SOME BEST PROXIMITY POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

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The purpose of this paper is to present some new results on the existence and convergence of best proximity points as well as fixed points for cyclic contractive mappings in a partially ordered metric spaces.

Keywords: ordered metric space; fixed point; best proximity point; cyclic mapping; property UC.

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1. Introduction and Preliminaries

The Banach contraction principle is a basic result in fixed point theory. Several extensions of this principle have been presented by many authors (see for instance [7]). An interesting extension of the Banach contraction principle was studied by Kirk, Srinivasan and Veeramani as follows.

Theorem 1.1. *([8]). Let A and B be two nonempty closed subsets of a complete metric space $X = (X, d)$. Suppose that $T : A \cup B \to A \cup B$ is a cyclic mapping i.e. $T(A) \subseteq B$, $T(B) \subseteq A$, such that*

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1.1}$$

for some $\alpha \in]0, 1[$ and for all $x \in A, y \in B$. Then $A \cap B \neq \emptyset$ and T has a unique fixed point in $A \cap B$.

If in the above theorem $A \cap B = \emptyset$, then we get the notion of best proximity point. A point $p \in A \cup B$ is called a best proximity point for the cyclic mapping T if $d(p, Tp) = dist(A, B)$ where

$$dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Existence and approximation of best proximity points is an interesting topic for which one can see [2, 3, 4, 6, 5] for more information.

Another extension of Banach contraction principle was given by Nieto and Rodriguez-Lopez in partially ordered metric spaces [9]. They proved some fixed point theorems in partially ordered sets in order to show the existence and uniqueness for a first-order ordinary differential equation with periodic boundary conditions

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admitting only the existence of a lower solution. In [1], the current authors established some theorems on the existence and convergence of fixed points, as well as best proximity points for cyclic mappings in partially ordered metric spaces. In this paper we aim to study the existence and convergence of fixed points as well as best proximity points for cyclic mappings; in this way we generalize the results of [1]. For this reason, we need to recall some results from [1]. Let us start with the following definition.

Definition 1.1. *Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a self mapping. We say that T is monotone nondecreasing if*

$$x, y \in X, x \preceq y \Rightarrow T(x) \preceq T(y).$$

Theorem 1.2. *([1]) Let A, B be two nonempty subsets of a metric space (X, d) such that A is complete, and let "$\preceq$" be a partially ordered relation on A. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that T is continuous on A and T^2 is nondecreasing on A and*

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) \tag{1.2}$$

for some $\alpha \in [0, 1[$ and for all $(x, \acute{x}) \in A \times A$ with $x \preceq \acute{x}$. If there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$, then $A \cap B \neq \emptyset$, hence T has a fixed point $p \in A \cap B$. Moreover, if $x_{n+1} = Tx_n$, then $x_{2n} \to p$.

Note that if in previous theorem A has the property that

$$\text{if a nondecreasing sequence } x_n \longrightarrow x \in A, \text{ then } x_n \preceq x \ \forall n, \tag{1.3}$$

we can omit the continuity assumption of T on A (see Theorem 2.3 of [1]). In the next section we state the generalized version of Theorem 1.3 and obtain new results on the existence and approximation of fixed points for generalized contractions.

2. Fixed Point Theorems

In this section we prove two extensions of Theorem 1.1.

Theorem 2.1. *Let A, B be two nonempty subsets of a metric space (X, d) such that A is complete, and let "$\preceq$" be a partially ordered relation on A. Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping such that T is continuous on A and T^2 is nondecreasing on A and*

$$d(T\acute{x}, T^2x) \leq \psi(d(\acute{x}, Tx)), \tag{2.1}$$

for all $(x, \acute{x}) \in A \times A$ with $x \preceq \acute{x}$, where $\psi : \mathbb{R}^+ \to [0, +\infty)$ is upper semi-continuous from the right such that $0 \leq \psi(t) < t$, for all $t > 0$. If there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$, then $A \cap B \neq \emptyset$, so that T has a fixed point $p \in A \cap B$. Moreover if $x_{n+1} = Tx_n$ then $x_{2n} \to p$.

Proof. If $T^2x_0 = x_0$, then

$$d(x_0, Tx_0) = d(T^2x_0, T(T^2x_0)) = d(T(T^2x_0), T^2x_0)$$

$$\leq \psi(d(T^2x_0, Tx_0)) = \psi(d(x_0, Tx_0)).$$

Thus $d(x_0, Tx_0) = 0$ and hence $Tx_0 = x_0$. Suppose that $T^2x_0 \neq x_0$. Since $x_0 \preceq T^2x_0$ and T^2 is nondecreasing on A,

$$x_0 \preceq x_2 \preceq ... \preceq x_{2n} \preceq \cdots .$$

We break the argument into two steps.

Step 1. $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$.

proof. The sequence $\{d(x_{2n}, x_{2n+1})\}$ is monotone decreasing. Indeed
$d(x_{2n+2}, x_{2n+3}) = d(T(x_{2n+2}), T^2(x_{2n}))$
$\leq \psi(d(x_{2n+2}, x_{2n+1})) < d(x_{2n+2}, x_{2n+1}) = d(T(x_{2n}), T^2(x_{2n})))$
$\leq \psi(d(x_{2n}, x_{2n+1})) < d(x_{2n}, x_{2n+1})$.
Let $d(x_{2n}, x_{2n+1}) \to r \geq 0$. Assume that $r > 0$. Then

$$r = \lim_{n\to\infty} d(x_{2n+2}, x_{2n+3}) \leq \overline{\lim_{n\to\infty}}\psi(d(x_{2n}, x_{2n+1})) \leq \psi(r),$$

a contradiction.

Step 2. $\{x_{2n}\}$ is a Cauchy sequence.

proof. Since $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$, it follows that for given $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that $d(x_{2n}, x_{2n+1}) < \varepsilon$ for all $n \geq N_1$. We claim that there exists $N_2 \in \mathbb{N}$ such that $d(x_{2m}, x_{2n+1}) < \varepsilon$ for all $m > n \geq N_2$. Suppose the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there is $m_k > n_k \geq k$ satisfying

$$d(x_{2m_k}, x_{2n_k+1}) \geq \varepsilon_0,\ \ d(x_{2m_k-2}, x_{2n_k+1}) < \varepsilon_0.$$

Therefore
$\varepsilon_0 \leq d(x_{2m_k}, x_{2n_k+1})$
$\leq d(x_{2m_k}, x_{2m_k-1}) + d(x_{2m_k-2}, x_{2m_k-1}) + d(x_{2m_k-2}, x_{2n_k+1})$
$< d(T(x_{2m_k-2}), T^2(x_{2m_k-2})) + d(x_{2m_k-2}, x_{2m_k-1}) + \varepsilon_0$
$\leq \psi(d(x_{2m_k-2}, x_{2m_k-1})) + d(x_{2m_k-2}, x_{2m_k-1}) + \varepsilon_0$
$< 2d(x_{2m_k-2}, x_{2m_k-1}) + \varepsilon_0$.
Letting $k \to \infty$ we obtain

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \varepsilon_0. \tag{2.2}$$

Triangle inequality implies that
$d(x_{2m_k}, x_{2n_k+1}) \leq d(x_{2m_k}, x_{2m_k+1}) + d(x_{2n_k+2}, x_{2m_k+1}) + d(x_{2n_k+2}, x_{2n_k+1})$
$= d(x_{2m_k}, x_{2m_k+1}) + d(T(x_{2m_k}), T^2(x_{2n_k})) + d(T(x_{2n_k}), T^2(x_{2n_k}))$
$\leq d(x_{2m_k}, x_{2m_k+1}) + \psi(d(x_{2m_k}, x_{2n_k+1})) + \psi(d(x_{2n_k}, x_{2n_k+1}))$
$< d(x_{2m_k}, x_{2m_k+1}) + \psi(d(x_{2m_k}, x_{2n_k+1})) + d(x_{2n_k}, x_{2n_k+1})$
$\leq 2d(x_{2k}, x_{2k+1}) + \psi(d(x_{2m_k}, x_{2n_k+1}))$.
Again letting $k \to \infty$ and using (5), we obtain $\varepsilon_0 \leq \psi(\varepsilon_0)$, which is a contradiction. Now if $N := \max\{N_1, N_2\}$, then for all $m > n \geq N$ we have

$$d(x_{2m}, x_{2n}) \leq d(x_{2m}, x_{2n+1}) + d(x_{2n}, x_{2n+1}) < 2\varepsilon.$$

Since A is complete, there exists $p \in A$ such that $x_{2n} \to p$. Now by the continuity of T on A we have $x_{2n+1} = T(x_{2n}) \to Tp$. This implies that $d(p, Tp) = \lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$ or $Tp = p$. □

The above theorem is still valid if T is not necessarily continuous, instead one should assume that the condition (3) holds.

Theorem 2.2. *Let A, B be two nonempty subsets of a metric space (X, d) such that A is complete and satisfies the condition (3), and let "$\preceq$" be a partially ordered relation on A. Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping such that T^2 is nondecreasing on A and*

$$d(T\acute{x}, T^2x) \leq \psi(d(\acute{x}, Tx)),$$

for all $(x, \acute{x}) \in A \times A$ with $x \preceq \acute{x}$, where $\psi : \mathbb{R}^+ \to [0, +\infty)$ is upper semi-continuous from the right such that $0 \leq \psi(t) < t$, for all $t > 0$. If there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$, then $A \cap B \neq \emptyset$ and so T has a fixed point $p \in A \cap B$. Moreover if $x_{n+1} = Tx_n$ then $x_{2n} \to p$.

Proof. By Theorem 2.1, $\{x_{2n}\}$ is a Cauchy sequence, so that there exists $p \in A$ such that $x_{2n} \to p$. Since T^2 is nondecreasing on A, and A satisfies the condition (3), we have $x_{2n} \preceq p$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{2n}) + d(Tp, T^2(x_{2n-2})) \\ &\leq d(p, x_{2n}) + \psi(d(p, x_{2n-1})) \\ &\leq d(p, x_{2n}) + d(p, x_{2n-1}) \\ &\leq d(p, x_{2n}) + d(p, x_{2n-2}) + d(x_{2n-2}, x_{2n-1}). \end{aligned}$$

Letting $n \to \infty$, we obtain $d(p, Tp) = 0$ or $Tp = p$. □

Remark 2.1. If in Theorem 2.1, $\varphi(t) = \alpha t$ for $t \geq 0$ and $0 \leq \alpha < 1$, then we obtain Theorem 1.3.

Example 2.1. Let $X = \mathbb{R}^2$ and $A = \{(x, 1-x) : 0 \leq x \leq \frac{1}{7}\}$, $B = \{(y, 1) : -1 \leq y \leq 1\}$, where the distance d in $\mathbb{R}^2$ is defined by $d((x, y), (\acute{x}, \acute{y})) = \max\{|x - \acute{x}|, |y - \acute{y}|\}$, for all $(x, y), (\acute{x}, \acute{y}) \in \mathbb{R}^2$. Consider the usual order $(x, y) \preceq (\acute{x}, \acute{y}) \Leftrightarrow x \leq \acute{x}, y \leq \acute{y}$. It is easy to see that A is a partially ordered set with the usual order and for all $(x, y), (\acute{x}, \acute{y}) \in A$ we have $(x, y) \preceq (\acute{x}, \acute{y}) \Leftrightarrow x = \acute{x}, y = \acute{y}$. Define $T : A \cup B \to A \cup B$ by $T(x, 1-x) = (-x, 1)$ for $0 \leq x \leq \frac{1}{7}$, and $T(y, 1) = (-\frac{y}{2}, 1 + \frac{y}{2})$ for $-1/7 \leq y \leq 0$, and $T(y, 1) = (\frac{1}{7}, \frac{6}{7})$ for $y \in [-1, 1] - [\frac{-1}{7}, 0]$. It is easy to see that T is cyclic on $A \cup B$. If $\psi(t) = \ln(1 + t)$ for $t \geq 0$ then the condition (4) holds. Indeed, we must check the relation

$$d(T\acute{x}, T^2\acute{x}) \leq \psi(d(\acute{x}, T\acute{x})) \tag{2.3}$$

for all $\acute{x} := (x, 1-x) \in A$. But $d(T\acute{x}, T^2\acute{x}) = \frac{3}{2}\, x$ and $d(\acute{x}, T\acute{x}) = 2x$, so that (6) is equivalent to

$$\frac{3}{2}x \leq \psi(2x) = \ln(1 + 2x), \quad 0 \leq x \leq \frac{1}{7}. \tag{2.4}$$

Indeed, if $h(x) := \ln(1 + 2x) - \frac{3}{2}x$, then $h(0) = 0$ and

$$h'(x) = \frac{2}{1 + 2x} - \frac{3}{2} = \frac{1 - 6x}{2 + 4x}.$$

This implies that $h'(x) > 0$, for $0 \leq x \leq \frac{1}{7}$, hence h is a monotone nondecreasing function. Thus $h(x) \geq 0$, for $0 \leq x \leq \frac{1}{7}$. Hence (7) holds. Similarly, we see that the other conditions of Theorem 2.1 hold. Therefore T has a fixed point in $A \cap B$, and this point is $p = (0, 1)$.

The following theorem is another version of Theorem 1.3.

Theorem 2.3. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Let A, B be two nonempty subsets of X such that A is complete. Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping such that T is continuous on A and $T|_A, T|_B$ are nondecreasing and*

$$d(T\acute{x}, T^2x) \leq \psi(d(\acute{x}, Tx)), \; d(T\acute{y}, T^2y) \leq \psi(d(\acute{y}, Ty) \tag{2.5}$$

for all $(x, \acute{x}) \in A \times A$, $(y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}, y \preceq \acute{y}$, where $\psi : (0, +\infty) \to (0, +\infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$, for all $t > 0$. If there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$, then $A \cap B \neq \emptyset$ and so T has a fixed point $p \in A \cap B$. Moreover if $x_{n+1} = Tx_n$ then $x_{2n} \to p$.

Proof. By assumption $\lim_{n\to\infty} \psi^n(t) = 0$. It follows that $\psi(\varepsilon) < \varepsilon$, for all $\varepsilon > 0$. If $T^2x_0 = x_0$, then by a similar argument as in the proof of Theorem 2.1, it can be seen that x_0 is a fixed point of T. Suppose that $T^2x_0 \neq x_0$. Again we break the proof into two steps.

Step 1. $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$.

proof. We have
$d(x_{2n}, x_{2n+1}) = d(T(x_{2n}), T^2(x_{2n-2})) \leq \psi(d(x_{2n}, x_{2n-1}))$
$= \psi(d(T(x_{2n-2}), T^2(x_{2n-2}))) \leq \psi^2(d(x_{2n-2}, x_{2n-1}))$
$\leq \psi^3(d(x_{2n-2}, x_{2n-3})) \leq ... \leq \psi^{2n}(d(x_0, x_1))$.
Thus $\overline{\lim}_{n\to\infty} d(x_{2n}, x_{2n+1}) \leq \lim_{n\to\infty} \psi^{2n}(d(x_0, x_1)) = 0$. It follows that $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$. Since
$d(x_{2n+2}, x_{2n+1}) = d(T(x_{2n}), T^2(x_{2n}))$
$\leq \psi(d(x_{2n}, x_{2n+1})) < d(x_{2n}, x_{2n+1})$. It follows that $d(x_{2n+2}, x_{2n+1}) \to 0$. Hence $d(x_{2n}, x_{2n+2} \to 0$. Thus for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\max\{d(x_{2N}, x_{2N+1}), d(x_{2N}, x_{2N+2})\} < \varepsilon - \psi(\varepsilon)$.

Step 2. $\{x_{2n}\}$ is a Cauchy sequence.

proof. Since $T|_A, T|_B$ are nondecreasing and $x_0 \preceq T^2x_0$, we see that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are nondecreasing sequences in A and B respectively. Put

$$G(x_{2N}, \varepsilon) = \{x \in A : x_{2N} \preceq x, d(x, x_{2N+1}) < \varepsilon\},$$

$$H(x_{2N+1}, \varepsilon) = \{y \in B : x_{2N+1} \preceq y, d(y, x_{2N}) < \varepsilon\}.$$

We show that T is cyclic on $G(x_{2N}, \varepsilon) \cup H(x_{2N+1}, \varepsilon)$. Let $z \in G(x_{2N}, \varepsilon)$. Then $z \in A$ and $d(z, x_{2N+1}) < \varepsilon$. It follows that $Tz \in B$ and $x_{2N+1} \preceq Tz$. Thus
$d(x_{2N}, Tz) \leq d(x_{2N}, x_{2N+2}) + d(Tz, T^2(x_{2N}))$
$\leq d(x_{2N}, x_{2N+2}) + \psi(d(z, x_{2N+1}))$
$< \psi(\varepsilon) - \varepsilon + \psi(\varepsilon) = \varepsilon$.

Hence $z \in H(x_{2N+1}, \varepsilon)$. Also if $w \in H(x_{2N+1}, \varepsilon)$, then $w \in B, x_{2N+1} \preceq w$ and $d(w, x_{2N}) < \varepsilon$. This implies that $Tw \in A$ and $x_{2N} \preceq x_{2N+2} \preceq Tw$. We now have
$d(Tw, x_{2N+1}) = d(Tw, T^2(x_{2N-1}))$
$\leq \psi(d(w, x_{2N})) \leq \psi(\varepsilon) < \varepsilon$.

Therefore $w \in G(x_{2N}, \varepsilon)$ and we conclude that T is cyclic on $G(x_{2N}, \varepsilon) \cup H(x_{2N+1}, \varepsilon)$. On the other hand since $x_{2N} \in G(x_{2N}, \varepsilon)$, it follows that $x_{2N+1} = T(x_{2N}) \in H(x_{2N+1}, \varepsilon)$ and $x_{2N+2} = T(x_{2N+1}) \in G(x_{2N}, \varepsilon)$. Now an appeal to induction reveals that $x_{2N+2n} \in G(x_{2N}, \varepsilon)$ or $d(x_{2N+2n}, x_{2N+1}) < \varepsilon$, for all $n \in \mathbb{N}$. Therefore

$$d(x_{2N+2n}, x_{2N}) \leq d(x_{2N+2n}, x_{2N+1}) + d(x_{2N}, x_{2N+1}) < \varepsilon + \varepsilon = 2\varepsilon,$$

for all $n \in \mathbb{N}$, from which it follows that $\{x_{2n}\}$ is a Cauchy sequence in A. The rest of proof is similar to that of Theorem 2.1. □

In the following theorem we replace the continuity of T on A by the condition (3).

Theorem 2.4. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Let A, B be two nonempty subsets of X such that A is complete and satisfies the condition (3). Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping such that $T|_A, T|_B$ are nondecreasing and*

$$d(T\acute{x}, T^2x) \le \psi(d(\acute{x}, Tx)),\ d(T\acute{y}, T^2y) \le \psi(d(\acute{y}, Ty))$$

for all $(x, \acute{x}) \in A \times A, (y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}, y \preceq \acute{y}$, where $\psi : (0, +\infty) \to (0, +\infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$, for all $t > 0$. If there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$, then $A \cap B \neq \emptyset$ and so T has a fixed point $p \in A \cap B$. Moreover if $x_{n+1} = Tx_n$ then $x_{2n} \to p$.

Proof. By Theorem 2.3, $\{x_{2n}\}$ is a nondecreasing sequence and $x_{2n} \to p \in A$. By the condition (3), we have $x_{2n} \preceq p$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} d(p, Tp) &\le d(p, x_{2n}) + d(x_{2n}, Tp) \\ &\le d(p, x_{2n}) + \psi(d(p, x_{2n-1})) \\ &< d(p, x_{2n}) + d(p, x_{2n-1}) \le 2d(p, x_{2n}) + d(x_{2n}, x_{2n-1}) \to 0. \end{aligned}$$

Hence $p = Tp$. □

Remark 2.2. If in Theorems 2.3, 2.4 the partial ordering relation "$\preceq$" on X has the property that

"every pair of elements of X has a lower bound or an upper bound" (9)

then the fixed point of T in A is unique.

Proof. Let p, q be two fixed points of T in A. If p is comparable to q, and $p \preceq q$ then

$$d(p, q) = d(Tq, T^2p) \le \psi(d(q, Tp)) < d(p, q).$$

Thus in this case we must have $p = q$. If p is not comparable to q, then there is either an upper or a lower bound for p and q, that is, there exists $r \in A$ such that p, q are comparable to r. Let $p \preceq r$ and $q \preceq r$. Therefore

$$\begin{aligned} d(p, q) &\le d(T^{2n}p, T^{2n+1}r) + d(T^{2n+1}r, T^{2n}q) \\ &= d(T(T^{2n}r), T^2(T^{2n-2}p) + d(T(T^{2n}r), T^2(T^{2n-2}q)) \\ &\le \psi(d(T^{2n}r, T^{2n-1}p)) + \psi(d(T^{2n}r, T^{2n-1}q)) \\ &= \psi(d(T(T^{2n-1}r), T^2(T^{2n-3}p))) + \psi(d(T(T^{2n-1}r), T^2(T^{2n-3}q))) \\ &\le \psi^2(d(T^{2n-1}r, T^{2n-2}p)) + \psi^2(d(T^{2n-1}r, T^{2n-2}q)) \\ &\le ... \le \psi^{2n}(d(p, Tr)) + \psi^{2n}(d(q, Tr)) \to 0. \end{aligned}$$

This implies that $p = q$. □

Remark 2.3. If in Theorems 2.3, 2.4 the partial ordering relation "$\preceq$" on X has the property (9), then $T^{2n}x \to p$ for all $x \in A$ where p is a fixed point of T in A.

Proof. If p is comparable to x and $p \preceq x$, then it is easy to see that

$$d(T^{2n}x, p) \le \psi^{2n}(d(x, p)) \to 0,$$

which implies that $x_{2n} \to p$. If p is not comparable to x, then by property (9), there exists $z \in A$ such that $p \preceq z, x \preceq z$. By a similar argument as in Remark 2.2, we see that

$$d(T^{2n}x, p) \leq \psi^{2n-1}(d(z, Tx)) + \psi^{2n-1}(d(z, Tp)) \to 0,$$

from which it follows that $x_{2n} \to p$. □

Example 2.2. Let $X = \mathbb{R}^2$ and $A = \{(x, 0) : 0 \leq x \leq 1\}$, $B = \{(0, y) : 0 \leq y \leq 1\}$, where the distance d in $\mathbb{R}^2$ is defined by $d((x, y), (\acute{x}, \acute{y})) = \max\{|x - \acute{x}|, |y - \acute{y}|\}$, for all $(x, y), (\acute{x}, \acute{y}) \in \mathbb{R}^2$. Consider the usual order $(x, y) \preceq (\acute{x}, \acute{y}) \Leftrightarrow x \leq \acute{x}, y \leq \acute{y}$. Define $T : A \cup B \to A \cup B$ by $T(x, 0) = (0, \frac{x}{x+1})$ for $0 \leq x \leq 1$, and $T(0, y) = (\frac{y}{y+1}, 0)$. We note that T is cyclic on $A \cup B$. If $\psi(t) = \frac{t}{t+1}$ for $t > 0$ then the condition (8) and the other conditions of Theorem 2.3 hold. Hence T has a fixed point and this point is $p = (0, 0)$.

3. Best Proximity Points

In this section we study existence and convergence of best proximity points for cyclic mappings and eventually obtain a new fixed point theorem for these mappings. The following theorem shows the existence of best proximity point for cyclic mappings under suitable conditions in a partially ordered metric space.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Let A, B be two nonempty subsets of X such that A is compact. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that $T|_A$ and $T^2|_A$ are continuous and T^2 is nondecreasing on A. Moreover,*

$$\begin{cases} d(T\acute{x}, T^2x) < d(\acute{x}, Tx) & \text{for } \ x \preceq \acute{x}, \quad d(\acute{x}, Tx) > dist(A, B), \\ d(T\acute{x}, T^2x) \leq d(\acute{x}, Tx) & \text{for } \ x \preceq \acute{x}, \end{cases}$$

for all $(x, \acute{x}) \in A \times A$. If there exists $x_0 \in A$ such that $x_0 \preceq T^2x_0$ then T has a best proximity point in A.

Proof. Since T, T^2 are nondecreasing on A, then $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are nondecreasing sequences in A, B, respectively. We show that $r_{2n} := d(x_{2n}, x_{2n+1}) \to dist(A, B)$. We note that

$$\begin{aligned} r_{2n} &= d(x_{2n}, x_{2n+1}) = d(T(x_{2n}), T^2(x_{2n-2})) \\ &\leq d(x_{2n}, x_{2n-1}) = d(T(x_{2n-2}), T^2(x_{2n-2})) \\ &\leq d(x_{2n-2}, x_{2n-1}) = r_{2n-2}. \end{aligned}$$

Therefore $\{r_{2n}\}$ is a decreasing sequence. Let $r_{2n} \to r^* \geq dist(A, B)$. Assume that $r^* > dist(A, B)$. Since A is compact, there exists a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ such that $x_{2n_k} \to p \in A$. By the continuity of T on A,

$$d(p, Tp) = \lim_{k \to \infty} d(x_{2n_k}, x_{2n_k+1}) = r^*.$$

Thus

$$\begin{aligned} r^* &= \lim_{k \to \infty} d(x_{2n_k+2}, x_{2n_k+3}) \leq \lim_{k \to \infty} d(x_{2n_k+2}, x_{2n_k+1}) \\ &= \lim_{k \to \infty} d(T(x_{2n_k}), T^2(x_{2n_k})) = d(Tp, T^2p) < d(p, Tp) = r^*, \end{aligned}$$

which is a contradiction. Hence $d(p, Tp) = dist(A, B)$ and the proof is complete. □

Now it is interesting to ask whether the approximation of best proximity points in the above theorem by an iterate sequence is possible or not. For this reason we need to recall a geometric property on a pair of subsets of a metric space.

Definition 3.1. *([10]) Let A, B be two nonempty subsets of a metric space (X, d). A pair (A, B) is said to satisfy property UC iff the following holds: If $\{x_n\}$ and $\{z_n\}$ are sequences in A, and $\{y_n\}$ is a sequence in B such that*

$$\lim_{n\longrightarrow\infty} d(x_n, y_n) = \lim_{n\longrightarrow\infty} d(z_n, y_n) = dist(A, B),$$

then $\lim_{n\longrightarrow\infty} d(x_n, z_n) = 0$.

For example if A, B are two nonempty subsets of a uniformly convex Banach space X such that A is closed and convex and B is closed, then (A, B) has the property UC (see Lemma 3.8 of [5]). Other examples can be found in [10].

In the following theorem we approximate the best proximity point which was found in Theorem 3.1.

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Let A, B be two nonempty subsets of X such that A is compact and (A, B) satisfies the property UC. Let the condition (3) hold on A, and let $T : A\cup B \to A\cup B$ be a cyclic mapping such that $T|_A$ and $T^2|_A$ are nondecreasing and continuous. Moreover,*

$$\begin{cases} d(T\acute{x}, T^2x) < d(\acute{x}, Tx) & for \;\; x \preceq \acute{x}, d(\acute{x}, Tx) > dist(A, B), \\ d(T\acute{x}, T^2x) \leq d(\acute{x}, Tx) & for \;\; x \preceq \acute{x}, \end{cases}$$

for all $(x, \acute{x}) \in A\times A$ and $(x, \acute{x}) \in B \times B$. If there exists $x_0 \in A$ such that $x_0 \leq T^2x_0$ and $x_{n+1} = Tx_n$, then T has a best proximity point $p \in A$ and $x_{2n} \to p$.

Proof. By Theorem 3.1 the existence of best proximity point $p \in A$ is guaranteed and there exists a subsequence $\{x_{2n_k}\}$ of the sequence $\{x_{2n}\}$ such that $x_{2n_k} \to p$. We must prove that $x_{2n} \to p$. Since the condition (3) holds on A and $\{x_{2n}\}, \{x_{2n+1}\}$ are nondecreasing sequences in A and B, respectively, it follows that $x_{2n_k} \preceq p$ and hence $x_{2n_k+1} \preceq Tp$ for all $k \in \mathbb{N}$. This implies that $x_{2n} \preceq p$ and $x_{2n+1} \preceq Tp$ for all $n \in \mathbb{N}$. On the other hand since (A, B) has the property UC and

$$\begin{cases} d(Tx, T^2x) < d(x, Tx) & \text{for all} \;\; x \in A\cup B \;\; \text{with} \;\; d(x, Tx) > dist(A, B), \\ d(Tx, T^2x) \leq d(x, Tx) & \text{for all} \;\; x \in A\cup B, \end{cases}$$

it follows from Lemma 3 of [10] that z is a best proximity point of T if and only if z is a fixed point of T^2. Therefore p is a fixed point of T^2. We now have

$$\begin{aligned} & d(p, x_{2n+1}) = d(T(Tp), T^2(x_{2n-1})) \\ & \leq d(Tp, x_{2n}) = d(Tp, T^2(x_{2n-2})) \leq d(p, x_{2n-1}). \end{aligned}$$

Thus $\{d(p, x_{2n+1})\}$ is a descending sequence. Let $d(p, x_{2n+1}) \to s^*$. Then

$$s^* = \lim_{n\to\infty} d(p, x_{2n+1}) = \lim_{k\to\infty} d(p, x_{2n_k+1}) = d(p, Tp) = dist(A, B).$$

We now have $d(x_{2n}, x_{2n+1}) \to dist(A, B)$ and $d(T^{2n}p, x_{2n+1}) \to dist(A, B)$. Again since (A, B) has the property UC, we obtain $d(x_{2n}, p) \to 0$ or $x_{2n} \to p$. □

Corollary 3.1. *Let X be a strictly convex Banach space and "$\preceq$" be a partially ordered relation on X. Let (A, B) be a nonempty pair of subsets of X such that A is convex, compact, and the closure of B is weakly compact. Assume that $T : A\cup B \to$*

$A \cup B$ is a cyclic mapping such that $T|_A$ and $T^2|_A$ are nondecreasing and continuous and

$$\begin{cases} d(T\acute{x}, T^2x) < d(\acute{x}, Tx) & for \;\; x \preceq \acute{x}, d(\acute{x}, Tx) > dist(A, B), \\ d(T\acute{x}, T^2x) \le d(\acute{x}, Tx) & for \;\; x \preceq \acute{x}, \end{cases}$$

for all $(x, \acute{x}) \in A \times A$ and $(x, \acute{x}) \in B \times B$. If there exists $x_0 \in A$ such that $x_0 \preceq T^2 x_0$ and $x_{n+1} = Tx_n$, then T has a best proximity point $p \in A$ and $x_{2n} \to p$.

Proof. By Proposition 5 of [10], (A, B) has the property UC. Now by Theorem 3.2 the result follows. □

As a result of Theorem 3.2 we obtain a new fixed point theorem in a partially ordered metric space.

Corollary 3.2. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that X is compact and satisfies the condition (3). Assume that $T : X \to X$ is a self mapping such that T is nondecreasing and continuous. Moreover,*

$$\begin{cases} d(T\acute{x}, T^2x) < d(\acute{x}, Tx) & for \;\; x \preceq \acute{x}, \quad \acute{x} \neq Tx, \\ d(T\acute{x}, T^2x) \le d(\acute{x}, Tx) & for \;\; x \preceq \acute{x}, \end{cases}$$

for all $x, \acute{x} \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $x_{n+1} = Tx_n$, then T has a fixed point $p \in X$ and $x_n \to p$.

Proof. It is sufficient to note that if in Theorem 3.2, $A = B = X$ then the conditions of Theorem 3.2 hold. Thus there exists $p \in X$ such that $d(p, Tp) = 0$ or $p = Tp$ and $x_n \to p$. □

Example 3.1. Let $X = \mathbb{R}^2$ and let the distance d in $\mathbb{R}^2$ is defined by $d((x, y), (\acute{x}, \acute{y})) = \max\{|x - \acute{x}|, |y - \acute{y}|\}$, for all $(x, y), (\acute{x}, \acute{y}) \in \mathbb{R}^2$. Consider the usual order $(x, y) \preceq (\acute{x}, \acute{y}) \Leftrightarrow x \le \acute{x}, y \le \acute{y}$ on $\mathbb{R}^2$. Assume that $A = \{(1, 0), (0, 1)\}$ and $B = \{(-1, 0), (0, -1)\}$. Define $T : A \cup B \to A \cup B$ by $T(1, 0) = (0, -1), T(0, 1) = (-1, 0), T(0, -1) = (1, 0)$ and $T(-1, 0) = (0, 1)$. It is easy to check that T satisfies the conditions of Theorem 3.1, and for $x_0 = (1, 0)$, we have $x_0 \preceq T^2 x_0$. Hence T has a best proximity point.

Example 3.2. Consider the space $C[0, \pi]$ with the supremum norm. For each $\alpha \in [0, \pi]$, let $f_\alpha : [0, \pi] \to [0, \pi]$ be defined by $f_\alpha(x) = \alpha \sin x$. Assume that $X = \{f_\alpha : 0 \le \alpha \le \pi\}$. By the Arzela-Ascoli Theorem X is compact in $C[0, \pi]$. Let the relation "$\preceq$" be defined as follows:

$$f_\alpha \preceq f_\beta \Leftrightarrow f_\alpha(x) \le f_\beta(x),$$

for all $x, \alpha, \beta \in [0, \pi]$. It is clear that $f_\alpha \preceq f_\beta \Leftrightarrow \alpha \le \beta$ and that "$\preceq$" is a partially ordered relation on X. Also $d(f_\alpha, f_\beta) = |\alpha - \beta|$, for all $\alpha, \beta \in [0, \pi]$. Suppose that the self mapping $T : X \to X$ is given by $T(f_\alpha) = f_{\frac{\alpha}{\alpha+1}}$. T is nondecreasing on X. Indeed if $f_\alpha \preceq f_\beta$ then $\alpha \le \beta$ and therefore $\frac{\alpha}{\alpha+1} \le \frac{\beta}{\beta+1}$. This implies that $f_{\frac{\alpha}{\alpha+1}} \preceq f_{\frac{\beta}{\beta+1}}$ or $T(f_\alpha) \preceq T(f_\beta)$. Moreover, T is continuous on X. We show that T satisfies the conditions of the theorem. Let $\alpha \le \beta$ and $\beta > 0$. Therefore $T(f_\alpha) \neq f_\beta$. We have to show that

$$d(T(f_\beta), T^2(f_\alpha)) < d(f_\beta, T(f_\alpha))$$

or equivalently

$$|\frac{\beta}{\beta+1}-\frac{\alpha}{2\alpha+1}|<|\beta-\frac{\alpha}{\alpha+1}|.$$

But since the function $g(t):=\frac{t}{t+1}$ is monotone nondecreasing on $[0,+\infty)$ and $\alpha\leq\beta$, it is easy to see that this latter inequality holds. On the other hand $f_0\preceq T(f_0)$. Hence by Corollary 3.5 T has a fixed point and this point is f_0.

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