ON THE KLEIN-GORDON EQUATION IN GRAVITATIONAL FIELD OF A MASSIVE POINT

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We consider for the Klein-Gordon equation in gravitational field of massive point source in general relativity. Using the D'Alembert operator and separation of variables, we work on the complicated differential equation governing the radial component. We report the quasi-exact analytical solutions by working on a corresponding Riccati differential equation. We also provide the numerical solutions via the Galerkin method.

Keywords: Klein-Gordon equation, gravitational field, quasi-exact solution, Galerkin method.

1. Introduction

Finding an acceptable consistent unification of theories of gravity and quantum mechanics has been an outstanding challenge in theoretical physics [1,2]. Although the related studies began many years ago, we have not been yet provided with a solid theory. Nevertheless, there have been motivating clues which connect the two theories. Perhaps, the most simple and primary example which might come into mind is the effect of gravitational field on the spectrum of a quantum particle [3,4]. The possible quantum effects on neutrons in earth's gravitational field were analyzed in Refs. [5,6]. Till now, various equations of quantum mechanics, both in nonrelativistic and relativistic regimes, have been considered in this field [1,2,7]. In particular, the study of Klein-Gordon equation in the gravitational field of a massive point was done in the interesting and instructive paper of Fiziev et al. [8] where they reported novel discrete spectra for Klein-Gordon test particles in the gravitational field of massive point. In our work, we first review the essential formulae from the work of Fiziev et al. [8] to preserve the continuity of the manuscript. Next, to solve the resulting radial equation and instead of working on the numerical basis, we introduce a quasi-exact solution.

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2. The Klein-Gordon Equation is Gravitational Field of a Massive Point

The regular solutions for the gravitational field outside of the massive point source can be obtained from [8]

$$ds^2 = e^{2\varphi_G}\left[dt^2 - \frac{dr^2}{N_G(r)^4}\right] - \rho(r)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),\tag{1}$$

where the radial variable lies in the interval $r \in (0, \infty)$ and

$$\varphi_G(r; M, M_0) := -\frac{G_N M}{r + G_N M \,/\ln\left(M/M_0\right)},\tag{2}$$

is the modified Newton potential and

$$N_G(r) = \left(2\varphi_G\right)^{-1}\left(e^{2\varphi_G} - 1\right).\tag{3}$$

The Hilbert luminosity variablemm is defined via [8]

$$\rho(r) = \frac{2G_N M}{1 - e^{2\varphi_G}} = \frac{r + G_N M \,/\ln\left(M_0/M\right)}{N_G(r)},\tag{4}$$

where G_N is the gravitational constant. In Hilbert space, for the outside region, the solution possesses the form

$$g = g_{tt}(\rho) = 1 - \rho_G/\rho, g_{\rho\rho}(\rho) = 1 - 1/g_{tt}(\rho),\tag{5}$$

where $\rho_G = 2G_N M$ denotes the Schwarzschild radius. We have to stress that the presence of the matter source forces us to consider this form of the solution only on the physical interval of the luminosity variable, i.e. $\rho \in (\rho_0, \infty)$, where

$$\rho_0 = 2G_N M \,/(1 - \tilde{n}^2) \geq \rho_G,\tag{6}$$

where $\tilde{n}^2$ is the squared mass ratio. We can now write

$$ds^2 = gdt^2 - \rho_G^2\left[\frac{dg^2}{g\left(1-g\right)^4} + \frac{d\theta^2 + \sin^2\theta d\phi^2}{\left(1-g\right)^2}\right].\tag{7}$$

On the other hand, the 4-dimensional D'Alembert operator possesses the form [7,8]

$$\Box = g^{-1}\partial_t^2 - \rho_G^{-2}\left(\left(1-g\right)^4\partial_g\left(g\partial_g\right) + \left(1-g\right)^2\Delta_{\theta\phi}\right),\tag{8}$$

where

$$\Delta_{\theta\phi} = \sin^{-1}\theta\partial_\theta\left(\sin\theta\partial_\theta\right) + \sin^{-2}\theta\partial_\phi^2,\tag{9}$$

and the Klein-Gordon equation is neatly written as

$$\left(g^{-1}\partial_t^2 - \rho_G^{-2}\left((1-g)^4\,\partial_g\left(g\,\partial_g\right) + (1-g)^2\sin^{-1}\theta\,\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \sin^{-2}\theta\,\partial_\phi^2\right)\right)\Phi$$
$$+m^2\Phi = 0. \tag{10}$$

Introducing [7,8]

$$\Phi(t,g,\theta,\phi) = \Psi_l(t,g)Y_{l,l_z}(\theta,\phi), \tag{11}$$

where $Y_{l,l_z}(\theta,\phi)$ are the standard hyperspherical harmonics satisfying

$$\Delta_{\theta\phi}Y_{l,l_z}(\theta,\phi) = -l(l+1)Y_{l,l_z}(\theta,\phi), \tag{12}$$

with $l = 0,1,2,...$ *and* $l_z = -l,...,0,...,l$, the angular components are separated and we have

$$g^{-1}\partial_t^2\Psi_l - \rho_G^{-2}\left\{(1-g)^4\,\partial_g\left(g\,\partial_g\Psi_l\right) + \left[l(l+1)(1-g)^2 + m^2\right]\Psi_l\right\} = 0. \tag{13}$$

As the final step, we use the well-known solution

$$\Psi_l(t,g) = e^{-iEt}R_l(g), \tag{14}$$

and write the radial equation in the form [8]

$$\frac{d^2R_l}{dg^2} + \frac{1}{g}\frac{dR_l}{dg} + \left[\frac{\varepsilon^2}{g^2(1-g)^4} - \frac{\mu^2}{g(1-g)^4} - \frac{l(l+1)}{g(1-g)^2}\right]R_l = 0, \qquad g \in [g_0, 1] \tag{15}$$

With ε and μ respectively being the dimensionless total energy and the mass of the particle. It should be noted that we have been working in units where $c = \hbar = 1$ and $l = L/m\rho_G$. Also $g_0 = \tilde{n}^2 > 0$. The point $g = 1$ corresponds to the physical infinity (with respect to the variables r, or ρ).

3. The Quasi-Exact Analytical Solution

As Eq. (15) has not analytically solved before, we intend to provide the problem with a quasi-exact analytical solution which provides us with a better insight into the solutions. As the first ste p, let us introduce the gauge transformation

$$R_l = g^{-\frac{1}{2}}U_l(g), \tag{16}$$

to remove the first order derivative;

$$U_l''(g) + \left[\frac{\varepsilon^2}{g^2(1-g)^4} - \frac{\mu^2}{g(1-g)^4} - \frac{l(l+1)}{g(1-g)^2} + \frac{1}{4g^2}\right]U_l(g) = 0. \tag{17}$$

We now use the simple idea of decomposition of fractions and write

$$U_l''(g) + \left[\frac{1}{(1-g)}\left(4\varepsilon^2 - \mu^2 - l(l+1)\right) + \frac{1}{g}\left(4\varepsilon^2 - \mu^2 - l(l+1)\right) \right.$$
$$+ \frac{1}{(1-g)^2}\left(3\varepsilon^2 - \mu^2 - l(l+1)\right) + \frac{1}{(1-g)^3}\left(2\varepsilon^2 - \mu^2\right)$$
$$\left. + \frac{1}{(1-g)^4}\left(\varepsilon^2 - \mu^2\right) + \frac{1}{g^2}\left(\varepsilon^2 - \frac{1}{4}\right) \right] U_l(g) = 0. \tag{18}$$

At this stage, we review the simple but powerful quasi-exact ansatz technique which is based on proposing a solution of the form [9,10]

$$U_l(g) = h(g)\exp(H(g)), \tag{19-a}$$

with

$$h(g) = \begin{cases} 1, & for \quad n = 0 \\ \displaystyle\prod_{i=1}^{n}(g - \alpha_i^n), & for \quad n \geq 1 \end{cases}. \tag{19-b}$$

The ansatz technique has provided us with analytical solutions to various differential equations of mathematical physics including the Schrödinger, semi-relativistic spinless Salpeter, and relativistic Dirac, Klein-Gordon and Duffin-Kemmer-Petiau (DKP) equations [9, 10] in many examples where powerful techniques such as supersymmetry quantum mechanics (SUSY), factorization, Nikiforov-Uvarov (NU) and Lie groups cannot help us. As the first part of the solution, we consider the case of $h(g) = 1$. In this case, the resulting Riccati equation gives the term in the exponent as

$$H = \alpha \ln g + \beta \ln(1-g) + \frac{\gamma}{(1-g)}, \tag{20}$$

which yields

$$U_l''(g) = \left[\frac{1}{g}\left[-2\alpha(\beta-\gamma)\right] + \frac{1}{g^2}\left[\alpha(\alpha-1)\right] \right.$$
$$+ \frac{1}{(1-g)}\left[-2\alpha(\beta-\gamma)\right] + \frac{1}{(1-g)^2}\left[\beta(\beta+1)+2\alpha\gamma\right]$$
$$\left. + \frac{1}{(1-g)^3}\left[-2\gamma(\beta+1)\right] + \frac{1}{(1-g)^4}\left(\gamma^2\right) \right]. \tag{21}$$

Equating the corresponding powers of Eqs. (21) and (18), gives

$$4\varepsilon^2 - \mu^2 - l(l+1) = 2\alpha(\beta-\gamma),$$
$$3\varepsilon^2 - \mu^2 - l(l+1) = -\beta(\beta+1) - 2\alpha\gamma,$$
$$2\varepsilon^2 - \mu^2 = 2\gamma(\beta+1),$$
$$\varepsilon^2 - \mu^2 = -\gamma^2,$$

$$\varepsilon^2 - \frac{1}{4} = -\alpha(\alpha - 1). \tag{22}$$

which can be solved to deter mine the unknown coefficients.

We can obtain the coefficients in term ε, μ and l as

$$\alpha = \frac{1}{2}\left(1 \pm \sqrt{2}\sqrt{1-2\varepsilon^2}\right) > 0 \; ; \quad \beta = \frac{\mu^2 - 2\varepsilon^2}{2\sqrt{\mu^2 - \varepsilon^2}} - 1 \; ; \quad \gamma = -\sqrt{\mu^2 - \varepsilon^2} < 0 \quad . \tag{23}$$

Here $\mu > \varepsilon$ and we take $\alpha = \frac{1}{2}\left(1 + \sqrt{2}\sqrt{1-2\varepsilon^2}\right)$. We have the restriction relations on the coefficients $\varepsilon, \alpha, \beta$ and γ. Due to these restrictions connection of the coefficients, we can plot ε in term one of the coefficients (here β coefficient), where it is shown in Fig (1).
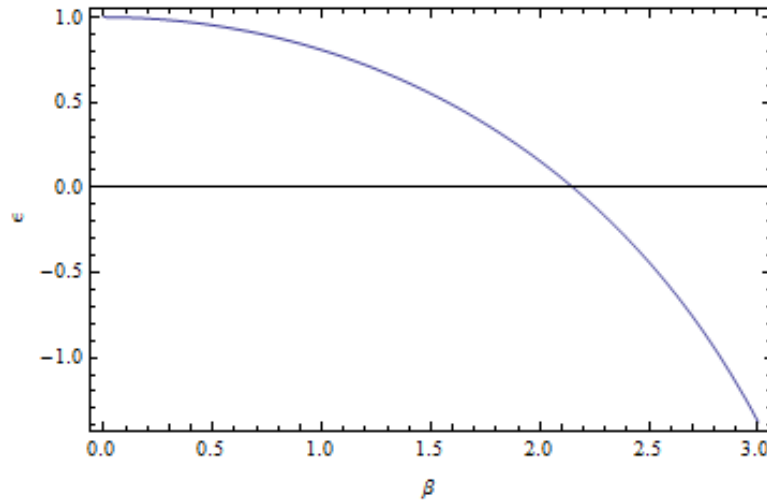

Fig. 1. ε vs. β

For the first excited-state, we have

$$U_l''(g) - \left[H'' + H'^2 + \frac{2Hf' + f''}{f}\right]U_l(g) = 0, \tag{24}$$

which corresponds to

$$H = \alpha \ln g + \beta \ln(1-g) + \frac{\gamma}{(1-g)}, \tag{25}$$

$$f = g - \alpha_1^1, \tag{26}$$

Substitution of the above terms in Eq. (18) gives

$$\left(g-\alpha_1^1\right)U_l''(g)+$$

$$\left[\left(g-\alpha_1^1\right)\left[\frac{1}{g}\left[-2\alpha\left(\beta-\gamma\right)\right]+\frac{1}{g^2}\left[\alpha\left(\alpha-1\right)\right]+\frac{1}{\left(1-g\right)}\left[-2\alpha\left(\beta-\gamma\right)\right]\right.\right.$$

$$+\frac{1}{\left(1-g\right)^2}\left[\beta\left(\beta+1\right)+2\alpha\gamma\right]+\frac{1}{\left(1-g\right)^3}\left[-2\gamma\left(\beta+1\right)\right]+\frac{1}{\left(1-g\right)^4}\left(\gamma^2\right)\right]$$

$$\left.+\frac{2\alpha}{g}-\frac{2\beta}{\left(1-g\right)}+\frac{2\gamma}{\left(1-g\right)^2}\right]U_l(g)=0. \tag{27}$$

Making a comparison between the latter and Eq. (18) gives the set of equations

$$4\varepsilon^2-\mu^2-l(l+1)+2\alpha=2\alpha\left(\beta-\gamma\right),$$

$$-4\varepsilon^2\alpha_1^1+\alpha_1^1\mu^2+\alpha_1^1 l(l+1)+\left(\varepsilon^2-\frac{1}{4}\right)=-2\alpha\alpha_1^1\beta+2\alpha\alpha_1^1\gamma+\alpha-\alpha^2,$$

$$4\varepsilon^2-\mu^2-l(l+1)=2\alpha\left(\beta-\gamma\right),$$

$$-4\varepsilon^2\alpha_1^1+\alpha_1^1\mu^2+\alpha_1^1 l(l+1)-2\beta=-2\alpha\alpha_1^1\beta+2\alpha\alpha_1^1\gamma,$$

$$3\varepsilon^2-\mu^2-l(l+1)=-\beta\left(\beta+1\right)-2\alpha\gamma,$$

$$-3\varepsilon^2\alpha_1^1+\alpha_1^1\mu^2+\alpha_1^1 l(l+1)+2\gamma=\beta\alpha_1^1+\beta^2\alpha_1^1+2\alpha_1^1\alpha\gamma,$$

$$2\varepsilon^2-\mu^2=2\gamma\left(\beta+1\right),$$

$$\varepsilon^2-\mu^2=-\gamma^2,$$

$$\varepsilon^2-\frac{1}{4}=-\alpha\left(\alpha-1\right). \tag{28}$$

which can determine the spectrum of the system.

The coefficients are determined as

$$\alpha=\frac{1}{2}\left(1+\sqrt{2}\sqrt{1-2\varepsilon^2}\right)\quad\alpha>0\ ;\quad\beta=\frac{\mu^2-2\varepsilon^2}{2\sqrt{\mu^2-\varepsilon^2}}-1\ ;\quad\gamma=-\sqrt{\mu^2-\varepsilon^2}\qquad\gamma<0$$

$$\alpha_1^1=\frac{2\beta}{2\alpha\beta-4\varepsilon^2-2\alpha\gamma+l(l+1)+\mu^2}=$$

$$=\frac{2\left(-1+\dfrac{\mu^2-2\varepsilon^2}{2\sqrt{\mu^2-\varepsilon^2}}\right)}{-4\varepsilon^2+l(l+1)+\mu^2-2\left(1+\sqrt{2}\sqrt{1-2\varepsilon^2}\right)\sqrt{\mu^2-\varepsilon^2}-2\left(1+\sqrt{2}\sqrt{1-2\varepsilon^2}\right)\left(-1+\dfrac{\mu^2-2\varepsilon^2}{2\sqrt{\varepsilon^2-\mu^2}}\right)}. \tag{29}$$

Here $\mu > \varepsilon$ and we take $\alpha = \frac{1}{2}\left(1 + \sqrt{2}\sqrt{1 - 2\varepsilon^2}\right)$. Finally, we have depicted the radial wavefunction and probability density by subsisting the coefficients α, β and γ in the Eq. (15) in terms of g for the ground state and the first excited-state in Figs. (2) and (3).
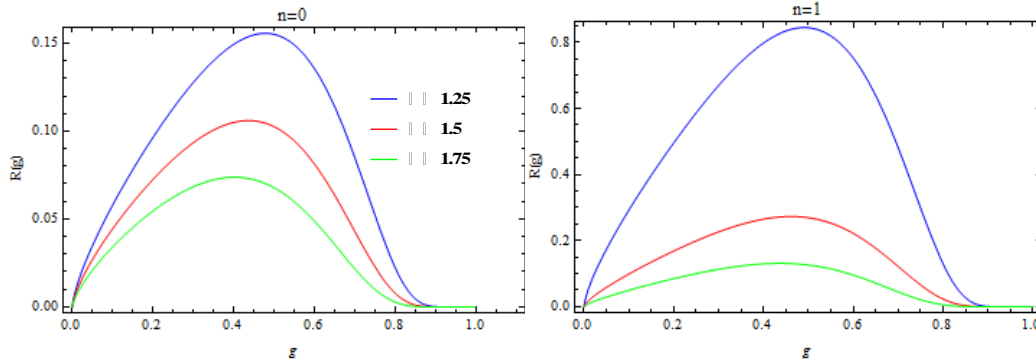

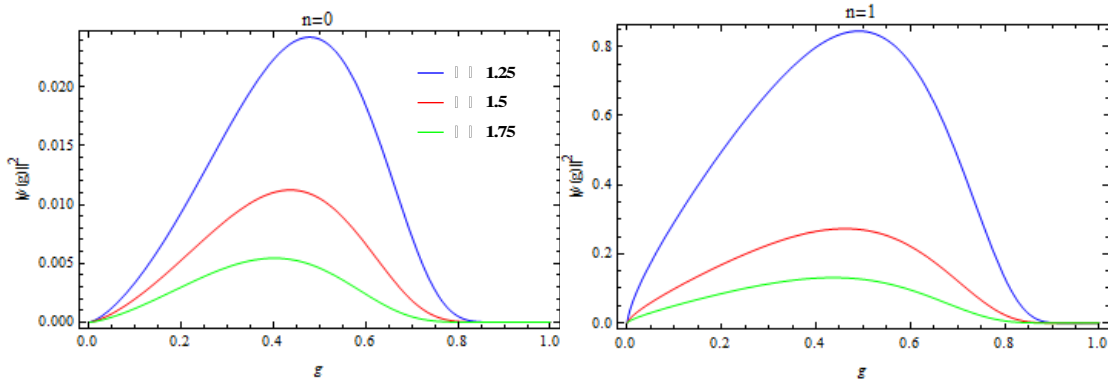Fig. 2. radial component for ground and first excited state

Fig. 3. probability densities for ground and first excited state

4. Numerical solution

Let us now check the validity of the result by a numerical solution. The eigenvalue problems are described by equation of the type [11]

$$LR(g) = \lambda MR(g), \tag{30}$$

where L and M are the differential operators. The problem is to determine the eigenvalues λ and corresponding eigenfunctions $R(g)$. Comparing Eqs. (15) and (30), one can easily find that L, M and λ can be chosen as

$$L = -g^2(1-g)^4 \frac{d^2}{dg^2} - g(1-g)^4 \frac{d}{dg} + \mu^2 g + l(l+1)g(1-g)^2,$$

$$M = 1,$$

$$\lambda = \varepsilon^2. \tag{31}$$

In order to solve Eq. (30), $R(g)$ is usually assumed as

$$R(g) = \sum_{i=1}^{N} a_i u_i, \tag{32}$$

where u_i are functions which satisfy the boundary conditions and a_i are constants. Substituting Eq. (32) into Eq. (30), we have

$$\sum_{i=1}^{N} a_i L u_i = \lambda \sum_{i=1}^{N} a_i M u_i, \tag{33}$$

Choosing the weighting functions w_j and taking the inner product of Eq. (33) with each w_j, we obtain

$$\sum_{i=1}^{N} [<w_j, Lu_i> - \lambda <w_j, Mu_i>]a_i = \sum_{i=1}^{N} (A_{ji} - \lambda B_{ji}) X_i = 0, \tag{34}$$

where $A_{ji} = <w_j, Lu_i>$, $B_{ji} = <w_j, Mu_i>$ and $X_i = a_i$. However in order to have nontrivial X_i solutions, the coefficients determinant must be zero, i.e., $|[A] - \lambda[B]| = 0$. By solving this equation, N approximate eigenvalues can be computed. Using the Galerkin method, $w_j = u_j$, and choosing $u_k = g(1 - g^k)$, which satisfy the boundary conditions, A_{ji} and B_{ji} can be easily obtained as

$$A_{ji} = -\frac{1}{105} + \frac{\mu^2}{4} + \frac{l(l+1)}{30} + \frac{(i+1)^2}{i+7} - \frac{4(i+1)^2 + l(l+1)}{i+6} + \frac{6(i+1)^2 + 2l(l+1)}{i+5} -$$
$$\frac{4(i+1)^2 + \mu^2 + l(l+1)}{i+4} + \frac{(i+1)^2}{i+3} + \frac{1}{j+7} - \frac{4 + l(l+1)}{j+6} + \frac{2(3 + l(l+1))}{j+5} -$$
$$\frac{4 + \mu^2 + l(l+1)}{j+4} + \frac{1}{j+3} - \frac{(i+1)^2}{i+j+7} + \frac{4(i+1)^2 + l(l+1)}{i+j+6} - \frac{6(i+1)^2 + 2l(l+1)}{i+j+5} +$$
$$\frac{4(i+1)^2 + \mu^2 + l(l+1)}{i+j+4} - \frac{(i+1)^2}{i+j+3},$$

$$B_{ji} = \frac{ji(i+j+6)}{3(j+3)(i+3)(i+j+3)}. \tag{35}$$

Choosing $N = 4$, ε and $R(g)$ can be achieved as follows:

$$\varepsilon = 0.92, \quad R(g) = 0.77\,g + 4.83\,g^2 + 3.22\,g^3 - 14.17\,g^4 + 5.35\,g^5,$$

$$\varepsilon = 1.09, \quad R(g) = 2.13\,g + 6.39\,g^2 + 17.74\,g^3 - 76.81\,g^4 + 50.56\,g^5,$$

$$\varepsilon = 10.64, \quad R(g) = 13.18\,g - 21.25\,g^2 - 42.5\,g^3 + 87.13\,g^4 - 36.55\,g^5. \tag{36}$$

It should be noted that all three above relations are normalized, $\int_0^1 |R(g)|^2 = 1$. Another important point is that the solutions have a degeneracy for $\varepsilon = 1.09$. In Figs. (4)-(6), $R(g)$ and $|R(g)|^2$ are plotted as a function of g for three different energies.
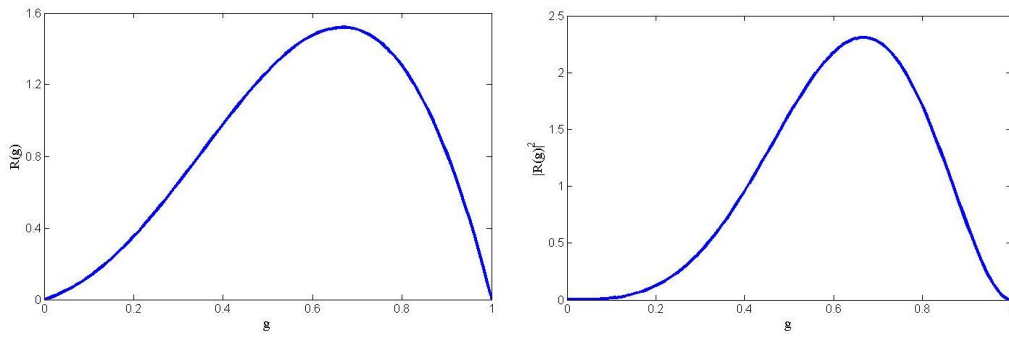

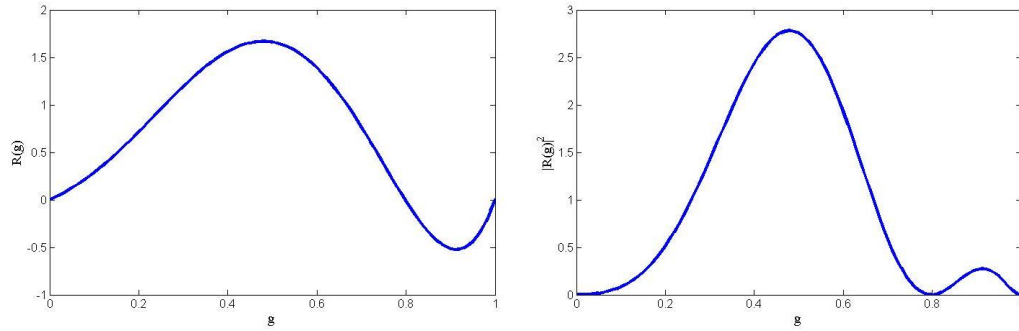Fig. 4. radial component and probability density for $\varepsilon = 0.92$.

Fig. 5. radial component and probability density for $\varepsilon = 1.09$.
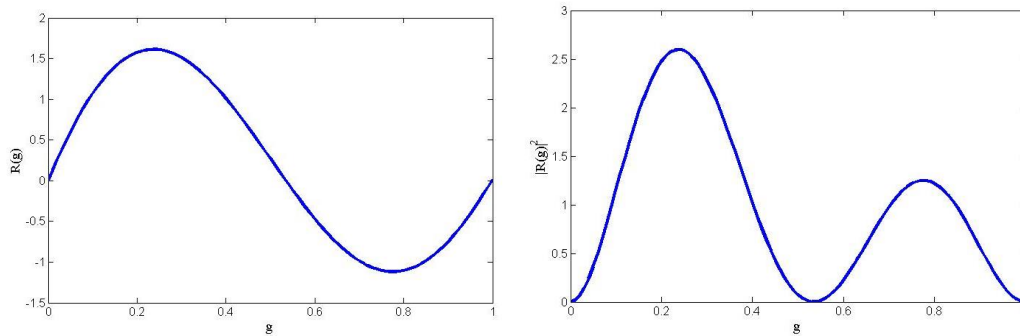

Fig. 6. radial component and probability density for $\varepsilon = 10.64$.

5. Conclusion

We solved the Klein-Gordon equation in the gravitational field of massive point source in general relativity. The arising equation, which was obtained in the interesting paper of Fiziev et al., to our very best knowledge, has not been analytically solved before. Therefore, the authors worked on the equation in numerical background. Here, however, we introduced a quasi-exact analytical solution by using the ansatz technique, which is based on finding the solution of a Riccati-like differential equation. It should be noted that, although we obtained the solution for the first two states, the higher states can be simply obtained by the same token via choosing $h(g)$ as, $(g - \alpha_1^2)(g - \alpha_2^2)$ for the first node, second node, etc. Nevertheless, this idea, just like any other quasi-exact technique, does have its limitations. In particular, finding the solution of the set of obtained equations becomes much complicated in higher states. At the last section, we provided the numerical counterparts obtained from Galerkin method to check the validity of analytical solutions.

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