(SEMI)PRIME BCI-ALGEBRAS: A CLASS OF BCI-ALGEBRAS

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The aim of the present paper is to define the prime BCI-algebra as a generalization of simple BCI-algebras with respect to prime ideals. The notions of semiprime ideals and semiprime BCI-algebras by using prime ideals, and some properties of these concepts are studied. Also we consider some relationship between this ideal and quotient algebras that are construct via this ideal. Finally, we use the concept of radical of ideal in order to construct the relationships between types of BCI-algebras.

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1. Introduction

BCK-algebras and BCI-algebras are two important classes of logical algebras introduced by Iséki in 1966 which have been extensively investigated by several researchers (see [10]). From then, some mathematicians studied and developed many concepts in this algebraic structures, for instance, K. Iséki in 1975 introduced the concept of ideals in BCI-algebras [8]. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. Iseki [8], introduced the concept of prime ideal in commutative BCK-algebras and Palasinski [14], generalized this definition for any lower BCK-semilattices. Then many authors have studied the properties of this ideal in lower BCK-semilattices [1, 11, 12, 13, 14]. They showed that this ideal is one of the most important ideals in lower BCK-semilattices. Any ideal I of a lower BCK-semilattices contained in a prime ideal, has prime and minimal prime decomposition. But prime and irreducible ideals are the same in lower BCK-semilattice. R. A. Borzooei and O. Zahiri generalized the concept of prime ideals for BCI-algebras. They verified some properties of this ideals in BCI-algebras such as the relationship between prime and maximal ideals [6].
In this paper, we present a definition for the semiprime ideal in BCI-algebras based on prime ideals. Also the notion of prime and semiprime BCI-algebras is defined, several characterizations of them are given. The class of prime BCI-algebras is a proper subclass of the class of semiprime BCI-algebras and illustrate also these notions by some examples. We use the notions of prime and semiprime BCI-algebras to develop other concepts such as prime radical in BCK and BCI-algebras, and to discuss further properties of these concepts. The main objectives for the introducing of this concept are to help on greater understanding of this structure and to provide a new way of categorising these algebra. We can also investigate the variety and some subvarieties of this specific type of BCI-algebras. Then we verify some properties of radical and use it for find a relationship between special types of BCI-algebras.

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2. Preliminaries

By a BCI-algebra, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms, for all $x, y, z \in X$ [5, 7],
(BCI1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI2) $(x * (x * y)) * y = 0$,
(BCI3) $x * x = 0$,
(BCI4) $x * y = y * x = 0$ implies $x = y$.
Recall that given a BCI-algebra X, the BCI-ordering $\leq$ on X is defined by $x \leq y$ if and only if $x * y = 0$ for any $x, y \in X$. The set $\mathbf{P} = \{x \in X : 0 * (0 * x) = x\}$ is called P-semisimple part of BCI-algebra X and X is called a P-semisimple BCI-algebra if $\mathbf{P} = X$. The set $\{x \in X : 0*x = 0\}$ is called BCK-part of BCI-algebra X and is denoted by $B(X)$. If $X = B(X)$, then we say X is a BCK-algebra. A BCK-algebra X is called to be a BCK chain if $x \leq y$ or $y \leq x$ for any $x, y \in X$. A BCK-algebra X is said to be a lower BCK-semilattice if X is lower semilattice with respect to BCK-order $\leq$. A BCI-algebra X has the following properties for all $x, y, z \in X$,
(BCI5) $x * 0 = x$,
(BCI6) $(x * y) * z = (x * z) * y$,
(BCI7) $0 * (x * y) = (0 * x) * (0 * y)$,
(BCI8) $x * (x * (x * y)) = x * y$.
If there exists $n \in N$ such that $0 * x^n = 0$, then x is called nilpotent, where $0 * x^n = (...(0 * x) * x) * ...) * x$ and x occurs n times. A BCI-algebra X is called nilpotent, if any $x \in X$ is nilpotent. Both BCK-algebras and finite BCI-algebras are nilpotent BCI-algebras. An ideal I of X is a subset of X such that (i) $0 \in I$ and (ii) $x, y * x \in I$ imply $y \in I$ for any $x, y \in X$. Sometimes, $\{0\}$ is called the zero ideal of X, denoted by O in brevity. A subalgebra Y of X is a nonempty subset of X such that Y is closed under the BCI-operation $*$ on X. If A is both an ideal and a subalgebra of X, we call it a closed ideal of X. An ideal I is called a maximal ideal of X if I is a proper ideal of X and it is not a proper subset of any proper ideal of X. Let I be an ideal of a BCI-algebra X, then the relation θ defined by $(x, y) \in \theta$ if and only if $x * y \in I$ and $y * x \in I$ is a congruence relation on X. We usually denote x/I for $[x] = \{y \in X : (x, y) \in \theta\}$. Moreover, $0/I$ is a closed ideal of BCI-algebra X. In fact, it is the greatest closed ideal contained in I. If I is a closed ideal, then $0/I = I$. Assume that $X/I = \{x/I : x \in X\}$. Then $(X/I, *, 0/I)$ is a BCI-algebra, where $x/I * y/I = (x * y)/I$, for all $x, y \in X$. Let $(X, *, 0)$ and $(Y, ., 0)$ be two BCI-algebras, the map $f : X \longrightarrow Y$ is called a homomorphism, if $f(x * y) = f(x).f(y)$ for all $x, y \in X$. A non zero BCI-algebra X is said to be a simple if O and X are the only ideal in X. A BCI-algebra X is called commutative BCI-algebra if $x \leq y$ implies $x = x \wedge y$, where $x \wedge y = y * (y * x)$. A BCK-algebra X is said to be implicative if $x * (y * x) = x$ for all $x, y \in X$. Let S be a subset of BCI-algebra X. We call the least ideal of X, containing S, the generated ideal of X by S, denoted by $(S]$. If I, J are ideals of X, then we denote $I + J$ by $(I \cup J]$. ([3, 7, 9])
Note: From now on, in this paper, we let $(X, *, 0)$ or simply X be a BCI-algebra, unless otherwise specified.

Definition 2.1. $[3, 7, 10, 13]$ *i) A proper ideal I of X is called an irreducible ideal if $A \cap B = I$ implies $A = I$ or $B = I$, for any ideals A and B of X.*
ii) A proper ideal P of X is called prime if $A \cap B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for all ideals A and B of X. If X is a lower BCK-semilattice, then this definition is equivalent with $x \wedge y \in P$ implying $x \in P$ or $y \in P$.
iii) If M is a maximal ideal of BCK-algebra X, then M is a prime ideal of X.
iv) A commutative BCK-algebra X is said to be cancellative if $x \wedge y = 0$ implying that $x = 0$ or $y = 0$ for any $x, y \in X$.

The set of all ideals of X is denoted by $Id(X)$, the set of all prime ideals of X is denoted by $Spec(X)$, called the spectrum of X (see [2]). A ring of sets is a nonempty set R of subsets of a set S such that if $A, B \in R$, then $A \cup B \in R$ and $A \cap B \in R$. (see [4,5])

Definition 2.2. $[7, 15-20]$ *i) A BCI-algebra X is called normal if for any positive element a of X, the right stabilizer of a is an ideal of X.*
ii) A BCI-algebra X is called J-semisimple if $J(X) = \{0\}$, where $J(X)$ is the intersection of the whole maximal ideals of X.
iii) A non zero BCI-algebra X is called subdirectly irreducible if the intersection of all non zero ideals of X is not equal to the zero ideal.
iv) A BCI-algebra X is said to be nilpotent of type 2 (respectively, solvable) if there exists $n \in \mathbb{N}$ such that $C^n(X) = \{0\}$ (respectively, $C_n(X) = \{0\}$).
v) An element x of X is said to be Engel if $[x,_k y] = 0$ and $[y,_k x] = 0$ for all $y \in X$ and for some $k \in \mathbb{N}$, where $[x,_k y] = [[x,_{k-1} y], y]$. A BCI-algebra X in which all elements are Engel is said to be an Engel BCI-algebra.

Lemma 2.1. [7] *Let I and J be two ideals of BCI-algebra X such that $I \subseteq J$.*
i) J/I is an ideal of X/I.
ii) Let I be closed and J be a prime ideal. Then J/I is a prime ideal of X/I.
iii) Let Y be a BCI-algebra and $f : X \longrightarrow Y$ be an onto BCI-homomorphism. If I is a prime ideal of X contain $Ker(f)$. Then $f(I)$ is a prime ideal of Y.

Theorem 2.1. [7] *X is nilpotent if and only if every ideal I of X is closed.*

Theorem 2.2. [3] *Let X be a BCK-algebra. Then*
i) Any proper ideal I of X can be expressed as the intersection of all prime ideals of X containing I.
ii) For any non zero element x, there exists a prime ideal P such that $x \notin P$.

3. Prime BCI-algebras

In this section, we present a definition for the prime BCI-algebras based on ideals. We show that the prime BCI-algebras are a special class of BCI-algebras, which play an important role in the theory of BCI-algebras and have close contacts with prime ideals.

Definition 3.1. *X is said to be a prime BCI-algebra if the zero ideal is a prime ideal, that is, $I \cap J = O$ implies that $I = O$ or $J = O$, for all proper ideals I and J of X.*

Example 3.1. *i) Let $X = \{0, a, b, c\}$ ($Y = \{0, a, b, c, d\}$) be a BCI-algebra in which $*$ ($*'$) operation is defined by the following table, respectively.*

$*$	0	a	b	c
0	0	0	b	b
a	a	0	b	b
b	b	b	0	0
c	c	b	a	0

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	a	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

By routine calculation, it was observed that $\{0\}$ and $\{0, a\}$ are all proper ideals of X. So, for ideals I, J of X if we have $I \cap J = \{0\}$, then $I = \{0\}$ or $J = \{0\}$. Hence X is a prime BCI-algebra.
Also, routine calculation shows that $\{\{0\}, \{0, c\}, \{0, a, b\}, \{0, a, b, c\}, \{0, a, b, d\}\}$ is the set of all proper ideals of Y. For ideals $I = \{0, a, b\}$ and $J = \{0, c\}$, note that $I \cap J = \{0\}$ but $I \neq \{0\}$ and $J \neq \{0\}$. Hence Y is not a prime BCI-algebra.
ii) The adjoint BCI-algebra $(G, *, e)$ of the Abelian group $(G, ., e)$ with $|G| > 2$, where $x * y =$

$x.y^{-1}$, for $x, y \in G$, is not a prime BCI-algebra. Because $I = \{e, x\}$ and $J = \{e, y\}$ are ideals of G and $I \cap J = \{e\} = O$, but $I, J \neq \{e\} = O$.

We know that simple BCI-algebras do not contain any nontrivial ideals. Therefore, we have the following assertion.

Proposition 3.1. *Every simple BCI-algebra X is a prime BCI-algebra.*

Remark 3.1. *The BCI-algebra $X = \{0, a, b, c\}$ in Example 3.1 is prime but is not simple, because $\{0, a\}$ is a non-trivial ideal of X. So the converse of Proposition 3.1 is not correct in general.*

The following points out that the prime of an ideal P of X can be used to characterize the primity of the quotient algebra X/P.

Proposition 3.2. *A closed ideal P of X is prime if and only if the quotient algebra X/P is a prime BCI-algebra.*

Proof. $\Longrightarrow$) Suppose that P is a prime ideal of X. Let I/P and J/P be ideals of X/P such that $I/P \cap J/P = O/P$. Hence $(I \cap J)/P = O/P$. Therefore $I \cap J \subseteq P$. Since P is a prime ideal of X, $I \subseteq P$ or $J \subseteq P$. If $I \subseteq P$, then $I/P = O/P$ and if $J \subseteq P$, then $J/P = O/P$. Thus X/P is a prime BCI-algebra.
$\Longleftarrow$) Let X/P be a prime BCI-algebra and $\pi : X \longrightarrow X/P$ be canonical homomorphism from X into X/P. If I and J are ideals of X such that $I \cap J \subseteq P$, then $\pi(I)$ and $\pi(J)$ are ideals of X/P such that $\pi(I \cap J) = (I \cap J)/P = I/P \cap J/P = \pi(I) \cap \pi(J)$. Since $I \cap J \subseteq P$, $\pi(I \cap J) = O/P$. But X/P is a prime BCI-algebra, so either $\pi(I) = O/P$ or $\pi(J) = O/P$; that is, $I/P = O/P$ or $J/P = O/P$. Hence $I \subseteq P$ or $J \subseteq P$. □

Corollary 3.1. *The following conditions are equivalent:*
i) X is prime,
ii) The zero ideal O is a prime ideal,
iii) O is an irreducible ideal.

Proof. *i* $\Longleftrightarrow$ *ii*) Since $X \equiv X/O$, by Proposition 3.2, it is clear.
ii $\Longleftrightarrow$ *iii*) Suppose that the zero ideal O is a prime ideal. Let I, J be proper ideals of X such that $I \cap J = O$. So $I \cap J \subseteq O$. Since O is a prime ideal, $I \subseteq O$ or $J \subseteq O$. Therefore $I = O$ or $J = O$. Hence O is a prime ideal of X.
Conversely, if the zero ideal O is an irreducible ideal, then for proper ideals I, J of X such that $I \cap J \subseteq O$ we obtain $I = O$ or $J = O$. Hence O is a prime ideal. □

Proposition 3.3. *i) If X is a BCI-chain, then X is prime.*
ii) X is prime if and only if every subalgebra Y of X is prime.

Proof. i) Let X be a BCI-chain, then for all ideals I and J of X either $I \subseteq J$ or $J \subseteq I$. So $I \cap J = I$ or $I \cap J = J$. Now, if $I \cap J = \{0\}$, then $I = \{0\}$ or $J = \{0\}$. Thus X is a prime BCI-algebra.
ii) As X is a subalgebra of itself, the sufficiency is obvious, and we only need to show the necessity. Assume that I and J are two ideals of the subalgebra Y such that $I \cap J = O$. Since impossible I and J are not ideals of X, we use $< I >$ and $< J >$. As $I \cap J = O$, $< I \cap J >= O$. Hence $< I > \cap < J >= O$. But X is prime so $< I >= O$ or $< J >= O$. Then $I = O$ or $J = O$. That means Y is a prime BCI-algebra. □

Proposition 3.4. *Let f be an isomorphism from X to BCI-algebra $(Y, *^{'}, 0^{'})$. Then X is prime if and only if Y is prime.*

Proof. Suppose that f be an isomorphism from prime BCI-algebra X to BCI-algebra $(Y, *^{'}, 0^{'})$. Let $I^{'}, J^{'}$ be proper ideals of Y such that $I^{'} \cap J^{'} = \{0^{'}\}$. Thus $f^{-1}(I^{'} \cap J^{'}) = f^{-1}(\{0^{'}\}) = \{0\}$.

Since $f^{-1}(I^{'} \cap J^{'}) = f^{-1}(I^{'}) \cap f^{-1}(J^{'})$, $f^{-1}(I^{'}) \cap f^{-1}(J^{'}) = \{0\}$. Then $f^{-1}(I^{'}) = \{0\}$ or $f^{-1}(J^{'}) = \{0\}$, as X is prime and $f^{-1}(I^{'})$ and $f^{-1}(J^{'})$ are ideals of X. If $f^{-1}(I^{'}) = \{0\}$, then $f(f^{-1}(I^{'})) = f(\{0\})$. As f is onto, $f(f^{-1}(I^{'})) = I^{'}$. Therefore $I^{'} = \{0^{'}\}$. Similarity, if $f^{-1}(J^{'}) = \{0\}$, then $J^{'} = \{0^{'}\}$. Hence Y is a prime BCI-algebra.
Conversely, let Y be a prime BCI-algebra and I, J be proper ideals of X such that $I \cap J = \{0\}$. So $f(I \cap J) = f(\{0\})$. Since f is surjective, $f(I \cap J) = f(I) \cap f(J)$. Therefore $f(I) \cap f(J) = f(\{0\}) = \{0^{'}\}$. But Y is a prime BCI-algebra, so $f(I) = f(\{0\})$ or $f(J) = f(\{0\})$. As f is one-to-one, $I = \{0\}$ or $J = \{0\}$. Hence X is prime. □

Theorem 3.1. *A product $\prod_{i\in I} X_i$ of BCI-algebras is prime if and only if X_i is nontrivial for precisely one $i \in I$, and moreover X_i is prime.*

Proof. $\Longrightarrow$) Obviously, If $\prod_{i\in I} X_i$ is nontrivial, Then for some $i \in I$, X_i is nontrivial. We shows that a product of BCI-algebras can only be prime if precisely one factor is nontrivial and prime. Let $\prod_{i\in I} X_i$ be a nontrivial prime BCI-algebra. Suppose that J_i, K_i are ideals of X_i such that $J_i \cap K_i = \{0_i\}$. Hence $\prod_{i\in I}(J_i \cap K_i) = \prod (0_i)_{i\in I} = O$ and so $(\prod_{i\in I} J_i) \cap (\prod_{i\in I} K_i) = O$. As $(\prod_{i\in I} J_i)$ and $(\prod_{i\in I} K_i)$ are ideals of $\prod_{i\in I} X_i$ and $\prod_{i\in I} X_i$ is prime, $\prod_{i\in I} J_i = O$ or $\prod_{i\in I} K_i = O$. If $\prod_{i\in I} J_i = O$, then $J_i = O$, for any $i \in I$, if $\prod_{i\in I} K_i = O$, then $K_i = O$, for any $i \in I$. Hence X_i is prime. Now, let X_i is nontrivial prime BCI-algebra for more then one $i \in I$. We consider subsets $I_i = \{(x_i)_{i\in I}\}$ where $(x_i) = (0, 0, ..., x_i, 0, ...)$ of $\prod_{i\in I} X_i$. Then I_i are proper ideals of $\prod_{i\in I} X_i$ and $\cap I_i = O$, so in this case the $\prod_{i\in I} X_i$ is not prime.
$\Longleftarrow$) Suppose that X_i is a nontrivial BCI-algebra, for precisely one $i \in I$, and moreover X_i is prime. Then $\prod_{i\in I} X_i \cong X_i$. With using Proposition 3.4 we deduce $\prod_{i\in I} X_i$ is prime. □

Proposition 3.5. *Every subdirectly irreducible BCI-algebra is a prime BCI-algebra.*

Proof. Let X be a subdirectly irreducible BCI-algebra, we have $X \neq \{0\}$. Therefore O is a proper ideal of X. Note that the intersection of the whole non zero ideals of X is not equal to $\{0\}$. For any non zero ideals A_1 and A_2 of X since $\bigcap_{i=1}^{n} A_i \subseteq A_1 \cap A_2$ we obtain $A_1 \cap A_2 \neq \{0\}$. Hence X is a prime BCI-algebra. □

Theorem 3.2. *Let I be a closed ideal of X. Then I is a prime ideal if and only if it is the kernel of a homomorphism of X onto a prime BCI-algebra.*

Proof. $\Longrightarrow$) Let I be a closed ideal of X such that is prime. We define $f : X \longrightarrow X/I$ by $f(x) = x/I$ for all $x \in X$. Obvious, f is a homomorphism of X onto BCI-algebra X/I and $Ker(f) = I$. By Proposition 3.2 since I is prime, X/I is a prime BCI-algebra.
$\Longleftarrow$) Let I be the kernel of a homomorphism f of X onto a prime BCI-algebra Y. By first isomorphism theorem we have $X/Ker(f) \cong Im(f)$. Hence $X/I \cong Y$. Then X/I is a prime BCI-algebra, as Y is prime. By Proposition 3.2, I is a prime ideal of X. □

We know that simple BCI-algebras do not from variety. Since every simple BCI-algebra is a prime BCI-algebra. Therefore, we have the following assertion.

Theorem 3.3. *Prime BCI-algebras do not form a variety.*

Definition 3.2. *For $S \subseteq X$, we define*

$$\sigma(S) = \{P \in Spec(X) : S \nsubseteq P\}.$$

Let X be a prime BCI-algebra. Since $O = \{0\} \subseteq I$, for all ideals I of X
$\sigma(O) = \{P \in Spec(X) : O \nsubseteq P\} = \emptyset$ and $\sigma(X) = \{P \in Spec(X) : X \nsubseteq P\} = Spec(X)$. For short denote $\sigma(a)$ instead of $\sigma(\{a\})$. Also $\sigma(I) = \sigma((I])$ and, in particular, $\sigma(a) = \sigma((a])$, for $a \in X$.

Lemma 3.1. *Let A, B are subsets of X and I, J two ideals of X.*
i) If $A \subseteq B$, then $\sigma(A) \subseteq \sigma(B)$.
ii) $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$.
iii) $\sigma(I \cup J) = \sigma(I) \cup \sigma(J)$.

Proof. i) Let $A \subseteq B$ and $P \in \sigma(A)$. Then $A \nsubseteq P$. As $A \subseteq B$, $B \nsubseteq P$. Hence $P \in \sigma(B)$. Thus $\sigma(A) \subseteq \sigma(B)$.
ii) We know that $I \cap J \subseteq I$ and $I \cap J \subseteq J$. Then $\sigma(I \cap J) \subseteq \sigma(I)$ and $\sigma(I \cap J) \subseteq \sigma(J)$. Hence $\sigma(I \cap J) \subseteq \sigma(I) \cap \sigma(J)$.
Conversely, let $P \in \sigma(I) \cap \sigma(J)$. Then $P \in \sigma(I)$ and $P \in \sigma(J)$. So $I \nsubseteq P$ and $J \nsubseteq P$. Since P is a prime ideal, $I \cap J \nsubseteq P$. Therefore $P \in \sigma(I \cap J)$.
iii) Since $I \subseteq I \cup J$ and $J \subseteq I \cup J$, $\sigma(I) \subseteq \sigma(I \cup J)$ and $\sigma(J) \subseteq \sigma(I \cup J)$. Hence $\sigma(I) \cup \sigma(J) \subseteq \sigma(I \cup J)$. Now, if $P \in \sigma(I \cup J)$, then $I \cup J \nsubseteq P$. So at least occurs one of the two states $I \nsubseteq P$ or $J \nsubseteq P$. If $I \nsubseteq P$, then $P \in \sigma(I)$ and if $J \nsubseteq P$, then $P \in \sigma(J)$. Therefore $P \in \sigma(I) \cup \sigma(J)$. □

Lemma 3.2. *For a prime BCI-algebra X the family $T(X) = \{\sigma(I) : I \in Id(X)\}$ satisfies in the following conditions:*
i) $T(X)$ is a ring of sets.
ii) $T(X)$ forms a topology on $Spec(X)$.

Proof. i) It is obvious by Lemma 3.1 parts (ii) and (iii).
ii) The empty set and $Spec(X)$ itself belong to $T(X)$. By Lemma 3.1 part (ii) the intersection of any finite number of members of $T(X)$ belongs to $T(X)$. Also by part (iii) any arbitrary (finite or infinite) union of members of $T(X)$ belongs to $T(X)$. So the ordered pair $(Spec(X), T(X))$ is a topological space of X. □

3.1. Prime BCK-algebras

Theorem 3.4. *i) Let X be implicative BCK-algebra. Then X is prime if and only if X is a chain.*
ii) If X is a finite lower BCK-semilattice and I is an ideal of X such that $|I| = |X| - 1$, then X/I is a prime BCI-algebra.

Proof. i) Let X be an implicative BCK-algebra such that is prime and let $x, y \in X$. By $(x * y) * z = (x * z) * y$, we obtain $(x * y) \wedge (y * x) = (y * x) * ((y * x) * (x * y)) = (y * x) * ((y * (x * y)) * x) = (y * x) * (y * x) = 0$. Since X is prime, $x * y = 0$ or $y * x = 0$.
Conversely, suppose that $x * y = 0$ or $y * x = 0$, for every $x, y \in X$. Let A and B be ideals of X such that $A \cap B = \{0\}$, but $A \neq \{0\}$ and $B \neq \{0\}$. Then there exist non zero elements x, y of X such that $x \in A$ and $y \in B$. Let $x * y = 0$. Then $x * y \in B$ and $y \in B$ implies that $x \in B$. Hence $x \in A \cap B = \{0\}$. Thus $x = 0$ and hence $A = \{0\}$. If $y * x = 0$, then $y * x \in A$ and $x \in A$ implies that $y \in A$. Therefore $y \in A \cap B = \{0\}$. That means $B = \{0\}$. Hence X is prime.
ii) Suppose that $x \wedge y \in I$, for $x, y \in X$ and $x \notin I$. Since $|I| = |X| - 1$, $y \in I$. Hence I is a prime ideal of X. Then by Proposition 3.2, X/I is a prime BCI-algebra. □

In this theorem the condition $|I| = |X| - 1$ is a necessary condition. For instance, if we choose the ideal $I = \{0, a, b\}$ of BCI-algebra $X = \{0, a, b, c, d\}$ in Example 3.1 we have $|I| = 3 \neq |X| - 1 = 4$ and I is not a prime ideal of X, because, for ideals $J = \{0, a, b, c\}$ and $K = \{0, a, b, d\}$ of X we obtain $J \cap K \subseteq I$, but $J, K \nsubseteq I$. By Proposition 3.2, the quotient algebra X/I is not a prime BCI-algebra.

Proposition 3.6. *Any cancellative BCK-algebra is a prime BCI-algebra.*

Proof. Let X be a cancellative BCK-algebra. We show that the zero ideal O is a prime ideal of X. Let $x \wedge y \in \{0\}$, for $x, y \in X$. Then $x \wedge y = 0$ and so by hypothesis, $x = 0$ or $y = 0$. It means that O is a prime ideal of X. □

Remark 3.2. *The BCI-algebra $X = \{0, a, b, c\}$ in Example 3.1 is a prime BCI-algebra which is not a cancellative BCK-algebra. Therefore, the converse of Proposition 3.6 does not hold, in general.*

4. Semiprime ideals in BCI-algebras

As a generalization of prime ideals, we introduce the concept of semiprime ideals in BCI-algebras. Some properties of semiprime ideals in BCI-algebra are studied.

Definition 4.1. *An ideal P of X is called semiprime if it is an intersection of prime ideals of X.*

It is clear that, if P is a prime ideal of X, then it is a semiprime ideal of X. But in the following example we show that O is semiprime, but is not a prime ideal. Also, it is known that intersection of any number of semiprime ideals of X is again a semiprime ideal of X. But the intersection of any number of prime ideals of X need not be a prime ideal of X. The set of all semiprime ideals of X is denoted by $SI(X)$.

Example 4.1. *i) For any propositional logic L with $L = [0, 1]$ and a continuous residuation operation $\rightarrow$ such that $x \rightarrow y = min\{1, 1 - x + y\}\}$, where $x, y \in [0, 1]$, $(L, \rightarrow, 1)$ is a BCI-Logic. If we define $x * y = y \rightarrow x$, then $(L, *, 0)$ is a BCI-algebra with only proper ideal O. Then O is a semiprime ideal of L.*
ii) Let $X = \{0, a, b, c, d, e, f, g\}$ be a BCI-algebra in which $*$ is defined by:

$*$	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Routin calculations shows that O, $I = \{0, a\}$, $J = \{0, d\}$, $K = \{0, a, b, c\}$, $L = \{0, a, d, e\}$ are proper ideals of X. O, I are not prime, but J, K, L are prime. As $O = J \cap K$, then O is a semiprime ideal of X. Also I is not prime, but since $K \cap L = I$, then I is a semiprime ideal of X.

Now, we want to study of the semiprime ideals of BCI-algebras and give some characterizations of these ideals.

Theorem 4.1. *Let I be a closed ideal of X. An ideal P of X containing I is semiprime if and only if P/I is a semiprime ideal of X/I.*

Proof. $\Longrightarrow$) If P is a semiprime ideal of X, then there exist prime ideals $P_1, ..., P_n$ of X such that $P = P_1 \cap P_2 \cap ... \cap P_n$. Since I is a closed ideal of X, by Lemma 2.1, P/I is an ideal of X/I. So $P/I = (P_1 \cap P_2 \cap ... \cap P_n)/I = P_1/I \cap P_2/I \cap ... \cap P_n/I$. Since I is a closed ideal and $P_1, ..., P_n$ are prime ideals of X, $P_1/I, P_2/I, ..., P_n/I$ are prime ideals of X/I. Hence P/I is an intersection of prime ideals of X/I. Therefore P/I is a semiprime ideal of X/I.
$\Longleftarrow$) Let P/I be a semiprime ideal of X/I. Then there exist prime ideals $P_1/I, ..., P_n/I$ of

X/I such that $P/I = P_1/I \cap P_2/I \cap ... \cap P_n/I$. Hence $P/I = (P_1 \cap P_2 \cap ... \cap P_n)/I$. Therefore $P = P_1 \cap P_2 \cap ... \cap P_n$ and hence P is a semiprime ideal of X. □

Proposition 4.1. *A semiprime ideal P is prime if and only if P is irreducible.*

Proof. $\Longrightarrow$) Let P be a semiprime ideal of X such that is a prime ideal. If I and J are ideals of X such that $P = I \cap J$, then $I \cap J \subseteq P$. Since P is a prime ideal of X, $I \subseteq P$ or $J \subseteq P$. If $I \subseteq P$, then $P = I \cap J \subseteq I$ and so in this case $I = P$. Also if $J \subseteq P$, then $P = I \cap J \subseteq J$ hence $J = P$. Thus P is a irreducible ideal.
$\Longleftarrow$) Suppose that P is a semiprime ideal of X such that is irreducible. Then there are prime ideals P_i $(i \in I)$ of X such that $P = \bigcap_{i \in I} P_i$. But P is irreducible, so there exists $i \in I$ such that $P = P_i$. Hence P is a prime ideal of X. □

Proposition 4.2. *Let I, J, K be closed ideals of X such that $J \subseteq I$. If I/J is semiprime ideal of X/J, so is $(I + K)/(J + K)$.*

Proof. Since $J \subseteq I$, we have $I+(J+K) = I+K$ and $J \subseteq I \cap (J+K)$. Therefore $(I+K)/(J+K) = (I + (J + K))/(J + K) \cong I/(I \cap (J + K))$. Applying Third isomorphism theorem and noticing $J \subseteq I \cap (J+K)$, we obtain $I/(I \cap (J+K)) \cong (I/J)/(I \cap (J+K))/J)$. Comparison gives $(I+K)/(J+K) \cong (I/J)/(I \cap (J+K))/J)$. By Theorem 4.1, $(I/J)/(I \cap (J+K))/J)$ is semiprime and so is $(I + K)/(J + K)$. □

In [2] for an ideal I of a lower BCK-semilattice X the *radical* of I is defined and some properties of these concept are studied. In the following definition, we generalize these concept to each BCI algebra X.

Definition 4.2. *Let I be a proper ideal of X. The radical of I, denoted Rad I, is the ideal $\bigcap P$, where the intersection is taken over all prime ideals P which contain I. If the set of prime ideals containing I is empty, then Rad I is defined to be X.*

Note that $Rad(I)$ is an ideal and $I \subseteq Rad(I)$. If I is a prime ideal of X, then $Rad(I) = I$. Despite the inconsistency of terminology, the radical of the zero ideal is sometimes called the nilradical or prime radical of X. We will study it in the next section.

Example 4.2. *i) Let $X = \{0, a, b, c\}$ be the BCI-algebra defined in Example* 4.1 *(ii). For proper ideals $I = \{0, a\}$ and $J = \{0, d\}$ of X, we obtain $Rad(I) = I$ and $Rad(J) = J$. Also $Rad(O) = I \cap K \cap L = O$.*
ii) $(\mathbb{Z}, -, 0)$ *is a BCI-algebra* ([4]). *Simple calculation show that $Rad(< 4 >) =< 2 >$, $Rad(< 8 >) =< 2 >$ and $Rad(O) = O$.*

Theorem 4.2. *Let I and J be ideals of X.*
(1) *If $I \subseteq J$, then $Rad(I) \subseteq Rad(J)$.*
(2) *$Rad(Rad(I)) = Rad(I)$, i.e. radicalization is an idempotent operation.*
(3) *If I and J are ideals of X such that I is closed and J is contains I, then $Rad(J/I) = Rad(J)/I$.*

Proof. (1) Let $I \subseteq J$. Since that the all prime ideals containing J are contain I, the intersection of all prime ideals containing I are contain the intersection of all prime ideals containing J. Therefore $Rad(I) \subseteq Rad(J)$.
(2) We know that $I \subseteq Rad(I)$. Hence by (1) we obtain $Rad(I) \subseteq Rad(Rad(I))$.
Conversely, let $x \in Rad(Rad(I))$. By definition of $Rad(I)$ we have $x \in P$ for all prime ideals containing $Rad(I)$. As $I \subseteq Rad(I)$, $x \in P$ for all prime ideals containing I. Then $x \in Rad(I)$. Hence $Rad(I) \subseteq Rad(Rad(I))$.
(3) Suppose x/I be arbitrary element in $Rad(J/I)$. Therefore $x/I \in P/I$ for all prime ideals P/I of BCI-algebra X/I which contain J/I. Hence $x \in P$ for all prime ideals P of X contain J. Then $x \in Rad(J)$ and hence $x/I \in Rad(J)/I$. So $Rad(J/I) \subseteq Rad(J)/I$.

Conversely, let x/I be an element in $Rad(J)/I$. Then $x \in Rad(J)$. Hence $x \in P$ for all prime ideals P contain J. Therefore $x/I \in P/I$ for all prime ideals P/I contain J/I. Thus $x/I \in Rad(J/I)$. Then $Rad(J)/I \subseteq Rad(J/I)$. □

Theorem 4.3. *Let $\{I_i\}_{i\in J}$ be a family of ideals of X. Then*
(1)$Rad(\bigcap_{i\in J} I_i) = \bigcap_{i\in J} Rad(I_i)$.
(2)$Rad(\bigcup_{i\in J} I_i) = \bigcup_{i\in J} Rad(I_i)$.
(3)If $\{I_i\}_{i\in J}$ is semiprime ideals of X, then $\bigcap_{i\in J} I_i$ is a semiprime ideal of X. Also, if $\bigcup_{i\in J} I_i$ is an ideal of X and $\{I_i\}_{i\in J}$ be semiprime ideals of X, then $\bigcup_{i\in J} I_i$ is a semiprime ideal of X.

Proof. (1) Since $\bigcap_{i\in J} I_i \subseteq I_i$ for any $i \in J$, by (1), $Rad(\bigcap_{i\in J} I_i) \subseteq Rad(I_i)$ for any $i \in J$. Then $Rad(\bigcap_{i\in J} I_i) \subseteq \bigcap_{i\in J}(Rad(I_i))$.
Conversely, let $x \in \bigcap_{i\in J}(Rad(I_i))$. Then $x \in Rad(I_i)$ for any $i \in J$. Thus $x \in P$ for all prime ideals P contain I_i and for any $i \in J$. Hence $x \in P$ for all prime ideals P contain $\bigcap_{i\in J} I_i$. Therefore $x \in Rad(\bigcap_{i\in J} I_i)$.
(2) Since $I_i \subseteq \bigcup_{i\in J} I_i$ for any $i \in J$, $Rad(I_i) \subseteq Rad(\bigcup_{i\in J} I_i)$ for any $i \in J$. Therefore $\bigcup_{i\in J} Rad(I_i) \subseteq Rad(\bigcup_{i\in J} I_i)$.
Conversely, let $x \in Rad(\bigcup_{i\in J} I_i)$. Then $x \in P$ for all prime ideals P contain $\bigcup_{i\in J} I_i$ and for any $i \in J$. Hence $x \in P$ for all prime ideals P containing I_i and for any $i \in J$. Therefore $x \in \bigcup_{i\in J} Rad(I_i)$.
(3) Let $\{I_i\}_{i\in J}$ be a family of semiprime ideals of X. Then for all $i \in J$, $Rad(I_i) = I_i$. Now $Rad(\bigcap_{i\in J} I_i) = \bigcap_{i\in J} Rad(I_i) = \bigcap_{i\in J} I_i$. Then $\bigcap I_i$ is a semiprime ideal of X. Also if $\bigcup_{i\in J} I_i$ is an ideal of X we obtain $Rad(\bigcup_{i\in J} I_i) = \bigcup_{i\in J} Rad(I_i) = \bigcup_{i\in J} I_i$. Then $\bigcup I_i$ is a semiprime ideal of X. □

Theorem 4.4. *An ideal I of X is semiprime if and only if $Rad(I) = I$.*

Proof. $\Longrightarrow$) Assume that $\{P_j : j \in J\}$ is the family of all prime ideals of X. If I is a semiprime ideal of X, then there are prime ideals P_i of X such that $I = \bigcap_{i\in J} P_i$. Therefore $Rad(I) = Rad(\bigcap_{i\in J} P_i) = \bigcap_{i\in J} Rad(P_i) = \bigcap_{i\in J} P_i = I$.
$\Longleftarrow$) Let $I = Rad(I)$. Then $I = \bigcap\{P_i : P_i$ is prime ideal of X containing $I\}$. Hence I is a semiprime ideal of X. □

Corollary 4.1. *Any proper ideal I of a BCK-algebra X is semiprime.*

Proof. By Theorem 2.2 we obtain $Rad(I) = I$ for any proper ideal I of a BCK-algebra X. Then by Theorem 4.4 any proper ideal I is semiprime. □

5. **Semiprime BCI-algebras**

In this section, we define the notion of the semiprime BCI-algebras and show that any BCK-algebra is a semiprime BCI-algebra.

Definition 5.1. *A BCI-algebra X is called semiprime BCI-algebra if the zero ideal O is a semiprime ideal of X.*

Now, we give examples to show that the semiprime BCI-algebra exist.

Example 5.1. *i) In BCI-algebra $(\mathbb{Z}, -, 0)$, if we put $P_1 = \mathbb{N} \cup \{0\}$ and $P_2 = \{-n : n \in \mathbb{N}\} \cup \{0\}$, then P_1, P_2 are maximal ideals of $\mathbb{Z}$. Therefore P_1, P_2 are prime ideals of $\mathbb{Z}$ ([3]). Then $(\mathbb{Z}, -, 0)$ is a semiprime BCI-algebra, because $P_1 \cap P_2 = \{0\}$.*
ii) Let $X = \{0, a, b, c, d\}$ be a BCI-algebra in which $*$ is defined by the following table:
$\{\{0\}, \{0, c\}, \{0, d\}, \{0, a, b\}, \{0, c, d\}, \{0, a, b, c\}, \{0, a, b, d\}\}$ is the set of all proper ideals of X. But only 3 ideals $\{0, c, d\}, \{0, a, b, c\}, \{0, a, b, d\}$ are prime. The zero idea O is semiprime, because $\{0, c, d\} \cap \{0, a, b, c\} \cap \{0, a, b, d\} = \{0\}$. Hence X is semiprime.

$*'$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	a
b	b	a	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

iii) The adjoint BCI-algebra of Klein's four group appear in the following table is not semiprime.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\{\{0\}, \{0, a\}, \{0, b\}, \{0, c\}\}$ *is the set of all proper ideals of* X. *By simple calculation we see that the none of them are not prime ideal. Hence* X *is not semiprime.*

In the following we study relationship between prime and semiprime BCK/BCI-algebras.

Theorem 5.1. *Any prime BCI-algebra is a semiprime BCI-algebra.*

Proof. Let X be a prime BCI-algebra. By definition of prime BCI-algebra the zero ideal O is prime and hence semiprime ideal of X. Thus X is semiprime. □

Remark 5.1. *The BCI-algebra in Example* 4.1 *is semiprime but is not prime, because* O *is semiprime ideal of* X, *but is not a prime ideal. Also, in this example* X *is a semiprime BCI-algebra, which is not a BCK-algebra.*

Corollary 5.1. *Semi prime BCI-algebras do not form a variety.*

Theorem 5.2. *Let* I *be a closed ideal of* X. *Then* X *is semiprime if and only if* X/I *is a semiprime BCI-algebra.*

Proof. $\Longrightarrow$) Let $\{P_j : j \in J\}$ is the set of all prime ideals of X. Suppose that X is semiprime. Then $\bigcap_{j\in J} P_j = O$. Hence $O/I = (\bigcap_{j\in J} P_j)/I = \bigcap_{j\in J}(P_j/I)$. Since I is a closed ideal and P_j for all $j \in J$ are prime ideals of X, P_j/I are prime ideals of X/I. Hence X/I is a semiprime BCI-algebra.
$\Longleftarrow$) Let X/I be a semiprime BCI-algebra, but X is not semiprime. Let $\{P_j : j \in J\}$ be the set of all prime ideals of X and any arbitrary intersection of these ideals is not zero. So $\bigcap_{j\in J} P_j \neq O$. Hence there exists non zero element $x \in X$ such that $x \in P_j$, for all $j \in J$. Therefore $x/I \in P_j/I$, for all $j \in J$. Hence $x/I \in \bigcap_{j\in J} P_j/I$ and so $\bigcap_{j\in J} P_j/I \neq O/I$. This contradicts our assumption and so X is semiprime. □

Corollary 5.2. *If for* $S \subseteq X$, *we define*

$$\sigma(S) = \{P \in SI(X) : S \nsubseteq P\}.$$

For a semiprime BCI-algebra X *the family* $T(X) = \{\sigma(I) : I \in Id(X)\}$ *forms a topology on* $SI(X)$ *and* $T(X)$ *is a ring of sets. Also* $\sigma(O) = \{P \in SI(X) : O \nsubseteq P\} = \emptyset$ *and* $\sigma(X) = \{P \in SI(X) : X \nsubseteq P\} = SI(X)$.

We now introduce the prime radical of a BCI-algebra and we prove that a BCI-algebra X is semiprime if it has zero prime radical (Theorem 5.3). We then develop the analogues of the results proved in the previous section for the radical of ideal and semisimple BCI-algebras. There is a strong analogy between the prime radical, prime ideals, semiprime BCI-algebras, prime BCI-algebras and the simple BCI-algebras respectively.

Definition 5.2. *The intersection of all prime ideals of X is called the prime radical of X, and denoted by $PR(X)$. If X does not contain any prime ideals, we provide $PR(X) = X$.*

So, the smallest of all the semiprime ideals

$$\cap\{P : P \text{ is a prime ideal of } X\}$$

is prime radical of X.
Obviously, $PR(X)$ is an ideal of X, which is not necessarily prime. Also, by Definition 4.2, the radical of zero ideal O is intersection of all prime ideals of X, including O, which is equal to $PR(X)$. So $PR(X) = Rad(O)$.

Example 5.2. *For BCI-algebra in Example 5.1(i) we obtain $PR(X) = \{0, c, d\} \cap \{0, a, b, c\} \cap \{0, a, b, d\} = \{0\}$. Also, for BCI-algebra in Example 5.1(ii), since X is not contains any prime ideal, we obtain $PR(X) = X$.*

Theorem 5.3. *X is semiprime if and only if $PR(X) = \{0\}$.*

Proof. $\Longrightarrow$) Let X be semiprime. Then the zero ideal O is a semiprime ideal of X and hence it is an intersection of prime ideals of X. Thus $PR(X) = \cap\{P : P$ is a prime ideal of $X\} = \{0\}$.
$\Longleftarrow$) If $PR(X) = \{0\}$, then $\cap\{P : P$ is a prime ideal of $X\} = \{0\}$. Hence O is a semiprime ideal of X. So X is semiprime. □

Proposition 5.1. *i) Let X be a nilpotent BCI-algebra. Then $PR(X) \subseteq \mathbf{P}$, where $\mathbf{P}$ is P-semisimple part of X.*
ii) If X is contain at least one prime ideal, then $PR(X) \subseteq J(X)$.

Proof. i) Let $x \notin \mathbf{P}$. Then $0 * (0 * x) \neq x$. But $0 * (0 * x) \leq x$, then $x \nleq 0 * (0 * x)$. Therefore $x * (0 * (0 * x)) \neq 0$. In the other hand, $0 * (x * (0 * (0 * x))) = (0 * x) * (0 * (0 * (0 * x))) = (0 * x) * (0 * x) = 0$ so $x * (0 * (0 * x))$ is a non zero element of BCK-part of X. Hence by Theorem 2.2, there is a prime ideal Q of X such that $x * (0 * (0 * x)) \notin Q$. In this case, we claim that $x \notin Q$. For proof of this claim we let $x \in Q$, then $(x * (0 * (0 * x))) * x = (x * x) * (0 * (0 * x)) = 0 * (0 * (0 * x)) = 0 * x \in Q$. Since X is nilpotent, by Theorem 2.1, Q is closed and so $x * (0 * (0 * x)) \in Q$, which is a contradiction. Therefore $x \notin Q$ for some prime ideal Q and hence $x \notin PR(X)$.
ii) At first by Zorn's Lemma we show that every prime ideal in X is contained in a maximal ideal. Let P be a prime ideal in X and $\mathbf{S}$ be the set of all ideals I such that $P \subseteq I$. $\mathbf{S}$ is nonempty since $P \in \mathbf{S}$. Partially order $\mathbf{S}$ by set theoretic inclusion. Let $\mathcal{C} = \{C_j : j \in J\}$ be an arbitrary chain of prime ideals in $\mathbf{S}$. We put $C = \bigcup_{j \in J} C_j$. We claim that C is a prime ideal. If $x * y, y \in C$, then for some $i, j \in J$, $x * y \in C_i$ and $y \in C_j$. Since $\mathcal{C}$ is a chain, either $C_i \subseteq C_j$ or $C_j \subseteq C_i$, say the latter. Hence $x * y, y \in C_i$. Since C_i is an ideal, $x \in C_i$. Therefore $x * y, y \in C$ imply $y \in C_i \subseteq C$. Consequently, C is an ideal of X. As $\mathcal{C}$ is a chain of prime ideals, there exists $k \in J$ such that $C = C_k$. Hence C is a prime ideal. Since $P \subseteq C_i$ for every $i \in J$, $P \subseteq \bigcup_{i \in J} C_i = C$. Since each C_i is in $\mathbf{S}$, $C \in \mathbf{S}$. Clearly C is an upper bound of the chain $\mathcal{C}$. Thus the hypotheses of Zorn's Lemma are satisfied and hence $\mathcal{C}$ contains a maximal element. But a maximal element of $\mathcal{C}$ is obviously a maximal ideal in X that contains P.
Now, suppose that $x \notin J(X)$. Then there exists maximal ideal M of X such that $x \notin M$. Since every prime ideal is contained in a maximal ideal, there exists prime ideal P such that $x \notin P$. Hence $x \notin PR(X)$. Thus $PR(X) \subseteq J(X)$. □

Theorem 5.4. *Any BCK-algebra is a semiprime BCI-algebra.*

Proof. Let X be a BCK-algebra. Then the P-semisimple part of $X = \{0\}$. Therefore $PR(X) = \{0\}$. □

Proposition 5.2. *If I is a closed ideal of X, then $PR(X/I) = PR(X)/I$. In particular, if $PR(X)$ is closed, then $PR(X/PR(X)) = 0$, whence $X/PR(X)$ is semiprime.*

Proof. Let $x/I \in PR(X/I)$. Then

$$\begin{aligned} x/I \in PR(X/I) &\Longleftrightarrow x/I \in P/I, \quad forallprimeidealsP/IofX/I \\ &\Longleftrightarrow x \in P, \quad foreveryprimeidealPofX \\ &\Longleftrightarrow x \in PR(X) \\ &\Longleftrightarrow x/I \in PR(X)/I \end{aligned}$$

□

Proposition 5.3. *A non zero BCI-algebra X is semiprime if and only if for any non zero element $x \in X$ there is a prime ideal P of X such that $x \notin P$.*

Proof. Suppose that X is semiprime, then $PR(X) = \{0\}$. Since $X \neq \{0\}$, by the definition of $PR(X)$, there is at least a prime ideal of X. Let $\{P_i : i \in I\}$ be the set of all prime ideals of X, then $\bigcap_{i\in I} P_i = PR(X) = \{0\}$. Therefore, for any non zero element $x \in X$ we have $x \notin \bigcap_{i\in I} P_i$. So there is $i \in I$ such that $x \notin P_i$.
Conversely, for any non zero element $x \in X$, letting P_x be a prime ideal of X such that $x \notin P_x$, we have $PR(X) \subseteq \bigcap_{x\in X-\{0\}} P_x = \{0\}$, then $PR(X) = \{0\}$. Hence X is semiprime. □

Proposition 5.4. *X is semiprime if and only if every subalgebra Y of X is semiprime.*

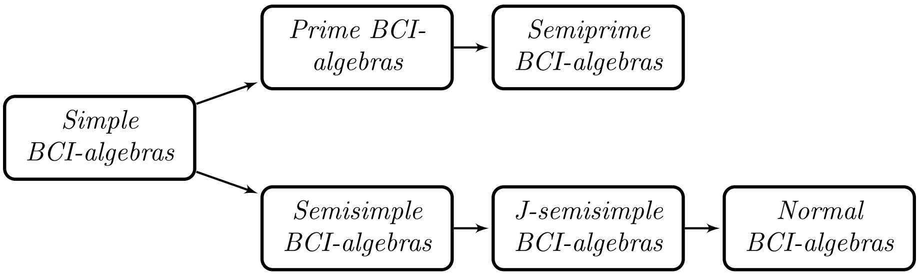
Proof. Since X is a subalgebra of itself, the sufficiency is obvious, and we only need to show the necessity. Let Y be any subalgebra of X. There is no harm in assuming $Y \neq \{0\}$. For any non zero element $x \in Y$ as X is semiprime, then by Proposition 5.3 there exists a prime ideal P of X such that $x \notin P$. So $x \notin Y \cap P$. Also, by routine verification $Y \cap P$ is a prime ideal of Y. Now, an application of Proposition 5.3 to Y gives that Y is semiprime. □

In the following remark we study relationship between prime and semiprime BCI-algebras with normal, semisimple, J-semisimple, solvable, nilpotent of type 2 and Engel BCI-algebras.

Remark 5.2. 1) [4] *The BCI-algebra $X = \{0, a, b, c\}$ in Example* 3.1 *is prime and semiprime, but is not normal and hence is neither J-semisimple and nor semisimple.*
2) *The BCI-algebra X in Example* 4.1 *is semisimple, J-semisimple and normal, but is not prime.*
3) *The BCI-algebra X in Example* 5.1(*iii*) *is semisimple, J-semisimple and normal, but is not semiprime.*
4) *The BCI-algebra X in Example* 5.1(*iii*) *is solvable, nilpotent of type 2 and Engel* [17], *but is neither semiprime nor prime.*

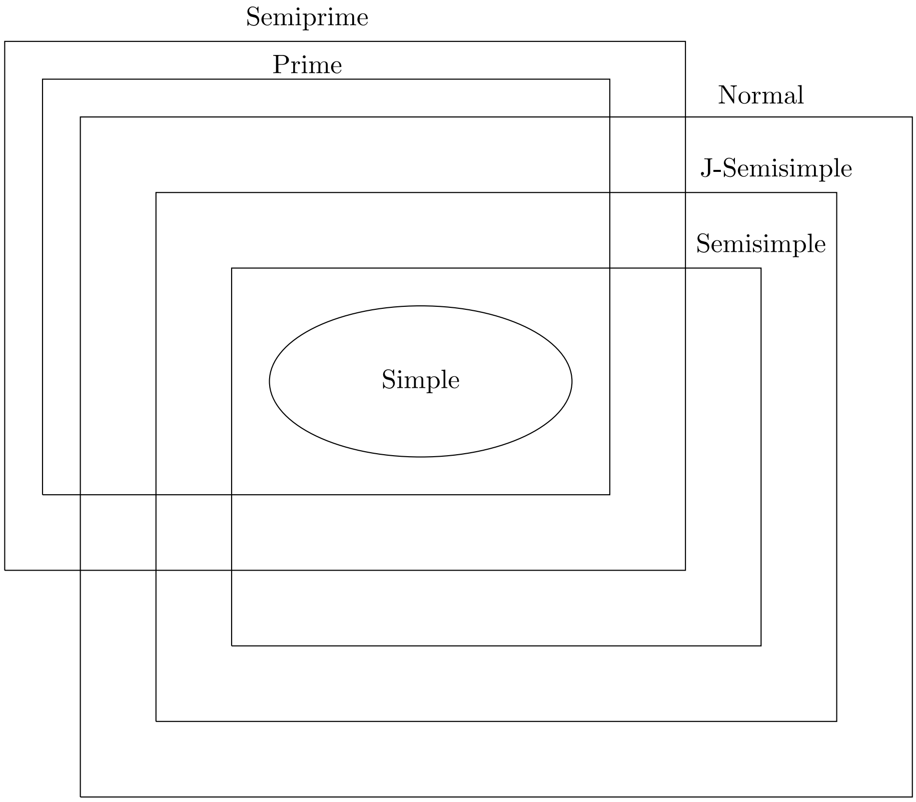
In general, the homomorphic image of a (Semi)prime BCI-algebra is not a BCI-algebra. Thus (Semi)prime BCI-algebras do not form a variety (for more details see Example 5.8 of [14]).

Corollary 5.3. *Putting Theorem* 5.1, *Remarks* 5.1 *and* 5.2 *together. Therefore so far we have clarified various relations among the six types of BCI-algebras described in the following diagram, for the whole BCI-algebras*



6. **Conclusions**

The results of this paper are be devoted to study prime ideals in BCI-algebras. We presented a characterization and several properties of the prime and semiprime ideals in BCI-algebras. Anyway, we also note that new fields like radical theory could find in this framework the more appropriate ground where to develop. Also, in this paper we consider the relation between some kinds of BCI-algebras and giving the following diagram.



Some important topics for future work are:
i) Determine relationships between prime (and also semiprime) BCI-algebras and other types of BCI-algebras.
ii) Checking the conditions under which the inverse of relations in diagram Corollary 5.3 are confirmed could be interesting subject for studies.
iii) Study topological properties of prime (and also semiprime) BCI-algebras X and $Spec(X)$.

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