ON THE BICOMPLEX VERSION OF THE WAVE EQUATION

Panagiotis N. Koumantos[1]

Considering the idempotent decomposition of bicomplex modules and bicomplex linear operators, we apply operator (C_0)*-group theory, analogous to the classical case of the problem, to achieve a unique solution for the bicomplex-valued wave equation. Also, we formulate the Laplacian and the corresponding problem of the wave equation in bicomplex Kähler manifolds.*

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1. Introduction

In the present work we study the wave equation in the context of generalization with values in the bicomplex numbers.

Let $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{BC} : (x,t) \to u(x,t)$, and the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t),\ t \in \mathbb{R}^+, \tag{1}$$

with initial data $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}\big|_{t=0} = g(x)$. Here Δ is the suitable bicomplex version of the Laplacian and for simplicity, in the standard form $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \Delta u$ of the wave equation, the real positive constant $c = 1$ is chosen for the propagation speed of the wave.

We also consider the above wave equation (1), more generally in bicomplex Kähler manifolds.

In the search for alternative algebras, and development of special algebras, bicomplex numbers were introduced in 1892 by Segre in [36]. Also, in the classic book by Price [30] there is a very comprehensive introduction to bicomplex numbers and their generalization, as well as an essential presentation of basic topological properties.

Differential equations such as the Schrödinger equation have already been considered in the context of bicomplex values in works by Rochon and Tremblay in [32], [33] and Rochon in [34], and by Theaker and Van Gorder in [39]. Also in the paper by Agarwal et al. [3] the Bochner theorem and applications of the bicomplex Fourier transform for the heat equation and the wave equation in the bicomplex setting have been studied. We also refer to Luna-Elizarrarás et al. in [24] and [25] for the Laplacian and derivatives of bicomplex functions, while for the study of differential equations in multicomplex spaces we refer to Struppa et al. in [38].

The topological bicomplex modules have been systematically studied by Kumar and Saini in [20]. Also, results have been presented regarding bicomplex C^*-algebras in article by Kumar et al. in [19], while by Kumar and Singh in [21] the bicomplex linear operators on bicomplex Hilbert spaces were studied, as well Littlewood's subordination theorem. We also

[1]Institute of Applied and Computational Mathematics, Foundation of Technology and Research (IACM-FORTH), 70013 Heraklion, Crete, Greece, e-mail: pnkoumantos@gmail.com; pkoumantos@iacm.forth.gr

refer to Guo in [16] for extension theory of Hilbert modules over semigroups, and to Alpay et al. [2] for slice hyperholomorphic fractional Hardy spaces, and to Ghiloni and Stoppato [15] for regularity in one hypercomplex variable. Further, we refer to the article by Colombo et al. [8] for the study of bicomplex linear and bounded operators and bicomplex functional calculus, as well as the article by Charak et al. [6] for infinite dimensional bicomplex spectral decomposition theorem.

In Section 2 we list the necessary terminology from bicomplex numbers, bicomplex Banach and Hilbert modules, and the (C_0)-group theory we need to develop the results. Further, in the case of the bicomplex Kähler manifold we extend the concept of the Laplacian according to Kodaira (cf. Morrow and Kodaira [28], Moroianu [27]) of the classical complex analysis to manifolds, again using the appropriate choice of the $*$-conjugate to the bicomplex numbers. We also refer to the article by Baird and Wood [4] on the concept of bicomplex manifold and applications. Also, we refer to the article by Chen and Millman [7] for classical results on the wave equation on a Riemannian manifold.

Then, in Section 3 we present the main results, where we apply the method of operator (C_0)-group theory to the case of the wave equation (1) for the construction of the infinitesimal generator of the bicomplex (C_0)-group obtained through the idempotent components and the existence of a generalized solution of the wave equation in the bicomplex version.

In addition to the books and articles, mentioned above for bicomplex analysis, we refer to the classical approaches of functional analysis and applications of semigroups to differential equations in the standard books of Pazy [29], Engel and Nagel [11] Hille and Phillips [17], and Yosida [40].

2. **Preliminaries**

2.1. **Bicomplex numbers**

The set $\mathbb{BC}$ of bicomplex numbers consists of the elements of the form $z_1 + jz_2$, where $z_1, z_2 \in \mathbb{C}(i)$, $i, j \notin \mathbb{R}$, with $i^2 = j^2 = -1$ and $ij = ji$.

Here $\mathbb{C}(i)$ and $\mathbb{C}(j)$ are two copies of complex numbers that coexist in set $\mathbb{BC}$, $k := ij$ is a hyperbolic unit and the set of hyperbolic numbers $\mathbb{D}$, *i.e.* the set of the elements $\alpha + \beta k$, $\alpha, \beta \in \mathbb{R}$ is contained in $\mathbb{BC}$.

The set $\mathbb{BC}$ is a commutative ring with the standard operations and with a unit element $1_{\mathbb{BC}} := 1$. Furthermore, the Euclidean-type norm $\|\cdot\|_{\mathbb{BC}}$ is defined on $\mathbb{BC}$. Then, $\|\zeta_1\zeta_2\|_{\mathbb{BC}} \leq \sqrt{2}\,\|\zeta_1\|_{\mathbb{BC}}\,\|\zeta_2\|_{\mathbb{BC}}$, for every $\zeta_1, \zeta_2 \in \mathbb{BC}$, and finally $\mathbb{BC}$ is a modified Banach algebra (cf. [30]).

The simplest elements that are singular and divisors of zero in $\mathbb{BC}$ are $e_1 := \frac{1+k}{2}$ and $e_2 := \frac{1-k}{2}$, for which: $e_1^2 = e_1$ and $e_2^2 = e_2$ are valid, *i.e.* they are idempotent elements, $\|e_1\|_{\mathbb{BC}} = \|e_2\|_{\mathbb{BC}} = \frac{\sqrt{2}}{2}$, $e_1 + e_2 = 1$, and additionally $e_1e_2 = 0$ while $e_1 \neq 0$ and $e_2 \neq 0$.

The role of the elements e_1 and e_2 is very important as they are also linearly independent in terms of complex linear combinations and each element in $\mathbb{BC}$ is written in a unique way in the the so-called idempotent representation.

Also, the set of positive hyperbolic numbers $\mathbb{D}^+$ is defined, and any hyperbolic number is written in the corresponding equivalent idempotent representation $\nu e_1 + \mu e_2$, with $\nu, \mu \geq 0$. Moreover, a binary relation $\preceq$ is defined on $\mathbb{D}$ as follows: $\alpha \preceq \beta \Leftrightarrow (\beta - \alpha) \in \mathbb{D}^+$. Then, $\mathbb{D}$ is a partially ordered set with $\mathbb{D}^+$ being convex and proper positive cone.

Let the projection maps $p_1 : \mathbb{BC} \to \mathbb{C}(i)$ and $p_2 : \mathbb{BC} \to \mathbb{C}(j)$ defined by $p_1(z_1 + jz_2) = z_1 - iz_2$ and $p_2(z_1 + jz_2) = z_1 + iz_2$.

For X_1 and X_2 be subsets of $\mathbb{C}(i)$, the set $X_1 \times_e X_2$ consists of all $\zeta = z_1 + jz_2 \in \mathbb{BC}$ such that $p_1(\zeta) \in X_1$ and $p_2(\zeta) \in X_2$, is called the bicomplex cartesian set determined by X_1 and X_2.

Also, it is well-known that if X_1 and X_2 are domains (i.e. open and connected) of $\mathbb{C}(i)$ then $X_1 \times_e X_2$ is also a domain in $\mathbb{BC}$. Moreover, if X_1 and X_2 are convex or closed or compact sets then $X_1 \times_e X_2$ is also a convex or closed or compact set in $\mathbb{BC}$.

For the standard properties, the expression of the Euclidean norm and for the principal ideals, as well as for detailed proofs of all the above standard concepts and results, refer to the books (respectively in the first Chapter) [1], [24] and [30].

2.2. Bicomplex version of the Laplacian

In order to formulate the appropriate choice in Equation (1) for the Laplacian in the bicomplex version, we recall that there are three standard types of conjugates defined and the corresponding modulus derived from them.

Specifically, for each element $\zeta = z_1 + jz_2 \in \mathbb{BC}$, bar-conjugate $\bar{\zeta} := \bar{z}_1 + j\bar{z}_2$, tilde-conjugate $\tilde{\zeta} := z_1 - jz_2$ and $*$-conjugate $\zeta^* := \bar{z}_1 - j\bar{z}_2$ are defined, where $\bar{z}_1$ and $\bar{z}_2$ are the usual conjugate complex numbers of z_1 and z_2 respectively. (cf. [1] and [24]).

There are several candidate forms for the bicomplex version of the Laplacian (see for example in El Gourari et al. [10]).

We note that for the bicomplex version of the Laplacian henceforth we consider the choice $\Delta := \frac{\partial^2}{\partial\zeta\partial\zeta^*}$, so we also write the corresponding idempotent representation $\Delta = e_1\Delta_1 + e_2\Delta_2$, with $\Delta_1 = e_1\Delta$ and $\Delta_2 = e_2\Delta$ due to the linearity of the operator.

This is the appropriate choice, because by writing $\zeta = z_1+jz_2 \in \mathbb{BC}$, in the idempotent representation $\zeta = e_1\alpha + e_2\beta$, where $\alpha = z_1 - iz_2$ and $\beta = z_1 + iz_2$, we have $\zeta^* = e_1\bar{\alpha} + e_2\bar{\beta}$.

Also, in this case $\frac{\partial}{\partial\zeta} := \frac{1}{2}\left(\frac{\partial}{\partial z_1} - j\frac{\partial}{\partial z_2}\right) = \frac{1}{2}\left(e_1\frac{\partial}{\partial\alpha} + e_2\frac{\partial}{\partial\beta}\right)$, and $\frac{\partial}{\partial\zeta^*} := \frac{1}{2}\left(\frac{\partial}{\partial\bar{z}_1} + j\frac{\partial}{\partial\bar{z}_2}\right)$ $= \frac{1}{2}\left(e_1\frac{\partial}{\partial\bar{\alpha}} + e_2\frac{\partial}{\partial\bar{\beta}}\right)$, applies.

For the other choices of the Laplacian related to the choice of the corresponding kind of conjugate bicomplex number, and how these operators are related to each other, we also refer to Luna-Elizarrarás et al. in [24] and [25].

2.3. Bicomplex Kähler manifolds and Laplacian

We will then define the notion of bicomplex Kähler manifolds and arrive at the extension to the bicomplex version of Kodaira's Laplacian known in the classical case.

Following Baird and Wood in [4] we give the following definition of a bicomplex manifold.

Definition 2.1. *A bicomplex manifold is a complex manifold endowed with a maximal differential structure, i.e. a complex atlas whose transition functions are bicomplex holomorphic functions.*

At this point we mention the Ringleb decomposition Lemma for bicomplex holomorphic functions (cf. Riley in [31], while for bicomplex meromorphic functions we refer to Charak, Rochon and Sharma in [5]).

Lemma 2.1. *Let $\Omega \subseteq \mathbb{BC}$ be an open set. Then, a function $f : \Omega \to \mathbb{BC}$ is bicomplex holomorphic (resp. meromorphic) on Ω if and only if the functions $f_{e1} : p_1(\Omega) \to \mathbb{C}(i)$ and $f_{e2} : p_2(\Omega) \to \mathbb{C}(j)$ are holomorphic (resp. meromorphic), where $f(\zeta) = f_{e1}(p_1(\zeta))e_1 + f_{e2}(p_2(\zeta))e_2$, for every $\zeta = z_1 + jz_2 \in \Omega$.*

Then, it is known that the next result about bicomplex manifolds holds (cf. [4]).

Proposition 2.1. *A bicomplex manifold of bicomplex dimension n is locally the product of complex manifolds of dimension n.*

Moreover, it is well-known that in addition to the complex viewpoint for Kähler manifolds, there are other compatible structures such that the symplectic and the Riemannian. Thus, following Baird and Wood in [4], and LeBrun's article [22] on which they are based, the next concept of the complex Riemannian manifold is defined. We also mention the article by Michelsohn [26] on Clifford and spinor cohomology of Kähler manifolds.

Definition 2.2. *Let X be a complex manifold, with complex dimension m, and TX its $(1,0)$-holomorphic tangent space, i.e. TX is spanned by $\left\{\frac{\partial}{\partial z^k}\right\}$, $k = 1, 2, \ldots, m$ for any complex coordinates $\left\{z^k\right\}$. Then, the pair (X, g) is called a complex-Riemannian manifold if g is a harmonic metric, i.e. g is a holomorhic section of $TX \otimes TX$ which is symmetric and non-degenerate.*

For U be an open set of $\mathbb{C}^m$ the complex Laplacian $\Delta_{\mathbb{C}}$ is defined as $\Delta_{\mathbb{C}} := \sum_{k=1}^{m} \frac{\partial^2}{\partial z_k^2}$, with $\Delta_{\mathbb{C}} f := \sum_{k=1}^{m} \frac{\partial^2 f}{\partial z_k^2}$, where $f : U \to \mathbb{C}$ and $(z_1, z_2, \cdots, z_m)$ are the standard coordinates on $\mathbb{C}^m$.

Also, a holomorphic function $f : X \to \mathbb{C}$ is said to be complex-harmonic if it satisfies the complex-Laplace equation $\Delta_{\mathbb{C}} f = 0$, where the complex-Laplace operator is defined by complexifying the formulae for the real case.

Then, in local complex coordinates $\left\{z^k\right\}$, defining the matrix (g_{ij}) by $g_{ab} = g\left(\frac{\partial}{\partial z_a}, \frac{\partial}{\partial z_b}\right)$ and for $\left(g^{ab}\right)$ its inverse, yields $\Delta_{\mathbb{C}} f = g^{ab}\left(\frac{\partial^2 f}{\partial z^a \partial z^b} - \Gamma_{ab}^k \frac{\partial f}{\partial z^k}\right)$, where the usual summation convention is assumed in terms of indices, and $\Gamma_{ab}^k = \frac{1}{2} g^{km}\left(\frac{\partial g_{bm}}{\partial z_a} + \frac{\partial g_{am}}{\partial z_b} - \frac{\partial g_{ab}}{\partial z_m}\right)$, are the corresponding Christoffel symbols.

In this direction, for examples of bicomplex manifolds by complexifying complex manifolds, and results on complex-harmonic morphisms and bicomplex manifolds, we refer to Baird and Wood in [4].

Now we give the next definition and keep in mind that from the three types of conjugates ($*$-, bar-, tilde-) in bicomplex numbers we need the concept of $*$-conjugate to define the concept of $*$-Hermitian form (cf. [1]).

Definition 2.3. *A bicomplex Kähler manifold is a bicomplex manifold X with a bicomplex $*$-Hermetian metric h whose associated 2-form ω is closed. That is h gives a positive definite $*$-Hermitian form on the tangent space $T_x X$ at each point $x \in X$, and the 2-form ω is defined by $\omega(u, v) = \mathrm{Re}h(iu, v) = \mathrm{Im}h(u, v)$, for tangent vectors u and v.*

A bicomplex Kähler manifold can also be viewed as a Riemannian manifold, with the Riemannian metric g defined by $g(u, v) = \mathrm{Re}h(u, v)$. Equivalently, a bicomplex Kähler manifold X is a bicomplex Hermitian manifold of complex dimension n such that for every point $x \in X$, there is a holomorphic coordinate chart around x in which the metric agrees with the standard metric on $\mathbb{C}^n$ to order 2 near x.

After the above, we have the optimal assumptions so that the next result holds.

Proposition 2.2. *A bicomplex manifold X is bicomplex Kähler manifold if and only if the idempotent components $X_\ell := e_\ell X$ are Kähler manifolds, for $\ell = 1, 2$ respectively.*

On a Riemannian manifold of dimension m, the Laplacian on smooth r-forms is defined by $\Delta_d = dd^* + d^* d$, where d is the exterior derivative and $d^* = -(-1)^{mr} \star d \star$, where

$\star$ is the Hodge star operator. Equivalently, d^* is the adjoint of d with respect to the L^2-inner product on r-forms with compact support.

Let $\Omega^{p,q}$ be the space of (p,q)-forms, E^k be the space of all complex differential forms of total degree k, so that $E^k = \Omega^{k,0} \oplus \Omega^{k-1,1} \oplus \cdots \oplus \Omega^{1,k-1} \oplus \Omega^{0,k} = \bigoplus_{p+q=k} \Omega^{p,q}$, and for each k and each p and q with $p+q=k$, the canonical projections of vector bundles $\pi^{p,q} : E^k \to \Omega^{p,q}$.

For a Hermitian manifold X, d and d^* are decomposed as $d = \partial + \bar{\partial}$ and $d^* = \partial^* + \bar{\partial}^*$, and the two Laplacians are defined: $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and $\Delta_{\partial} = \partial\partial^* + \partial^*\partial$.

Here ∂ and $\bar{\partial}$ are the Dolbeault operators: $\partial := \pi^{p+1,q} \circ d : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar{\partial} := \pi^{p,q+1} \circ d : \Omega^{p,q} \to \Omega^{p,q+1}$, while ∂^* and $\bar{\partial}^*$ are the (formal) adjoints of ∂ and $\bar{\partial}$, i.e. $\partial^* : \Omega^{p,q} \to \Omega^{p-1,q}$, with $\partial^* := - \star \bar{\partial} \star$, and $\bar{\partial}^* : \Omega^{p,q} \to \Omega^{p,q-1}$, with $\bar{\partial}^* := - \star \partial \star$ respectively.

At this point we also refer to the work of Ghiloni and Perotti in [14], where global differential equations for slice regular functions are studied, and the global differential operators ∂ and $\bar{\partial}$ are particularly studied. Also, in the same article [14] examples are given in quaternions, octonions and Clifford algebras for the form of the differential operators ∂ and $\bar{\partial}$, while the characterization of $\mathrm{Ker}(\bar{\partial})$ via semi-step functions is given.

Since X is Kähler, the Kähler identities imply these Laplacians are all the same up to a constant, $\Delta := \Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$.

Hence, on a Kähler manifold X, $\mathcal{H}^r(X) = \bigoplus_{p+q=r} \mathcal{H}^{p,q}(X)$, where $\mathcal{H}^r$ is the space of harmonic r-forms on X, i.e. forms α with $\Delta\alpha = 0$, and $\mathcal{H}^{p,q}$ is the space of harmonic (p,q)-forms. Therefore, a differential form α is harmonic if and only if each of its (p,q)-component is harmonic.

Further, in order to ensure self-adjointness results (cf. Morrow and Kodaira in [28]) for the Lapalacian, and subsequently to be a semigroup infinitesimal generator, we consider a compact Kähler manifold. So we will have a compact bicomplex Kähler manifold if and only if the idempotent components are compact Kähler manifolds.

For pointwise gradient estimates results and poinwise semigroup domination inequalities on E-valued (r,k)-forms over a complete Kähler manifold X we refer to Li [23], while the Kodaira Laplacian is essentially a Schrödinger type operator acting on $C^\infty\left(\Omega^{r,k}T^*X \otimes E\right)$, where E is a holomorphic Hermitian vector bundle on X.

Therefore, after the above, and due to the linearity of the operators, we write in the idempotent representation: $d = e_1 d_1 + e_2 d_2 = e_1(\partial_1 + \bar{\partial}_1) + e_2(\partial_2 + \bar{\partial}_2)$ and $d^* = e_1 d_1^* + e_2 d_2^* = e_1(\partial_1^* + \bar{\partial}_1^*) + e_2(\partial_2^* + \bar{\partial}_2^*)$,

Also, $\Delta_{\bar{\partial}} = e_1 \Delta_{\bar{\partial};1} + e_2 \Delta_{\bar{\partial};2} = e_1\left(\bar{\partial}_1\bar{\partial}_1^* + \bar{\partial}_1^*\bar{\partial}_1\right) + e_2\left(\bar{\partial}_2\bar{\partial}_2^* + \bar{\partial}_2^*\bar{\partial}_2\right)$ and $\Delta_{\partial} = e_1\Delta_{\partial;1} + e_2\Delta_{\partial;2} = e_1\left(\partial_1\partial_1^* + \partial_1^*\partial_1\right) + e_2\left(\partial_2\partial_2^* + \partial_2^*\partial_2\right)$.

Consequently, we consider the best possible situation, so that the compact bicomplex Kähler manifold X is written in the idempotent representation $X = e_1 X_1 + e_2 X_2$, where $X_1 = e_1 X$, $X_2 = e_2 X$ are compact Kähler manifolds.

2.4. Bicomplex Banach and Hilbert Modules

For the concept of topological bicomplex modules we refer to Kumar and Saini [20], and for standard results of functional analysis with bicomplex scalars to Saini et al. in [35].

Let X be a $\mathbb{BC}$-module with the idempotent decomposition $X = e_1 X_1 + e_2 X_2$, or equivalents $X = e_1 X + e_2 X$, where $X_1 := e_1 X$ and $X_2 := e_2 X$.

If we assume that X_1 and X_2 are spaces with norm $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$ respectively, then $\|\cdot\|_X : X \to \mathbb{R} : x \to \|x\|_X := \frac{1}{\sqrt{2}}\sqrt{\|x_1\|_{X_1}^2 + \|x_2\|_{X_2}^2}$, where $x = e_1 x_1 + e_2 x_2$, with $x_1 \in X_1$

and $x_2 \in X_2$ is the idempotent representation of $x \in X$, is a norm in X, the so-called Euclidean-type norm in X.

In fact, then it applies $\|\zeta x\|_X \le \sqrt{2}\,\|\zeta\|_{\mathbb{BC}}\,\|x\|_X$, for every $\zeta \in \mathbb{BC}$, for every $x \in X$.

Moreover, X can also be equipped with the hyperbolic-valued norm or $\mathbb{D}$-norm $\|x\|_{\mathbb{D}} = e_1\,\|x_1\|_{X_1} + e_2\,\|x_2\|_{X_2}$, and for every $x \in X$ it holds that $|\|x\|_{\mathbb{D}}| = \|x\|_X$.

Further, we refer to the books [1] and [24] for the standard notions of sequence convergence in terms of the above norm types, as well as for Cauchy sequences.

The following result is well-known (cf. [21]).

Proposition 2.3. *The pair $(X, \|\cdot\|_X)$ is a bicomplex Banach module, if and only if the pairs $\left(X_1, \|\cdot\|_{X_1}\right)$ and $\left(X_2, \|\cdot\|_{X_2}\right)$ are complex Banach spaces.*

Respectively, if we have in X_1 and X_2 the inner products $\langle\cdot,\cdot\rangle_{X_1}$ and $\langle\cdot,\cdot\rangle_{X_2}$, then the formula: $\langle x, y\rangle_X := \langle e_1x_1 + e_2x_2, e_1y_1 + e_2y_2\rangle_X = e_1\,\langle x_1, y_1\rangle_{X_1} + e_2\,\langle x_2, y_2\rangle_{X_2}$, defines an inner product in X.

Then the hyperbolic norm in X is defined as $\|x\|_{\mathbb{D}} = \|e_1x_1 + e_2x_2\|_{\mathbb{D}} = \sqrt{\langle x, x\rangle_X}$, and the real-valued norm $\|x\|_X = \frac{1}{\sqrt{2}}\sqrt{\langle x_1, x_1\rangle_{X_1} + \langle x_2, x_2\rangle_{X_2}} = \frac{1}{\sqrt{2}}\sqrt{\|x_1\|_{X_1}^2 + \|x_2\|_{X_2}^2}$.

Thus, a $\mathbb{BC}$-module X with inner product $\langle\cdot,\cdot\rangle_X$ is said to be a bicomplex Hilbert module if it is complete with respect to the hyperbolic norm induced by inner product square.

The following result is valid (cf. [1]).

Proposition 2.4. *The pair $(X, \langle\cdot,\cdot\rangle_X)$ is a bicomplex Hilbert module, if and only if the pairs $\left(X_1, \langle\cdot,\cdot\rangle_{X_1}\right)$ and $\left(X_2, \langle\cdot,\cdot\rangle_{X_2}\right)$ are complex Hilbert spaces.*

Applying the above, we formulate the following bicomplex Banach modules by analogy with the classical infinitely differentiable functions with a compact support C_c^∞ and Sobolev spaces $W^{m,p}$ of functional analysis.

More specifically we have $C_c^\infty(\mathbb{R}^n, \mathbb{BC})$ is the set of all $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{BC})$ such that $\text{supp}(\varphi)$ is compact, where $\text{supp}(\varphi) := \overline{\{x : \varphi(x) \neq 0\}}$ is the support of φ.

Moreover, for $f : C_c^\infty(\mathbb{R}^n, \mathbb{BC}) \to \mathbb{BC}$, $s \in \mathbb{N}_0^n$, the Schwartz generalized $|s|$-order distributional derivative in the bicomplex setting is $(\partial^s f) : C_c^\infty(\mathbb{R}^n, \mathbb{BC}) \to \mathbb{BC}$, with $(\partial^s f)(\varphi) := (-1)^{|s|} f(\partial^s \varphi)$, for every $s \in \mathbb{N}_0^n$, for every $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{BC})$, where $(\partial^s\varphi)(x) := \frac{\partial^{|s|}\varphi(x)}{\partial x_1^{s_1}\partial x_2^{s_2}\cdots\partial x_n^{s_n}}$, where $s = (s_1, s_2, \ldots, s_n)$, $|s| := s_1 + s_2 + \cdots + s_n$.

Then, for $1 \le p < \infty$ and $m \in \mathbb{N}_0$, we have the bicomplex Banach modules $W^{m,p}(\mathbb{R}^n, \mathbb{BC})$ consisting of all $g \in L^p(\mathbb{R}^n, \mathbb{BC})$ such that exists $(\partial^s g) \in L^p(\mathbb{R}^n, \mathbb{BC})$, for every $s \in \mathbb{N}_0^n$, $|s| \le m$, with the norm

$||g||_{m,p} := \left(\sum_{0\le|s|\le m} ||\partial^s g||_{p;\mathbb{BC}}^p\right)^{\frac{1}{p}} = \left(\sum_{0\le|s|\le m}\int_{\mathbb{R}^n} ||\partial^s g(x)||_{\mathbb{BC}}^p dx\right)^{\frac{1}{p}}$, where $L^p(\mathbb{R}^n, \mathbb{BC}) := \left\{f : \mathbb{R}^n \to \mathbb{BC} : \exists \int_{\mathbb{R}^n} ||f(x)||_{\mathbb{BC}}^p dx < +\infty\right\}$, under the norm

$||f||_{p;\mathbb{BC}} = \left(\int_{\mathbb{R}^n} ||f(x)||_{\mathbb{BC}}^p dx\right)^{\frac{1}{p}}$.

Also, $W^{0,p}(\mathbb{R}^n, \mathbb{BC}) = L^p(\mathbb{R}^n, \mathbb{BC})$, and for $s = (0, 0, \ldots, 0)$, we have $\partial^0 g = g$. For $p = 2$ as usual we write $W^{m,2}(\mathbb{R}^n, \mathbb{BC}) := H^m(\mathbb{R}^n, \mathbb{BC})$ and $W_c^{m,2}(\mathbb{R}^n, \mathbb{BC}) := H_c^m(\mathbb{R}^n, \mathbb{BC})$, for the corresponding bicomplex Hilbert modules.

In the idempotent decomposition, for $\ell = 1, 2$, we write:
$C_c^\infty(\mathbb{R}^n, \mathbb{BC})_\ell := e_\ell C_c^\infty(\mathbb{R}^n, \mathbb{BC}) := C_c^\infty(\mathbb{R}^n, e_\ell\mathbb{BC})$,
$W_\ell^{m,p}(\mathbb{R}^n, \mathbb{BC}) := e_\ell W^{m,p}(\mathbb{R}^n, \mathbb{BC}) := W^{m,p}(\mathbb{R}^n, e_\ell\mathbb{BC})$, and
$L_\ell^p(\mathbb{R}^n, \mathbb{BC}) := e_\ell L^p(\mathbb{R}^n, \mathbb{BC}) := L^p(\mathbb{R}^n, e_\ell\mathbb{BC})$,

so that in the idempotent representation:
$C_c^\infty(\mathbb{R}^n, \mathbb{BC}) = e_1 C_c^\infty(\mathbb{R}^n, e_1\mathbb{BC}) + e_2 C_c^\infty(\mathbb{R}^n, e_2\mathbb{BC})$,

$W^{m,p}(\mathbb{R}^n, \mathbb{BC}) = e_1 W_1^{m,p}(\mathbb{R}^n, \mathbb{BC}) + e_2 W_2^{m,p}(\mathbb{R}^n, \mathbb{BC})$, and
$L^p(\mathbb{R}^n, \mathbb{BC}) = e_1 L_1^p(\mathbb{R}^n, \mathbb{BC}) + e_2 L_2^p(\mathbb{R}^n, \mathbb{BC})$.

We can also consider the corresponding functional bicomplex modules with X instead of $\mathbb{R}^n$ in the left hand sides and $X_\ell = e_\ell X$, $\ell = 1, 2$, in the corresponding right hand sides of the above idempotent decompositions, where $X = e_1 X_1 + e_2 X_2$ is a bicomplex Kähler manifold.

For differential-difference bicomplex we refer to Zharinov [41]. We also cite the papers by Değirmen and Sağir [9] for bicomplex $l^p_{\mathbb{BC}}$ sequence spaces, and Eryilmaz [12], [13] (and references therein) on bicomplex versions of function spaces and in particular $L^p_{\mathbb{BC}}$ bicomplex Lebesgue spaces.

For more details, corresponding results in bicomplex Hilbert modules, the bicomplex vesrion of the parallelogram law and for bicomplex polarizarion identities we refer to the book of Alpay et al. in [1] (especially Chapters 3 and 4) and the article of Kumar and Singh in [21].

2.5. Bicomplex groups of operators

Let $(X, \|\cdot\|_X)$ be a bicomplex Banach module and let $\{T_t\}_{t\in\mathbb{R}}$ be a family of $\mathbb{BC}$-linear and $\mathbb{D}$-bounded operators, where $T_t : X \to X : x \to T_t(x)$. Then, the definition of a bicomplex (C$_0$)-group $\{T_t\}_{t\in\mathbb{R}}$ and its infinitesimal generator A is typically the same as in the classical case (cf.[17] and [40]).

Furthermore, considering the idempotent representation of the linear operator T_t, *i.e.* for $-\infty < t < \infty$ writing $T_t = e_1 (T_1)_t + e_2 (T_2)_t$, where $(T_\ell)_t : e_\ell X \to e_\ell X$ with $(T_\ell)_t(x) := e_\ell T_t(e_\ell x)$, for $\ell = 1, 2$ are linear operators respectively, the conditions of a bicomplex (C$_0$)-group (or of class (C$_0$) or strongly continuous group) are reworded in the idempotent components as follows: $(T_\ell)_t \circ (T_\ell)_s = (T_\ell)_{t+s}$, $\forall -\infty < t, s < \infty$, $\text{s}-\lim_{t\to 0} (T_\ell)_t(x) = (T_\ell)_0(x) = x$, $\forall x \in e_\ell X$, and $(T_\ell)_0 = I_\ell$, for $\ell = 1, 2$ respectively, where I_1 the identity operator on $e_1 X$ and I_2 the identity operator on $e_2 X$.

Also, for the infinitesimal generator A, if the limits exist, we will have the idempotent representation

$$A = e_1 A_1 + e_2 A_2,$$

with the idempotent components

$$A_\ell := \text{s}-\lim_{h\to 0^+} h^{-1}\left((T_\ell)_h - I_\ell\right)$$

defined pointwise. As usual, the corresponding domains of definition will be the sets $\mathrm{D}(A_\ell)$, consisting of all elements $x \in e_\ell X$ such that the limit $\lim_{h\to 0^+} h^{-1}\left((T_\ell)_h(x) - I_\ell(x)\right)$ exists, for $\ell = 1, 2$ respectively. Further, we will have that $\mathrm{D}(A) = e_1 \mathrm{D}(A_1) + e_2 \mathrm{D}(A_2)$.

We also refer to our paper [18] where we have studied linear operators and semigroups of linear operators on bicomplex Banach modules, with results on the mean ergodic theorem.

At this point we recall the norm of a bounded bicomplex linear operator T and for standard concepts and results we also refer in [1].

Also, the operator T is called $\mathbb{D}$-bounded, if there is a $\Lambda \in \mathbb{D}^+$ so that for every $x \in X$ it holds, $\|T(x)\|_{\mathbb{D}} \preceq \Lambda \|x\|_{\mathbb{D}}$.

The $\mathbb{D}$-infimum with respect to these Λ is called the $\mathbb{D}$-valued norm of the operator T, and is denoted by $\|T\|_{\mathbb{D}}$. It follows that the operator T is $\mathbb{D}$-bounded if and only if the operators T_1 and T_2 are bounded.

Then, combining and applying the known results for bounded and bicomplex linear operators (cf. [1], p. 75), we have that for the case of a bicomplex group $\{T_t\}_{t\in\mathbb{R}}$ of class

(C_0) yields $\|T_t\|_{\mathbb{D}} = e_1 \|(T_1)_t\|_1 + e_2 \|(T_2)_t\|_2$, where $\|(T_1)_t\|_1$ and $\|(T_2)_t\|_2$ are the usual norms of the idempotent components $(T_1)_t$ and $(T_2)_t$ respectively.

3. Solution for the wave equation in the bicomplex setting

3.1. The case of bicomplex-valued wave equation

Following the usual practice (cf. Yosida [40], Pazy [29]), denoting by O and I the zero and identity operators, we observe that also in the bicomplex case the wave equation (1) is rewritten equivalently in the form:

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} O & I \\ \Delta & O \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \; x \in \mathbb{R}^n, \; t > 0, \tag{2}$$

where $v := \frac{\partial u}{\partial t}$, with the initial condition $\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$.

Then, in the idempotent representation for the problem (2), writing $u = e_1u_1 + e_2u_2$, $f = e_1f_1 + e_2f_2$ and $g = e_1g_1 + e_2g_2$, and due to the linearity of the operators, we will have:

$$\frac{\partial}{\partial t}\begin{pmatrix} e_1u_1 + e_2u_2 \\ e_1v_1 + e_2v_2 \end{pmatrix} = \begin{pmatrix} e_1O_1 + e_2O_2 & e_1I_1 + e_2I_2 \\ e_1\Delta_1 + e_2\Delta_2 & e_1O_1 + e_2O_2 \end{pmatrix}\begin{pmatrix} e_1u_1 + e_2u_2 \\ e_1v_1 + e_2v_2 \end{pmatrix},$$

or equivalent

$$e_1\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + e_2\frac{\partial}{\partial t}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = e_1\begin{pmatrix} O_1 & I_1 \\ \Delta_1 & O_1 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + e_2\begin{pmatrix} O_2 & I_2 \\ \Delta_2 & O_2 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}.$$

Thus,

$$\frac{\partial}{\partial t}\begin{pmatrix} u_\ell \\ v_\ell \end{pmatrix} = \begin{pmatrix} O_\ell & I_\ell \\ \Delta_\ell & O_\ell \end{pmatrix}\begin{pmatrix} u_\ell \\ v_\ell \end{pmatrix}, \tag{3}$$

where $v_\ell = \frac{\partial u_\ell}{\partial t}$, with the initial condition $\begin{pmatrix} u_\ell(x,0) \\ v_\ell(x,0) \end{pmatrix} = \begin{pmatrix} f_\ell(x) \\ g_\ell(x) \end{pmatrix}$, $\Delta_\ell = e_\ell\Delta$, $I_\ell = e_\ell I$, and $O_\ell = e_\ell O$, for $\ell = 1,2$ respectively.

Then, as in the classical case following Yosida [40], Pazy [29], Engel and Nagel [11], we will have in our case as well that an operator A_ℓ can be associated with the differential operator $\begin{pmatrix} O_\ell & e_\ell I \\ \Delta_\ell & O_\ell \end{pmatrix}$, for $\ell = 1,2$ respectively, as follows.

Let $\mathrm{D}(A_\ell) = H^2(\mathbb{R}^n, e_\ell\mathbb{BC}) \times H^1(\mathbb{R}^n, e_\ell\mathbb{BC})$ and for $U_\ell = (f_\ell, g_\ell) \in \mathrm{D}(A_\ell)$ with $A_\ell U_\ell = A_\ell(f_\ell, g_\ell) = (g_\ell, \Delta_\ell f_\ell)$.

Then, we will show that the operator A_ℓ is the infinitesimal generator of a (C_0)-group on the Hilbert space $\mathfrak{H}_\ell = H^1(\mathbb{R}^n, e_\ell\mathbb{BC}) \times L^2(\mathbb{R}^n, e_\ell\mathbb{BC})$ which is the completion of $C_c^\infty(\mathbb{R}^n, e_\ell\mathbb{BC}) \times C_c^\infty(\mathbb{R}^n, e_\ell\mathbb{BC})$ under the norm

$$|||U_\ell|||_\ell := |||(f_\ell, g_\ell)|||_\ell = \left(\int_{\mathbb{R}^n} \left(||f_\ell||^2_{e_\ell\mathbb{BC}} + ||\nabla f_\ell||^2_{e_\ell\mathbb{BC}} + ||g_\ell||^2_{e_\ell\mathbb{BC}}\right) dx\right)^{\frac{1}{2}},$$

for $\ell = 1,2$ respectively.

We start with the next Proposition, which collects the preliminary results, in order to arrive at the infinitesimal generator A_ℓ, for $\ell = 1,2$ respectively.

Proposition 3.1. (i) *Let $\nu > 0$ and $f_\ell \in H^k(\mathbb{R}^n, e_\ell\mathbb{BC})$, $k \geq 0$. Then, there is a unique function $u_\ell \in H^{k+2}(\mathbb{R}^n, e_\ell\mathbb{BC})$ satisfying $u_\ell - \nu\Delta_\ell u_\ell = f_\ell$, for $\ell = 1,2$ respectively.*

(ii) *For every $F_\ell = (\phi_\ell, \theta_\ell) \in C_c^\infty(\mathbb{R}^n, e_\ell\mathbb{BC}) \times C_c^\infty(\mathbb{R}^n, e_\ell\mathbb{BC})$ and real $\lambda \neq 0$ the equation $U_\ell - \lambda A_\ell U_\ell = F_\ell$ has a unique solution $U_\ell = (f_\ell, g_\ell) \in H^k(\mathbb{R}^n, e_\ell\mathbb{BC}) \times H^{k-2}(\mathbb{R}^n, e_\ell\mathbb{BC})$, for every $k \geq 2$. Moreover, $|||U_\ell|||_\ell \leq (1 - 2|\lambda|)^{-1}|||F_\ell|||_\ell$, for $\ell = 1,2$ respectively.*

(iii) *For every* $F_\ell = (\phi_\ell, \theta_\ell) \in H^1(\mathbb{R}^n, e_\ell\mathbb{BC}) \times L^2(\mathbb{R}^n, e_\ell\mathbb{BC})$ *and real* λ *satisfying* $0 < \lambda < \frac{1}{2}$ *the equation* $U_\ell - \lambda A_\ell U_\ell = F_\ell$ *has a unique solution* $U_\ell = (f_\ell, g_\ell) \in H^2(\mathbb{R}^n, e_\ell\mathbb{BC}) \times H^1(\mathbb{R}^n, e_\ell\mathbb{BC})$*, and* $|||U_\ell|||_\ell \leq (1-2\lambda)^{-1}|||F_\ell|||_\ell$*, for* $\ell = 1, 2$ *respectively.*

Proof. (i) Let $\hat{f}_\ell = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} f_\ell(x)dx$ be the Fourier transform of f_ℓ and $\tilde{u}_\ell(\xi) = (1+\nu|\xi|^2)^{-1}\hat{f}_\ell$. Since $f_\ell \in H^k(\mathbb{R}^n, e_\ell\mathbb{BC})$ and $(1+|\xi|^2)^{\frac{k}{2}}\hat{f}_\ell \in L^2(\mathbb{R}^n, e_\ell\mathbb{BC})$ follows $(1+|\xi|^2)^{\frac{k+2}{2}}\hat{u}_\ell \in L^2(\mathbb{R}^n, e_\ell\mathbb{BC})$. Then, for $u_\ell(x) := (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi}\tilde{u}_\ell(\xi)d\xi$ follows that $u_\ell \in H^{k+2}(\mathbb{R}^n, e_\ell\mathbb{BC})$ and u_ℓ is a solution of $u_\ell - \nu\Delta_\ell u_\ell = f_\ell$, for $\ell = 1, 2$ respectively.

(ii) Let $\lambda \neq 0$ be real and $w_{\ell;1}$, $w_{\ell;2}$ be solutions of $w_{\ell;1} - \lambda^2\Delta_\ell w_{\ell;1} = \phi_\ell$ and $w_{\ell;2} - \lambda^2\Delta_\ell w_{\ell;2} = \theta_\ell$, for $\ell = 1, 2$ respectively.Then, $w_{\ell;1}, w_{\ell;2} \in H^k(\mathbb{R}^n, e_\ell\mathbb{BC})$ for every $k \geq 0$, and setting $f_\ell = w_{\ell;1} + \lambda w_{\ell;2}$ and $g_\ell = w_{\ell;2} + \lambda\Delta_\ell w_{\ell;1}$ we verify that $U_\ell = (f_\ell, g_\ell)$ is a solution of $U_\ell - \lambda A_\ell U_\ell = F_\ell$. Hence, $f_\ell - \lambda g_\ell = \phi_\ell$ and $g_\ell - \lambda\Delta_\ell f_\ell = \theta_\ell$.

Moreover, $U_\ell \in H^k(\mathbb{R}^n, e_\ell\mathbb{BC}) \times H^{k-2}(\mathbb{R}^n, e_\ell\mathbb{BC})$, for every $k \geq 2$, and for $\langle\langle\langle \cdot, \cdot \rangle\rangle\rangle_{L^2(\mathbb{R}^n, e_\ell\mathbb{BC})}$ the scalar product in $L^2(\mathbb{R}^n, e_\ell\mathbb{BC})$ we have:

$$|||F_\ell|||_\ell = \langle\langle\langle \phi_\ell - \Delta_\ell\phi_\ell, \phi_\ell \rangle\rangle\rangle_{L^2(\mathbb{R}^n, e_\ell\mathbb{BC})} + \langle\langle\langle \theta_\ell, \theta_\ell \rangle\rangle\rangle_{L^2(\mathbb{R}^n, e_\ell\mathbb{BC})}$$

$$\begin{aligned}
&= \langle\langle\langle f_\ell - \lambda g_\ell - \Delta_\ell f_\ell + \lambda\Delta_\ell g_\ell, f_\ell - \lambda g_\ell \rangle\rangle\rangle_{L^2(\mathbb{R}^n, e_\ell\mathbb{BC})} \\
&\quad + \langle\langle\langle g_\ell - \lambda\Delta_\ell f_\ell, g_\ell - \lambda\Delta_\ell f_\ell \rangle\rangle\rangle_{L^2(\mathbb{R}^n, e_\ell\mathbb{BC})} \\
&\geq \langle\langle\langle f_\ell - \Delta_\ell f_\ell, f_\ell \rangle\rangle\rangle_{L^2(\mathbb{R}^n, e_\ell\mathbb{BC})} + ||g_\ell||^2_{0,2;\ell} - 2|\lambda|\mathrm{Re}(\langle\langle\langle f_\ell, g_\ell \rangle\rangle\rangle_{L^2(\mathbb{R}^n, e_\ell\mathbb{BC})}).
\end{aligned}$$

Finally, for $0 < \lambda < \frac{1}{2}$ follows $|||F_\ell|||^2_\ell \geq (1-2|\lambda|)^2|||U_\ell|||^2_\ell$, that is, it holds $|||U_\ell|||_\ell \leq (1-2|\lambda|)^{-1}|||F_\ell|||_\ell$.

(iii) For every $F_\ell = (\phi_\ell, \theta_\ell) \in H^1(\mathbb{R}^n, e_\ell\mathbb{BC}) \times L^2(\mathbb{R}^n, e_\ell\mathbb{BC})$ and real λ satisfying $0 < \lambda < \frac{1}{2}$ the equation $U_\ell - \lambda A_\ell U_\ell = F_\ell$ has a unique solution $U_\ell = (f_\ell, g_\ell) \in H^2(\mathbb{R}^n, e_\ell\mathbb{BC}) \times H^1(\mathbb{R}^n, e_\ell\mathbb{BC})$, and $|||U_\ell|||_\ell \leq (1-2\lambda)^{-1}|||F_\ell|||_\ell$, for $\ell = 1, 2$ respectively. □

Theorem 3.1. *The operator* A_ℓ *is the infinitesimal generator of a* (C_0)*-group on* $\mathfrak{H}_\ell = H^1(\mathbb{R}^n, e_\ell\mathbb{BC}) \times L^2(\mathbb{R}^n, e_\ell\mathbb{BC})$ *satisfying* $||(T_\ell)_t||_\ell \leq e^{2|t|}$*, for* $\ell = 1, 2$ *respectively.*

Proof. The domain of A_ℓ, $H^2(\mathbb{R}^n, e_\ell\mathbb{BC}) \times H^1(\mathbb{R}^n, e_\ell\mathbb{BC})$ is dense in $\mathfrak{H}_\ell$. Also, $(\mu I_\ell - A_\ell)^{-1}$ exists for $|\mu| > 2$ and holds $||(\mu I_\ell - A_\ell)^{-1}||_\ell \leq \frac{1}{|\mu|-2}$. Hence, A_ℓ (applying Theorem 6.3, p. 23 in [29]) is the infinitesimal generator of a (C_0)-group $(T_t)_\ell$ satisfying $||(T_\ell)_t||_\ell \leq e^{2|t|}$. □

Proposition 3.2. *For every* $f_\ell \in H^2(\mathbb{R}^n, e_\ell\mathbb{BC})$ *and* $g_\ell \in H^1(\mathbb{R}^n, e_\ell\mathbb{BC})$ *there exists a unique* $u_\ell \in C^1([0, +\infty), H^2(\mathbb{R}^n, e_\ell\mathbb{BC}))$ *satisfying the initial value problem* $\frac{\partial^2 u_\ell}{\partial t^2} = \Delta_\ell u_\ell$*,* $u_\ell(x, 0) = f_\ell(x)$*,* $\frac{\partial u_\ell}{\partial t}\Big|_{t=0} = g_\ell(x)$*, for* $\ell = 1, 2$ *respectively.*

Proof. Let $(T_\ell)_t$ be the (C_0)-group generated by A_ℓ and set $(f_\ell(x,t), g_\ell(x,t)) = (T_\ell)_t(\phi_\ell, \theta_\ell)$. Then, $\frac{\partial}{\partial t}(f_\ell, g_\ell) = A_\ell(f_\ell, g_\ell) = (g_\ell, \Delta_\ell f_\ell)$, and thus we have the desired solution. □

In closing, we mention that by standard arguments and Sobolev's Theorem (cf. p. 222; Theorem 4.7 in [29]), if the initial data f_ℓ, g_ℓ in the above initial value problem are smooth for $\ell = 1, 2$ respectively so is the solution.

Finally, we have that the operator $A = e_1A_1 + e_2A_2$ is the infinitesimal generator of a bicomplex (C_0)-group T_t, with $T_t = e_1(T_1)_t + e_2(T_2)_t$ on the bicomplex Hilbert module $\mathfrak{H} = e_1\mathfrak{H}_1 + e_2\mathfrak{H}_2$, and the idempotent combination of the solutions u_ℓ, $\ell = 1, 2$ respectively, i.e. $u := e_1u_1 + e_2u_2$, is the desired unique solution of the original problem on the bicomplex module $C^1([0, +\infty), H^2(\mathbb{R}^n, \mathbb{BC}))$ with idempotent components for $\ell = 1, 2$ $e_\ell C^1([0, +\infty), H^2(\mathbb{R}^n, e_\ell\mathbb{BC}))$.

3.2. The case of the wave equation in compact bicomplex Kähler manifolds

Let X be a compact bicomplex Kähler manifold. Then, we have the wave equation written in the compact bicomplex Kähler manifold X, in the homogeneous form $\frac{\partial^2}{\partial t^2}u(x,t) = \Delta u(x,t),\ \ x \in X,\ \ t \in \mathbb{R}^+$, and with the corresponding conditions we have set.

Equivalently, we then have the idempotent decomposition $\frac{\partial^2}{\partial t^2}u_\ell(x,t) = \Delta_\ell u_\ell(x_\ell,t) = 0,\ \ x_\ell \in X_\ell,\ \ t \in \mathbb{R}^+$, for the homogeneous problem, formulated in the compact Kähler manifolds $X_\ell := e_\ell X$, where $\Delta_\ell = e_\ell \Delta$, for $\ell = 1,2$ respectively.

That is, again we can consider the corresponding forms of equations (2) and (3) for the problem, now formulated in the compact bicomplex Kähler manifold $X = e_1X_1 + e_2X_2$.

Then, corresponding to the previous case, we have the following results.

Proposition 3.3. *The operator A_ℓ is the infinitesimal generator of a* (C_0)*-group on $\mathfrak{H}_\ell = H^1(X_\ell, e_\ell\mathbb{BC}) \times L^2(X_\ell, e_\ell\mathbb{BC})$ satisfying $||(T_\ell)_t||_\ell \leq e^{2|t|}$, for $\ell = 1,2$ respectively.*

Proposition 3.4. *For every $f_\ell \in H^2(X_\ell, e_\ell\mathbb{BC})$ and $g_\ell \in H^1(X_\ell, e_\ell\mathbb{BC})$ there exists a unique $u_\ell \in C^1([0,+\infty), H^2(X_\ell, e_\ell\mathbb{BC}))$ satisfying the initial value problem $\frac{\partial^2 u_\ell}{\partial t^2} = \Delta_\ell u_\ell$, $u_\ell(x,0) = f_\ell(x)$, $\frac{\partial u_\ell}{\partial t}\Big|_{t=0} = g_\ell(x)$, for $\ell = 1,2$ respectively.*

Corollary 3.1. *The operator $A = e_1A_1 + e_2A_2$ is the infinitesimal generator of a bicomplex* (C_0)*-group T_t, with $T_t = e_1(T_1)_t + e_2(T_2)_t$ on the bicomplex Hilbert module $\mathfrak{H} = e_1\mathfrak{H}_1 + e_2\mathfrak{H}_2$, and $u := e_1u_1 + e_2u_2$, is the unique solution of the problem on the bicomplex module $C^1([0,+\infty), H^2(X,\mathbb{BC}))$ with idempotent components $e_\ell C^1([0,+\infty), H^2(X_\ell, e_\ell\mathbb{BC}))$, $\ell = 1,2$.*

In conclusion, under the conditions we have set, we can propose new results of non-trivial and generalized solutions in this case for the wave equation formulated in the bicomplex setting.

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