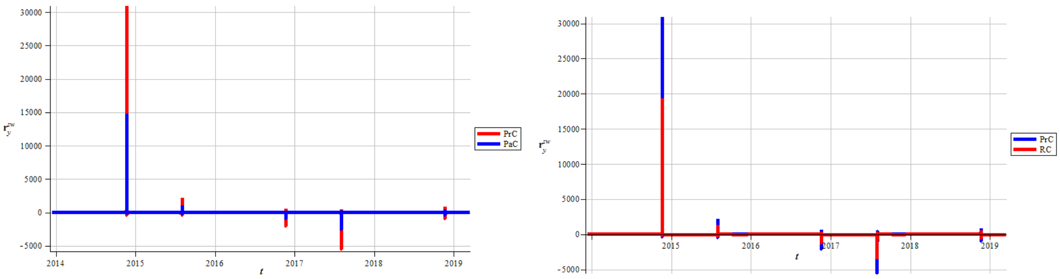


FIGURE 7. Shannon wavelet solution $z^{SW}(t)$ for significant values of δ.



(A) PrC & PaC (B) PrC & RC

FIGURE 8. The net ratio of migration $r_y^{SW}(t)$ compared between the significant instances as following (**a**)Present Candidates (PrC) and Passed Candidates (PaC). (**b**) Present Candidates (PrC) and Rejected Candidates (RC) with average mark 5.5-5.98.

5. **Conclusions**

It was followed the mathematical modeling of two types of populations in connection with wide applicability in the analysis of the evolution of educational management factors.

Different independent models have been customized and certain parameters that describe the model have been determined. Based on these parameters, the forecast values for the following year were specified, observing the increase of inequality in rural and urban education. With increased attention on the category of students with grades between 5 and 6 to raise their level, this inequality can be alleviated.

The study was made at national level, as a closed system ($\Phi = \phi$). The same study can be performed regionally considering also the area from which the candidate comes from and the high school graduated for rural / urban cases.

Further studies may follow directions related to the function of utility, [12], or a report of the school results against the poverty of the population, [14], as well as a territorial study using a spatial distribution of the high schools and an analysis of the results of the candidates on this distribution, [20]. A bifurcation analysis as in [2, 1] could also be made. From the perspective of multiresolution analysis, the wavelet representation involves the decomposition of a signal (function) into subsignals and each level of resolution represents a more refined or coarse version of the original signal. The wavelet representations of functions are more accurate and require less computation time, because offer the possibility to reconstruction a function with a relatively small number of terms, saving storage space and time, while increasing the accuracy of the reconstruction.

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