WEIGHTED SEMIGROUP MEASURE ALGEBRA AS A WAP-ALGEBRA

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A Banach algebra $\mathfrak{A}$ for which the natural embedding from $\mathfrak{A}$ into $WAP(\mathfrak{A})^*$ is bounded below is called a WAP-algebra. We study those conditions under which the weighted semigroup measure algebra $M_b(S,\omega)$ is a WAP-algebra or a dual Banach algebra. In particular, we show that the semigroup measure algebra $M_b(S)$ is a WAP-algebra (resp. dual Banach algebra) if and only if $wap(S)$ separates the points of S (resp. S is compactly cancellative semigroup). Some older results, in the case where S is discrete, are also improved.

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1. Introduction and Preliminaries

The dual $\mathfrak{A}^*$ of a Banach algebra $\mathfrak{A}$ can be turned into a Banach $\mathfrak{A}$-module equipped with the natural module operations

$$\langle f\cdot a, b\rangle = \langle f, ab\rangle \quad \text{and} \quad \langle a\cdot f, b\rangle = \langle f, ba\rangle \quad (a,b\in\mathfrak{A}, f\in\mathfrak{A}^*).$$

A dual Banach algebra is a Banach algebra $\mathfrak{A}$ enjoying a predual $\mathfrak{A}_*$ such that $\mathfrak{A}_*$, as a Banach space is a closed $\mathfrak{A}$-submodule of $\mathfrak{A}^*$; or equivalently, the multiplication on $\mathfrak{A}$ is separately weak*-continuous. It should be remarked that the predual of a dual Banach algebra need not be unique, in general (see [5, 10]); so we usually point to the involved predual of a dual Banach algebra.

A functional $f\in\mathfrak{A}^*$ is said to be weakly almost periodic if $\{f\cdot a : \|a\|\le 1\}$ is relatively weakly compact in $\mathfrak{A}^*$. We denote by $WAP(\mathfrak{A})$ the set of all weakly

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almost periodic elements of $\mathfrak{A}^*$. It is easy to verify that, $WAP(\mathfrak{A})$ is a (norm) closed subspace of $\mathfrak{A}^*$.

It is known that the multiplication of a Banach algebra $\mathfrak{A}$ has two natural but, in general, different extensions (called Arens products) to the second dual $\mathfrak{A}^{**}$ each turning $\mathfrak{A}^{**}$ into a Banach algebra. When these products are equal, $\mathfrak{A}$ is said to be (Arens) regular. It can be verified that $\mathfrak{A}$ is Arens regular if and only if $WAP(\mathfrak{A}) = \mathfrak{A}^*$. Further information for the Arens regularity of Banach algebras can be found in [5, 6].

WAP-algebras, as a generalization of the Arens regular algebras, have been introduced and intensively studied in [9]. A Banach algebra $\mathfrak{A}$ for which the natural embedding $x \mapsto \hat{x}$ of $\mathfrak{A}$ into $WAP(\mathfrak{A})^*$, where $\hat{x}(f) = f(x)$ for $f \in WAP(\mathfrak{A})$, is bounded below, is called a WAP-algebra. When $\mathfrak{A}$ is either Arens regular or a dual Banach algebra, then natural embedding of $\mathfrak{A}$ into $WAP(\mathfrak{A})^*$ is an isometry [16, Corollary 4.6]. It has also known that $\mathfrak{A}$ is a WAP-algebra if and only if it admits an isomorphic representation on a reflexive Banach space. Convolution group algebras are the main examples of WAP-algebras; however; they are neither dual nor Arens regular in general, see [17]. For more information about WAP-algebras one may consult to the impressive paper [9].

The main aim of this paper is to investigate those conditions under which the weighted measure algebra $M_b(S,\omega)$ is either a WAP-algebra or a dual Banach algebra, where ω is a weight on a locally compact semigroup S.

First we recall some preliminaries about the (weighted) measure algebras. Let S be a locally compact semitopological semigroup. Let $M_b(S)$ be the space of all complex regular Borel measures on S, which is known as a Banach algebra under the convolution product $*$ defined by the equation $\langle \mu * \nu, f\rangle = \int_S \int_S f(xy) d\mu(x) d\nu(y) \quad (f \in C_0(S))$. Our mean by a weight ω on S is a Borel measurable function $\omega : S \to (0,\infty)$ such that $\omega(st) \leq \omega(s)\omega(t), \quad (s,t \in S)$. For $\mu \in M_b(S)$ we define $(\mu\omega)(E) = \int_E \omega d\mu$, ($E \subseteq S$ is Borel set). If $\omega \geq 1$, then $M_b(S,\omega) = \{\mu \in M_b(S) : \mu\omega \in M_b(S)\}$ is known as a Banach algebra which is called the weighted semigroup measure algebra (see [6, 12, 13, 14]). In the case where S is discrete we write $\ell_1(S,\omega)$ instead of $M_b(S,\omega)$ and $c_0(S,1/\omega)$ instead of $C_0(S,1/\omega)$. Then the Banach algebra $\ell_1(S,\omega) = \{f : f = \sum_{s\in S} f(s)\delta_s, \quad ||f||_{1,\omega} = \sum_{s\in S} |f(s)|\omega(s) < \infty\}$ (where, $\delta_s \in \ell_1(S,\omega)$ is the point mass at s) equipped with the convolution product is called a weighted semigroup algebra. We also suppress 1 from the notation whenever $w = 1$.

Let $B(S)$ denote the space of all bounded Borel measurable functions on S. Set $B(S,1/\omega) = \{f : S \to \mathbb{C} : f/\omega \in B(S)\}$. Let $f \in C(S,1/\omega)$ then f is called ω-weakly almost periodic if the set $\{\frac{R_s f}{\omega(s)\omega} : s \in S\}$ is relatively weakly compact in $C(S)$. The set of all ω-weakly almost periodic functions on S is denoted by

$wap(S, 1/\omega)$. The space $wap(S)$ of 1-weakly almost periodic functions on S is a C^*-subalgebra of $C(S)$ and its character space S^{wap}, endowed with the Gelfand topology, enjoys a (Arens type) multiplication that turns it into a compact semi-topological semigroup. Many other properties of $wap(S)$ and its inclusion relations among other function algebras are completely explored in [3].

The paper is organized as follows. In section 2 we study the weighted measure algebra $M_b(S, \omega)$ from the dual Banach algebra point of view. In this respect, we shall show that, $M_b(S, \omega)$is a dual Banach algebra with respect to the predual $C_0(S, 1/\omega)$ if and only if for all compact subsets F and K of S , the maps $\frac{\chi_{F^{-1}K}}{\omega}$ and $\frac{\chi_{KF^{-1}}}{\omega}$ vanishes at infinity. This extends an earliear result of Abolghasemi, Rejali, and Ebrahimi Vishki [1]. We also conclude that, the measure algebra $M_b(S)$ is a dual Banach algebra with respect to $C_0(S)$ if and only if S is a compactly cancellative semigroup. The later result is an extension of a known result of Dales, Lau and Strauss [7, Theorem 4.6] stating that, $\ell_1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if S is a weakly cancellative semigroup.

Section 3 is devoted to the study of $M_b(S, \omega)$ from the WAP-algebra point of view. We shall prove that, $M_b(S, \omega)$ is a WAP-algebra if and only if the evaluation map $\epsilon : S \longrightarrow \tilde{X}$ is one to one, where $\tilde{X} = MM(wap(S, 1/\omega))$. The main result of this section is that $M_b(S)$ is WAP-algebra if and only if $wap(S)$ separate the points of S. We conclude the paper with some illuminating examples.

2. **Semigroup Measure Algebras as Dual Banach Algebras**

It is known that the (discrete) semigroup algebra $\ell_1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if S is a weakly cancellative semigroup, see [7, Theorem 4.6].This result has been extended to the weighted semigroup algebras; [1, 8]. In this section we extend the aforementioned results to the non-discrete case. More precisely, we provide some necessary and sufficient conditions that the measure algebra $M_b(S, \omega)$ becomes a dual Banach algebra with respect to the predual $C_0(S, 1/\omega)$.

Let F and K be nonempty subsets of a semigroup S and $s \in S$. We set $s^{-1}F = \{t \in S : st \in F\}$, and $Fs^{-1} = \{t \in S : ts \in F\}$. We also write $s^{-1}t$ for $s^{-1}\{t\}$, FK^{-1} for $\cup_{s\in K} Fs^{-1}$ and $K^{-1}F$ for $\cup_{s\in K} s^{-1}F$.

A semigroup S is called left (respectively, right) zero semigroup if $xy = x$ (respectively, $xy = y$), for all $x, y \in S$. A semigroup S is called zero semigroup if there exist $z \in S$ such that $xy = z$ for all $x, y \in S$. A semigroup S is said to be left (respectively, right) weakly cancellative semigroup if $s^{-1}F$ (respectively, Fs^{-1}) is finite for each $s \in S$ and each finite subset F of S. A semigroup S is said to be weakly cancellative semigroup if it is both left and right weakly cancellative semigroup.

A semi-topological semigroup S is said to be compactly cancellative semigroup if for every compact subsets F and K of S the sets $F^{-1}K$ and KF^{-1} are compact set.

The following lemma needs a routine argument.

Lemma 2.1. Let S be a topological semigroup. For every compact subsets F and K of S the sets $F^{-1}K$ and KF^{-1} are closed.

In the next result we study $M_b(S,\omega)$ from the dual Banach algebra point of view.

Theorem 2.1. Let S be a locally compact topological semigroup and ω be a continuous weight on S. Then the measure algebra $M_b(S,\omega)$ is a dual Banach algebra with respect to the predual $C_0(S,1/\omega)$ if and only if for all compact subsets F and K of S , the maps $\frac{\chi_{F^{-1}K}}{\omega}$ and $\frac{\chi_{KF^{-1}}}{\omega}$ vanishes at infinity.

Proof. Suppose that $M_b(S,\omega)$ is a dual Banach algebra with respect to $C_0(S,1/\omega)$ and let $\varepsilon > 0$. Let K, F be nonempty compact subsets of S with a net (x_α) in $\{t \in F^{-1}K : 1/\omega(t) \geq \varepsilon\}$. Let $C_{00}^{+}(S)$ denote the non-negative continuous functions with compact support on S and set $C_{00}^{+}(S,1/\omega) = \{f \in C_0(S,1/\omega) : f/\omega \in C_{00}^{+}(S)\}$. Since ω is continuous we may choose $f \in C_{00}^{+}(S,1/\omega)$ with $f(K) = 1$. There exists a net $(t_\alpha) \in F$ such that $t_\alpha x_\alpha \in K$ and the compactness of F guaranties the existence of a subnet (t_γ) of (t_α) such that $t_\gamma \to t_0$ for some t_0 in S. Indeed, since for each $s \in S$,

$$\lim_\gamma (\frac{\delta_{t_\gamma}.f}{\omega})(s) = \lim_\gamma \frac{f(t_\gamma s)}{\omega(s)} = \frac{f(t_0 s)}{\omega(s)} = \frac{\delta_{t_0}.f}{\omega}(s),$$

there exists a γ_0 such that

$$\begin{aligned} \{t \in \cup_{\gamma\geq\gamma_0} t_\gamma^{-1}K : 1/\omega(t) \geq \varepsilon\} &\subseteq \cup_{\gamma\geq\gamma_0}\{r \in S : (\frac{\delta_{t_\gamma}.f}{\omega})(r) \geq \varepsilon\} \\ &\subseteq \{r \in S : (\frac{\delta_{t_0}.f}{\omega})(r) \geq \frac{\varepsilon}{2}\}. \end{aligned}$$

Let $H = \{t_\gamma : \gamma \geq \gamma_0\} \cup \{t_0\}$. Then

$$\{t \in H^{-1}K : 1/\omega(t) \geq \varepsilon\} = \{t \in \cup_{\gamma\geq\gamma_0} t_\gamma^{-1}K \cup t_0^{-1}K : 1/\omega(t) \geq \varepsilon\}$$

as a closed subset of $\{r \in S : (\frac{\delta_{t_0}.f}{\omega})(r) \geq \frac{\varepsilon}{2}\}$ is compact. It follows that the net (x_γ) in $\{t \in H^{-1}K : 1/\omega(t) \geq \varepsilon\}$ has a convergent subnet. Thus $\{t \in F^{-1}K : 1/\omega(t) \geq \varepsilon\}$ is compact and that $\frac{\chi_{F^{-1}K}}{\omega}$ vanishes at infinity. Similarly $\frac{\chi_{KF^{-1}}}{\omega}$ vanishes at infinity.

The proof of sufficiency can be adopted from [1, Proposition 3.1]. Let $f \in C_0(S,1/\omega)$, $\mu \in M_b(S,\omega)$ and $\varepsilon > 0$ be arbitrary. Then there exist compact subsets F and K of S such that $|\frac{f}{\omega}(s)| < \varepsilon$ for all $s \notin K$ and $|(\mu\omega)|(S \setminus F) < \varepsilon$. Let

$s \notin \{t \in F^{-1}K : \omega(t) \leq \frac{1}{\varepsilon}\}$. Then

$$\begin{aligned} |\frac{\mu.f}{\omega}(s)| &= |\int_S \frac{f(ts)}{\omega(s)} d\mu(t)| \leq |\int_F \frac{f(ts)}{\omega(s)} d\mu(t)| + |\int_{S\setminus F} \frac{f(ts)}{\omega(s)} d\mu(t)| \\ &\leq \int_F |\frac{f(ts)}{\omega(ts)}| \omega(t) d|\mu|(t) + \int_{S\setminus F} |\frac{f(ts)}{\omega(ts)}| \omega(t) d|\mu|(t) \\ &\leq \varepsilon \int_S \omega(t) d|\mu|(t) + \|f\|_{\omega,\infty} \int_{S\setminus F} \omega(t) d|\mu|(t) \leq \varepsilon \|\mu\|_\omega + \varepsilon \|f\|_{\omega,\infty}. \end{aligned}$$

That is, $\mu.f \in C_0(S, 1/\omega)$ and so $M_b(S,\omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$. □

As immediate consequences of Theorem 2.1 we have the next corollary.

Corollary 2.1. Let S be a locally compact topological semigroup.

(1) The measure algebra $M_b(S)$ is a dual Banach algebra with respect to $C_0(S)$ if and only if S is compactly cancellative.
(2) If $M_b(S)$ is a dual Banach algebra with respect to $C_0(S)$ then $M_b(S,\omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$.

Applying Theorem 2.1 for a discrete semigroup, we arrive at the next result.

Corollary 2.2 ([1, Theorem 2.2])**.** For a semigroup S, the weighted semigroup algebra $\ell_1(S,\omega)$ is a dual Banach algebra with respect to the predual $c_0(S, 1/\omega)$ if and only if the maps $\frac{\chi_{t^{-1}s}}{\omega}$ and $\frac{\chi_{st^{-1}}}{\omega}$ are in $c_0(S)$ for all $s, t \in S$.

We have also the next result as an application of Theorem 2.1.

Corollary 2.3. Let S be either a left zero, a right zero or a zero locally compact semigroup. Then there exists a weight ω on S such that $M_b(S,\omega)$ is a dual Banach algebra with respect to $C_0(S, 1/\omega)$ if and only if S is σ-compact.

Proof. Let K and F be compact subsets of S. It can be readily verified that in either cases (being left zero, right zero or zero) the sets $F^{-1}K$ and KF^{-1} are either empty or the whole S. For each $m \in \mathbb{N}$ we set $S_m = \{t \in F^{-1}K : \omega(t) \leq m\} = \{t \in S : \omega(t) \leq m\}$. Then $S = \cup_{m\in\mathbb{N}} S_m$ and so S is σ-compact. For the converse let $S = \cup_{n\in\mathbb{N}} S_n$ as a disjoint union of compact sets and let z be a (left or right) zero for S. Define $\omega(z) = 1$ and $\omega(x) = 1 + n$ for $x \in S_n$ then ω is a weight on S and $M_b(S,\omega)$ is a dual Banach algebra. □

Examples 2.1. (1) *The set $S = \mathbb{R}^+ \times \mathbb{R}$ equipped with the multiplication*

$$(x, y).(x', y') = (x + x', y') \quad ((x, y), (x', y') \in S)$$

and the weight $\omega(x,y)=e^{-x}(1+|y|)$ *is a weighted semigroup. Set* $F=[a,b]\times[c,d]$ *and* $K=[e,f]\times[g,h]$*. Then* $F^{-1}K=[e-b,f-a]\times[g,h]$ *and* $KF^{-1}=\begin{cases}[e-b,f-a]\times\mathbb{R} & \text{if } [c,d]\cap[g,h]\neq\emptyset\\ \emptyset & \text{if } [c,d]\cap[g,h]=\emptyset.\end{cases}$
Thus $M_b(S)$ *is not a dual Banach algebra by Corollary 2.1 (1). However, for all compact subsets* F *and* K *of* S*, the maps* $\frac{\chi_{F^{-1}K}}{\omega}$ *and* $\frac{\chi_{KF^{-1}}}{\omega}$ *vanishes at infinity. So* $M_b(S,\omega)$ *is a dual Banach algebra with respect to* $C_0(S,1/\omega)$*. This shows that the converse of Corollary 2.1 (2) may not be valid.*

(2) *For the semigroup* $S=[0,\infty)$ *endowed with the zero multiplication, neither* $M_b(S)$ *nor* $\ell_1(S)$ *is a dual Banach algebra. In fact,* S *is neither compactly nor weakly cancellative semigroup.*

3. Semigroup Measure Algebras as WAP-Algebras

In this section we study some conditions under which the weighted measure algebra $M_b(S,\omega)$ is a WAP-algebra. First, we provide some preliminaries.

Definition 3.1. Let $\tilde{\mathfrak{F}}$ be a linear subspace of $B(S,1/\omega)$, and let $\tilde{\mathfrak{F}}_r$ denote the set of all real-valued members of $\tilde{\mathfrak{F}}$. A mean on $\tilde{\mathfrak{F}}$ is a linear functional $\tilde{\mu}$ on $\tilde{\mathfrak{F}}$ with the property that $\inf_{s\in S}\frac{f}{\omega}(s)\le\tilde{\mu}(f)\le\sup_{s\in S}\frac{f}{\omega}(s)\quad(f\in\tilde{\mathfrak{F}}_r)$. The set of all means on $\tilde{\mathfrak{F}}$ is denoted by $M(\tilde{\mathfrak{F}})$. If $\tilde{\mathfrak{F}}$ is also an algebra with the multiplication given by $f\odot g:=(f.g)/\omega\quad(f,g\in\tilde{\mathfrak{F}})$ and if $\tilde{\mu}\in M(\tilde{\mathfrak{F}})$ satisfies $\tilde{\mu}(f\odot g)=\tilde{\mu}(f)\tilde{\mu}(g)\quad(f,g\in\tilde{\mathfrak{F}})$, then $\tilde{\mu}$ is said to be multiplicative. The set of all multiplicative means on $\tilde{\mathfrak{F}}$ will be denoted by $MM(\tilde{\mathfrak{F}})$.

Let $\tilde{\mathfrak{F}}$ be a conjugate closed, linear subspace of $B(S,1/\omega)$ such that $\omega\in\tilde{\mathfrak{F}}$.

(i) For each $s\in S$ define $\epsilon(s)\in M(\tilde{\mathfrak{F}})$ by $\epsilon(s)(f)=(f/\omega)(s)\quad(f\in\tilde{\mathfrak{F}})$. The mapping $\epsilon:S\longrightarrow M(\tilde{\mathfrak{F}})$ is called the evaluation mapping. If $\tilde{\mathfrak{F}}$ is also an algebra, then $\epsilon(S)\subseteq MM(\tilde{\mathfrak{F}})$.

(ii) Let $\tilde{X}=M(\tilde{\mathfrak{F}})$ (resp. $\tilde{X}=MM(\tilde{\mathfrak{F}})$, if $\tilde{\mathfrak{F}}$ is a subalgebra) be endowed with the relative weak* topology. For each $f\in\tilde{\mathfrak{F}}$ the function $\hat{f}\in C(\tilde{X})$ is defined by $\hat{f}(\tilde{\mu}):=\tilde{\mu}(f)\quad(\tilde{\mu}\in\tilde{X})$.

Furthermore, we define $\hat{\tilde{\mathfrak{F}}}:=\{\hat{f}:f\in\tilde{\mathfrak{F}}\}$.

Remark 3.1. (i) *The mapping* $f\longrightarrow\hat{f}:\tilde{\mathfrak{F}}\longrightarrow C(\tilde{X})$ *is clearly linear and multiplicative if* $\tilde{\mathfrak{F}}$ *is an algebra and* $\tilde{X}=MM(\tilde{\mathfrak{F}})$*. Also it preserves complex* *conjugation, and is an isometry, since for any* $f \in \tilde{\mathfrak{F}}$

$$\begin{aligned} ||\hat{f}|| &= \sup\{|\mu(\frac{f}{\omega})| : \mu \in X\} \leq \sup\{|\mu(\frac{f}{\omega})| : \mu \in C(X)^*, ||\mu|| \leq 1\} \\ &= ||\frac{f}{\omega}|| = \sup\{|\frac{f}{\omega}(s)| : s \in S\} = \sup\{|\epsilon(s)(f)| : s \in S\} \\ &= \sup\{|\hat{f}(\epsilon(s))| : s \in S\} \leq ||\hat{f}||, \end{aligned}$$

where $X = M(\mathfrak{F})$ *and* $\mathfrak{F} = \{f/\omega : f \in \tilde{\mathfrak{F}}\}$*. Note that* $\hat{f}(\epsilon(s)) = \epsilon(s)(f) = (\frac{f}{\omega})(s)(f \in \tilde{F}, s \in S)$*. This identity may be written in terms of dual map* $\tilde{\epsilon}^* : C(\tilde{X}) \longrightarrow C(S,\omega)$ *as* $\epsilon^*(\hat{f}) = f$ *for* $f \in \tilde{F}$*.*

(ii) *Let* $\tilde{\mathfrak{F}}$ *be a conjugate closed linear subspace of* $B(S, 1/\omega)$*, containing* ω*. Then* $M(\tilde{\mathfrak{F}})$ *is convex and weak* compact,* $co(\epsilon(S))$ *is weak* dense in* $M(\tilde{\mathfrak{F}})$*,* $\tilde{\mathfrak{F}}^*$ *is the weak* closed linear span of* $\epsilon(S)$*,* $\epsilon : S \longrightarrow M(\tilde{\mathfrak{F}})$ *is weak* continuous, and if* $\tilde{\mathfrak{F}}$ *is also an algebra, then* $MM(\tilde{\mathfrak{F}})$ *is weak* compact and* $\epsilon(S)$ *is weak* dense in* $MM(\tilde{\mathfrak{F}})$*.*

(iii) *Let* $\tilde{\mathfrak{F}}$ *be a* C^**-subalgebra of* $B(S, 1/\omega)$*, containing* ω*. If* $\tilde{X}$ *denotes the space* $MM(\tilde{\mathfrak{F}})$ *with the relative weak* topology, and if* $\epsilon : S \longrightarrow \tilde{X}$ *denotes the evaluation mapping, then the mapping* $f \longrightarrow \hat{f} : \tilde{\mathfrak{F}} \longrightarrow C(\tilde{X})$ *is an isometric isomorphism with the inverse* $\epsilon^* : C(\tilde{X}) \longrightarrow \tilde{\mathfrak{F}}$*.*

Let $\tilde{\mathfrak{F}} = wap(S, 1/\omega)$. Then $\tilde{\mathfrak{F}}$ is a C^*-subalgebra of $WAP(M_b(S,\omega))$, see [11, Theorem1.6, Theorem 3.3]. Set $\tilde{X} = MM(\tilde{\mathfrak{F}})$. By the above remark $wap(S, 1/\omega) \cong C(\tilde{X})$ and so

$$M_b(\tilde{X}) \cong C(\tilde{X})^* \cong wap(S, 1/\omega)^* \subseteq WAP(M_b(S,\omega))^*.$$

Let $\epsilon : S \longrightarrow \tilde{X}$ be the evaluation mapping. We also define

$$\bar{\epsilon} : M_b(S,\omega) \longrightarrow M_b(\tilde{X}) \quad \text{by} \quad \langle \bar{\epsilon}(\mu), f\rangle = \int_S f\omega d\mu$$

for $f \in wap(S, 1/\omega) \cong C(\tilde{X})$. Then for every Borel set B in $\tilde{X}$, we have $\bar{\epsilon}(\mu)(B) = (\mu\omega)(\epsilon^{-1}(B))$. In particular, $\bar{\epsilon}(\frac{\delta_x}{\omega(x)}) = \delta_{\epsilon(x)}$.

The next theorem is the main result of this section.

Theorem 3.1. For every weighted locally compact semi-topological semigroup (S,ω) the following statements are equivalent:

(1) The map $\epsilon : S \longrightarrow \tilde{X}$ is one to one, where $\tilde{X} = MM(wap(S, 1/\omega))$;
(2) $\bar{\epsilon} : M_b(S,\omega) \longrightarrow M_b(\tilde{X})$ is an isometric isomorphism;
(3) $M_b(S,\omega)$ is a WAP-algebra.

Proof. (1) ⇒ (2). Take $\mu \in M_b(S,\omega)$, say $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$, where $\mu_j \in M_b(S,\omega)^+$ for each $j = 1, 2, 3, 4$. Set $\nu_j = \bar{\epsilon}(\mu_j) \in M_b(\tilde{X})^+$, and $\nu = \bar{\epsilon}(\mu) = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$. Take $\delta > 0$. For each j, there exists a Borel set B_j in $\tilde{X}$ such that $\nu_j(B) \geq 0$ for each Borel subset B of B_j with $\sum_{j=1}^4 \nu_j(B_j) > ||\nu|| - \delta$. In fact, by the Hahn decomposition theorem for the signed measures $\lambda_1 = \nu_1 - \nu_2$ and $\lambda_2 = \nu_3 - \nu_4$, there exist four Borel sets P_1, P_2, N_1 and N_2 in $\tilde{X}$ such that $P_1 \cup N_1 = \tilde{X}$, $P_1 \cap N_1 = \emptyset$, $P_2 \cup N_2 = \tilde{X}$, $P_2 \cap N_2 = \emptyset$, and $\nu_1(E) = \lambda_1(P_1 \cap E)$, $\nu_2(E) = -\lambda_1(N_1 \cap E)$, $\nu_3(E) = \lambda_2(P_2 \cap E)$, $\nu_4(E) = -\lambda_2(N_2 \cap E)$, for every Borel set E of $\tilde{X}$. That is, the measures $\nu_1, \nu_2, \nu_3, \nu_4$ are concentrated on P_1, N_1, P_2, N_2, respectively.

Set $D_1 := P_1 \cap N_2, D_2 := N_1 \cap P_2, D_3 := P_2 \cap P_1, D_4 := N_2 \cap N_1$. Then the family $\{D_1, D_2, D_3, D_4\}$ is a partition for $\tilde{X}$. Further, there exists a compact set K for which $||\nu|| - \delta \leq \sum_{j=1}^{4} ||\nu_{j_{|D_j}}|| - \delta \leq \sum_{j=1}^{4} \nu_{j_{|D_j}}(K) = \sum_{j=1}^{4} \nu_j(D_j \cap K)$. Set $B_j = D_j \cap K$. Then the sets B_1, B_2, B_3, B_4 are pairwise disjoint.

For each j, set $C_j = (\epsilon)^{-1}(B_j)$, a Borel set in S. Then $(\mu_j\omega)(C_j) = \nu_j(B_j)$. Since ϵ is injection, the sets C_1, C_2, C_3, C_4 are pairwise disjoint, and so $||\mu||_\omega \geq \sum_{j=1}^{4} |\mu\omega(C_j)| \geq \sum_{j=1}^{4} (\mu_j\omega)(C_j) = \sum_{j=1}^{4} \nu_j(B_j) > ||\nu|| - \delta$ This holds for each $\delta > 0$, so $||\mu||_\omega \geq ||\nu||$. A similar argument shows that $||\mu||_\omega \leq ||\nu||$. Thus $||\mu||_\omega = ||\nu||$.

(2)⇒(1). Let $P(S,\omega)$ denote the subspace of all probability measures of $M_b(S,\omega)$ and $ext(P(S,\omega))$ the extreme points of unit ball of $P(S,\omega)$. Then $ext(P(S,\omega)) = \{\frac{\delta_x}{\omega(x)} : x \in S\} \cong S$ and $ext(P(\tilde{X})) \cong \tilde{X}$, see [4, p.151]. By the injectivity of $\bar{\epsilon}$, it maps the extreme points of the unit ball onto the extreme points of the unit ball, thus $\epsilon : S \longrightarrow \tilde{X}$ is one to one.

(2)⇒(3). Since $\tilde{X}$ is compact, $M_b(\tilde{X})$ is a dual Banach algebra with respect to $C(\tilde{X})$, so it has an isometric representation ψ on a reflexive Banach space E, see [9]. In the following commutative diagram,

$$\begin{array}{ccc} M_b(S,\omega) & \xrightarrow{\bar{\epsilon}} & M_b(\tilde{X}) \\ & \phi & \downarrow \psi \\ & & B(E) \end{array}$$

If $\bar{\epsilon}$ is isometric, then so is ϕ. Thus $M_b(S,\omega)$ has an isometric representation on a reflexive Banach space E if $\bar{\epsilon}$ is an isometric isomorphism. So $M_b(S,\omega)$ is a WAP-algebra if $\bar{\epsilon}$ is an isometric isomorphism.

(3)⇒(1). Let $M_b(S,\omega)$ be a WAP-algebra. Since $\ell_1(S,\omega)$ is a norm closed subalgebra of $M_b(S,\omega)$, the weighted semigroup algebra $\ell_1(S,\omega)$ is a WAP-algebra. Using the double limit criterion, it is easy to check that $wap(S, 1/\omega) = WAP(\ell_1(S,\omega))$ (see also [11, Theorem 3.7]) where we treat $\ell^\infty(S, 1/\omega)$ as an $\ell_1(S,\omega)$-bimodule. Then $\bar{\epsilon} : \ell_1(S,\omega) \longrightarrow wap(S, 1/\omega)^*$ is an isometric isomorphism. Since $wap(S, 1/\omega)$ is a C^*-algebra, as (2)⇒(1), $\epsilon : S \longrightarrow \tilde{X}$ is one to one. □

Corollary 3.1. For a locally compact semi-topological semigroup S, $M_b(S,\omega)$ is a WAP-algebra if and only if $\ell_1(S,\omega)$ is a WAP-algebra.

For $\omega = 1$, it is clear that $\tilde{X} = S^{wap}$, and the map $\epsilon : S \longrightarrow S^{wap}$ is one to one if and only if $wap(S)$ separates the points of S, see [3].

Corollary 3.2. For a locally compact semi-topological semigroup S, the following statements are equivalent:

(1) $M_b(S)$ is a WAP-algebra;
(2) $\ell_1(S)$ is a WAP-algebra;
(3) The evaluation map $\epsilon : S \longrightarrow S^{wap}$ is one to one;
(4) $wap(S)$ separates the points of S.

Illustrating our results, we conclude the paper with the following examples.

Examples 3.1.

(i) *We examine the semigroup algebra $\ell_1(S)$ for $S = \mathbb{N}$ equipped with various multiplications. When S is equipped with the* min *multiplication, the semigroup algebra $\ell_1(S)$ is a WAP-algebra, while, is not neither Arens regular nor a dual Banach algebra. If we furnish S with the* max *multiplication, then $\ell_1(S)$ is a dual Banach algebra (and so a WAP-algebra) which is not Arens regular. If we change the multiplication of S to the zero multiplication then the resulted semigroup algebra is Arens regular (so a WAP-algebra) which is not a dual Banach algebra. This describes the interrelation between the concepts of being Arens regular algebra, dual Banach algebra and WAP-algebra.*

(ii) *Let S be the set of all sequences with* $0, 1$ *values. We equip S with pointwise multiplication. We denote by e_n the characteristic of n. Let $s = \{x_n\} \in S$, and let $F_w(S)$ be the set of all elements of S such that $x_i = 0$ for only finitely index i. It is easy to see that $F_w(S)$ is countable. Let $F_w(S) = \{s_1, s_2, \cdots\}$. Recall that, each element $g \in \ell^\infty(S)$ has the presentation as $g = \sum_{s\in S} g(s)\chi_s$, see* [6, p.65]. *Suppose $g = \sum_{s\in S\setminus F_w(S)} g(s)\chi_s$ be in $wap(S)$, we show that $g = 0$. Let $s = \{x_n\} \in S$, and $\{k \in \mathbb{N} : x_k = 0\} = \{k_1, k_2, \cdots\}$ be an infinite set. Put $a_n = s + \sum_{j=1}^{n} e_{k_j}$ and $b_m = s + \sum_{i=m}^{\infty} e_{k_i}$. Then*

$$a_n b_m = \begin{cases} \sum_{j=m}^{n} e_{k_j} + s & \text{if} \quad m \le n \\ s & \text{if} \quad m > n. \end{cases}$$

Thus

$$g(s) = \lim_n \lim_m g(a_n b_m) = \lim_m \lim_n g(a_n b_m) = \lim_m g(s + \sum_{i=m}^{\infty} e_{k_i}) = 0.$$

Indeed,

$$wap(S) = \{f \in \ell^\infty(S) : f = \sum_{i=1}^{\infty} f(s_i)\chi_{s_i}, \quad s_i \in F_w(S)\} \oplus \mathbb{C}$$

It is also clear that $F_w(S)$ is the subsemigroup of S with$wap(F_w(S)) = \ell_\infty(F_w(S))$. So $\ell_1(F_w(S))$ is Arens regular. Let T consist of those sequences $s = \{x_n\} \in S$

such that $x_i = 0$ *for infinitely index* i*, then* T *is a subsemigroup of* S *and* $wap(T) = \mathbb{C}$*. Since* $\epsilon_{|_T} : T \longrightarrow S^{wap}$ *is not one to one,* $\ell_1(S)$ *is not a* WAP-*algebra. This shows that* $\ell_1(S)$ *need not be a* WAP-*algebra.*

(iii) *If we equip* $S = \mathbb{R}^2$ *with the multiplication* $(x, y).(x', y') = (xx', x'y + y')$*, then* $M_b(S)$ *is not a WAP-algebra. Indeed, every non-constant function* f *over* x*-axis is not in* $wap(S)$*. Let* $f(0, z_1) \neq f(0, z_2)$ *and* $\{x_m\}$*,*$\{y_m\}$*,* $\{\beta_n\}$ *be sequences with distinct elements satisfying the recursive equation*

$$\beta_n x_m + y_m = \frac{mz_1 + nz_2}{m + n}.$$

Then

$$\begin{aligned} \lim_n \lim_m f((0, \beta_n).(x_m, y_m)) &= \lim_n \lim_m f(0, \beta_n x_m + y_m) \\ &= \lim_n \lim_m f(0, \frac{mz_1 + nz_2}{m + n}) = f(0, z_1), \end{aligned}$$

and similarly

$$\lim_m \lim_n f((0, \beta_n).(x_m, y_m)) = f(0, z_2).$$

Thus the map $\epsilon : S \longrightarrow S^{wap}$ *is not one to one, so* $M_b(S)$ *is not a WAP-algebra.*

(iv) *Let* S *be the interval* $[\frac{1}{2}, 1]$ *with the multiplication* $x.y = \max\{\frac{1}{2}, xy\}$*, where* xy *is the ordinary multiplication on* $\mathbb{R}$*. Then for each* $s \in S \setminus \{\frac{1}{2}\}$*,* $x \in S$*, the set* $x^{-1}s$ *is finite. But* $x^{-1}\frac{1}{2} = [\frac{1}{2}, \frac{1}{2x}]$*. Let* $B = [\frac{1}{2}, \frac{3}{4})$*. Then for every finite subset* F *of* B*,*

$$\bigcap_{x\in F} x^{-1}\frac{1}{2} \setminus \bigcap_{x\in B\setminus F} x^{-1}\frac{1}{2} = [\frac{2}{3}, \frac{1}{2x_F}],$$

where $x_F = \max F$*. By* [15, Theorem 4], $\chi_{\frac{1}{2}} \notin wap(S)$*. So* $c_0(S \setminus \{\frac{1}{2}\}) \oplus \mathbb{C} \subsetneqq wap(S)$*. It can be readily verified that* $\epsilon : S \longrightarrow S^{wap}$ *is one to one, so* $\ell_1(S)$ *is a WAP-algebra but* $c_0(S) \nsubseteq wap(S)$*.*

(v) *Take* $T = (\mathbb{N} \cup \{0\}, .)$ *with 0 as zero of* T *and the multiplication defined by*

$$n.m = \begin{cases} n & \text{if} \quad n = m \\ 0 & \text{otherwise.} \end{cases}$$

Set $S = T \times T$ *equipped with the pointwise product. Now let* $X = \{(k, 0) : k \in T\}$*,* $Y = \{(0, k) : k \in T\}$ *and* $Z = X \cup Y$*. We use the Ruppert criterion* [15] *to show that* $\chi_z \notin wap(S)$*, for each* $z \in Z$*. Let* $B = \{(k, n) : k, n \in T\}$*, then* $(k, n)^{-1}(k, 0) = \{(k, m) : m \neq n\} = B \setminus \{(k, n)\}$*. Thus for each finite subsets*

F of B,

$$\begin{aligned}\big(\cap\{(k,n)^{-1}(k,0):(k,n)\in F\}\big) &\setminus (\cap\{(k,n)^{-1}(k,0):(k,n)\in B\setminus F\})\\ &= \big(\cap\{(k,0)(k,n)^{-1}:(k,n)\in F\}\big)\\ &\setminus (\cap\{(k,n)^{-1}(k,0):(k,n)\in B\setminus F\})\\ &= (B\setminus F)\setminus F = B\setminus F\end{aligned}$$

and the last set is infinite. This means that $\chi_{(k,0)} \notin wap(S)$. Similarly $\chi_{(0,k)} \notin wap(S)$. Let $f = \sum_{n=0} f(0,n)\chi_{(0,n)} + \sum_{m=1}^{\infty} f(m,0)\chi_{(m,0)}$ be in $wap(S)$. Then for each fixed n and the sequence $\{(n,k)\}$ in S, we have $\lim_k f(n,k) = \lim_k \lim_l f(n,l.k) = \lim_l \lim_k f(n,l.k) = f(n,0)$, which implies that $f(n,0) = 0$. Similarly $f(0,n) = 0$ and $f(0,0) = 0$. Thus $f = 0$. Since $wap(S)$ can not separate the points of S so $\ell_1(S)$ is not a WAP-algebra. Let $\omega(n,m) = 2^n 3^m$ for $(n,m) \in S$. Then ω is a weight on S such that $\omega \in wap(S, 1/\omega)$. Then the evaluation mapping $\epsilon : S \longrightarrow \tilde{X}$ is one to one. This means that $\ell_1(S,\omega)$ is a WAP-algebra while $\ell_1(S)$ is not !

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REFERENCES

[1] *M. Abolghasemi, A. Rejali and H.R. Ebrahimi Vishki*, Weighted semigroup algebras as dual Banach algebras, arXiv:0808.1404 [math.FA].

[2] *J.W. Baker and A. Rejali*, On the Arens regularity of weighted convolution algebras, J. London Math. Soc. (2) **40**, 535-546, (1989).

[3] *J.F. Berglund, H.D. Junghenn and P. Milnes,* Analysis on Semigroups, Wiley, New York, 1989.

[4] *J.B. Conway*, A Course in Functional Analysis, Springer-Verlag, Berlin, 1985.

[5] *H.G. Dales*, Banach Algebras and Automatic Continuity, Clarendon Press, Oxford, 2000.

[6] *H.G. Dales and A.T.-M. Lau*, The Second Dual of Beurling Algebras, Mem. Amer. Math. Soc. **177** (2005), no. 836.

[7] *H.G. Dales, A.T.-M. Lau, and D. Strauss*, Banach algebras on semigroups and on their compactification, Mem. Amer. Math. Soc. **205** (2010), 1-165.

[8] *M. Daws*, Connes-amenability of bidual and weighted semigroup algebras, Math. Scand. **99**, no. 2 (2006), 217-246.

[9] *M. Daws,* Dual Banach algebras: Representation and injectivity, Studia Math. **178** (2007), 231-275.

[10] *M. Daws, H.L. Pham and S. White*, Conditions implying the uniqeness of the weak$^*-$topology on certain group algebras, Houston J. Math. **35**, no. 1 (2009), 253-276.

[11] *M. Lashkarizadeh Bami*, Function algebras on weighted topological semigroups, Math. Japon. **47**, no. 2 (1998), 217-227.

[12] *A. Rejali*, The analogue of weighted group algebra for semitopological semigroups, J. Sci. I.R. Iran **6**, no. 2 (1995), 113-120.

[13] *A. Rejali*, The Arens regularity of weighted semigroup algebras, Sci. Math. Jpn. **60**, no.1 (2004), 129-137.

[14] *A. Rejali and H.R.E. Vishki*, Weighted convolution measure algebras characterized by convolution algebras, J. Sci. I. Iran **18**, (4) (2007), 345-349 .

[15] *W.A.F. Ruppert*, On weakly almost periodic sets, Semigroup Fourm **32**, no.3 (1985), 267-281.

[16] *V.Runde*, Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule, Math. Scand. **95**, (2004), 124-144.

[17] *N.J.Young*, Periodicity of functionals and representations of normed algebras on reflexive spaces, Proc. Edinburgh Math. Soc. (2) **20**, (1976/77), 99-120.