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PROPER PROJECTIVE SYMMETRY IN SOME WELL KNOWN CONFORMALLY FLAT SPACE-TIMES

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A study of conformally flat- but non flat Bianchi type I and cylindrically symmetric static space-times according to proper projective symmetry is given by using some algebraic and direct integration techniques. It is shown that the special class of the above space-times admit proper projective vector fields.

Keywords: conformally flat space-times, projective vector field, direct integration technique

1. Introduction

Through out the paper M is representing the four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (-, +, +, +). The curvature tensor associated with g_{ab}, of the Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$, the Weyl tensor components are $C^a{}_{bcd}$, and the Ricci tensor components are $R_{ab} = R^c{}_{acb}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. A Space-Time is said to be conformally flat if $C^a{}_{bcd} = 0$ everywhere on M. Finally, M is assumed to be non-flat in the sense that the curvature tensor does not vanish over a non-empty open subset of M, and is not of constant curvature.

Any vector field X on M can be decomposed as

$$X_{a;b} = \frac{1}{2}h_{ab} + F_{ab}, \tag{1}$$

where $h_{ab}(= h_{ba}) = L_X g_{ab}$ and $F_{ab}(= -F_{ba})$ are symmetric and skew symmetric tensors on M, respectively. Such a vector field X is called projective if the local diffeomorphisms ψ_t (for appropriate t) associated with X maps geodesics into geodesics. This is equivalent to the condition that h_{ab} satisfies

$$h_{ab;c} = 2g_{ab}\phi_c + g_{ac}\phi_b + g_{bc}\phi_a \tag{2}$$

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for some smooth closed 1-form on M with local components ϕ_a. Thus ϕ_a is locally gradient and will, where appropriate, be written as $\phi_a = \phi_{,a}$ for some function ϕ on some open subset of M. If X is a projective vector field and $\phi_{a;b} = 0$, then X is called a special projective vector field on M. If $h_{ab;c} = 0$ on M is, from (2), equivalent to ϕ_a being zero on M and is, in turn equivalent to X being an affine vector field on M (so that the local diffeomorphisms ψ_t preserve not only geodesics but also their affine parameters). If X is projective but not affine, then it is called proper projective [1]. Further, if X is affine and $h_{ab} = 2cg_{ab}, c \in R$ then X is homothetic (otherwise proper affine). If X is homothetic, and $c \neq 0$ it is proper homothetic while if $c = 0$ it is Killing.

2. Projective symmetry

Let X be a projective vector field on M. Then from (1) and (2) [2]

$$L_X R^a{}_{bcd} = \delta^a_d \phi_{b;c} - \delta^a_c \phi_{a;b}, \qquad L_X R_{ab} = -3\phi_{a;b}.$$

Also the Ricci identity on h gives

$$h_{ae} R^e{}_{bcd} + h_{be} R^e{}_{acd} = g_{ac}\phi_{b;d} - g_{ad}\phi_{b;c} + g_{bc}\phi_{a;d} - g_{bd}\phi_{a;c}.$$

Let X be a projective vector field on M such that (1) and (2) holds and let F be a real curvature eigenbivector at $p \in M$ with eigenvalue $\lambda \in R$ (such that $R^{ab}{}_{cd} F^{cd} = \lambda F^{ab}$ at p); then at p one has [1]

$$P_{ac} F^c{}_b + P_{bc} F^c{}_a = 0 \qquad (P_{ab} = \lambda h_{ab} + 2\phi_{a;b}) \tag{3}$$

Equation (3) gives a relation between $F^a{}_b$ and P_{ab} (which is a second order symmetric tensor) at p and reflects the close connection between $h_{ab}, \phi_{a;b}$ and the algebraic structure of the curvature at p. If F is simple, then the blade of F (a two dimensional subspace of T_pM) consists of eigenvectors of P with same eigenvalue. Similarly, if F is non-simple then it has two well defined orthogonal timelike and spacelike blades at p each of which consists of eigenvectors of P with same eigenvalue but with the possibly different eigenvalue for the two blades [3].

2.1 Existence of Projective vector field in non flat conformally flat cylindrically symmetric static space-times

Consider a cylindrically symmetric static space-time in the usual coordinate system (t,r,θ,ϕ) (labeled by (x^0,x^1,x^2,x^3), respectively) with first fundamental form [4]

$$ds^2 = -e^{v(r)}dt^2 + dr^2 + e^{u(r)}d\theta^2 + e^{w(r)}d\phi^2. \qquad (4)$$

Since we are interested in those cases when the above space-time (4) becomes conformally flat but non flat, I follows from [5,6] there exists only one possibility namely:

(P1) $v(r) = u(r) = w(r)$.

Case P1

In this case the above Space-Times becomes

$$ds^2 = -e^{v(r)}dt^2 + dr^2 + e^{v(r)}(d\theta^2 + d\phi^2). \qquad (5)$$

The above Space-Times (5) admits six independent Killing vector fields, which are

$$\frac{\partial}{\partial t},\ \frac{\partial}{\partial \theta},\ \frac{\partial}{\partial \phi},\ \theta\frac{\partial}{\partial \phi} - \phi\frac{\partial}{\partial \theta},\ \theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta},\ \phi\frac{\partial}{\partial t} + t\frac{\partial}{\partial \phi}.$$

These six Killing vector fields are clearly tangent to the family of three dimensional timelike hypersurfaces of constant r. Consequently, these hypersurfaces are of constant (zero) curvature. The Ricci tensor Segre of the above Space-Times is $\{(1,11)1\}$ or $\{(1,111)\}$. If the Segre is $\{(1,111)\}$ then the space-time is of constant curvature and the projective vector fields are given in [2]. Here it is assumed that the Space-Times is not of constant curvature. The non-zero independent components of the Riemann tensor are

$$\begin{aligned} R^{21}{}_{21} = R^{31}{}_{31} = R^{10}{}_{10} &= -\frac{1}{4}(2v'' + v'^2) \equiv \beta_2, \\ R^{30}{}_{30} = R^{20}{}_{20} = R^{32}{}_{32} &= -\frac{1}{4}v'^2 \equiv \beta_1. \end{aligned} \qquad (6)$$

One can write the curvature tensor with the components $R^{ab}{}_{cd}$ at p as a 6×6 matrix in a well known way [7]

$R^{ab}{}_{cd} = diag(\beta_2, \beta_1, \beta_1, \beta_2, \beta_2, \beta_1)$

where β_1 and β_2 are real functions of r only, and where the 6-dimensional labelling is in the order $01, 02, 03, 12, 13, 23$ with $x^0 = t$. Here, at $p \in M$ one may choose a tetrad (t,r,θ,ϕ) satisfying $-t^a t_a = r^a r_a = \theta^a\theta_a = \phi^a\phi_a = 1$ (with all others inner products zero) such that the eigenbivector of the curvature tensor at

p are all simple with blades spanned by the vector pairs (t,r), $(r,\theta),(r,\phi)$ each with eigenvalue $\beta_2(p)$, and (t,θ), (t,ϕ), (θ,ϕ) each with eigenvalue $\beta_1(p)$. Here we are considering the open subregion where β_2 and β_1 are nowhere equal (if $\beta_1=\beta_2$ then it follws from (6) that the above Space-Times (5) becomes of constant curvature, which our assumtion; hence $\beta_1 \neq \beta_2$) and $\beta_2 \neq 0$. If $\beta_2 = 0$, then the rank of the 6×6 Riemann matrix becomes three and it follows from [8] that no proper projective vector fieldwill exist. So $\beta_2 \neq 0$. Thus, at p, the tensor $P_{ab} = \beta_2 h_{ab} + 2\psi_{a;b}$ has eigenvectors t,r,θ,ϕ with same eigenvalue, say, δ_1 and $P_{ab} = \beta_1 h_{ab} + 2\psi_{a;b}$ has eigenvectors t,θ,ϕ with same eigenvalue, say, δ_2. Hence on M one has after using the completeness relation

$$\beta_2 h_{ab} + 2\psi_{a;b} = \delta_1 g_{ab},\ \ \beta_1 h_{ab} + 2\psi_{a;b} = \delta_2 g_{ab} + \delta_4 r_a r_b, \qquad (7)$$

where δ_1, δ_2 and δ_4 are some real functions on M. Since $\beta_2 \neq \beta_1$, then it follows from (7) that

$$h_{ab} = C\,g_{ab} + D\,r_a r_b,\ \ \psi_{a;b} = F\,g_{ab} + F\,r_a r_b \qquad (8)$$

for some real functions C, D, E *and* F on M. Next one substitutes the first equation of (8) in (2) and contracts the resulting expression first with $t^a\theta^b$ and then with $t^a\phi^b$, to get $\psi_a x^a = \psi_a\theta^a = \psi_a\phi^a = 0$ and hence $\psi_a = \eta\, r_a$ for some function η. The same expression transvected with $t^a t^b$ gives $C_c = 2\psi_c \Rightarrow C = C(r)$. Now again the same expression transvected with $r^a r^b$ and using the above information gives $D_c = 2\eta\, r_c$ and hence $D = D(r)$. Consider the equation $\psi_a = \eta\, r_a$ and after taking the covariant derivative we get $\psi_{a;b} = \eta\, r_{a;b} + \eta_b\, r_a$. Next consider the second equation of (8) and use $\psi_{a;b} = \eta\, r_{a;b} + \eta_b\, r_a$ and then contract with r^a to get $\eta_a \propto r_a$ so that $\eta = \eta(r)$. Consider the first equation of (8) and use (5) one has the following non-zero components of h_{ab}

$$h_{00} = -Ce^{v},\ h_{11} = (C+D),\ h_{22} = Ce^{v}\ \ and\ \ h_{33} = Ce^{v}. \qquad (9)$$

Now we are interested in finding projective vector fields by using the following relation

$$L_X\, g_{ab} = h_{ab}. \qquad (10)$$

Using equation (9) and (5) in (10) and writing out explicitly we get

$$v' X^1 + 2X^0_{,0} = C \qquad (11)$$

$$X^1_{,0} - e^{v} X^0_{,1} = 0 \qquad (12)$$

$$X^2_{,0} - X^0_{,2} = 0 \qquad (13)$$

$$X^3_{,0} - X^0_{,3} = 0 \tag{14}$$

$$X^1_{,1} = \frac{1}{2}(C + D) \tag{15}$$

$$e^{\nu} X^2_{,1} + X^1_{,2} = 0 \tag{16}$$

$$e^{\nu} X^3_{,1} + X^1_{,3} = 0 \tag{17}$$

$$\nu' X^1 + 2X^2_{,2} = C \tag{18}$$

$$X^3_{,2} + X^2_{,3} = 0 \tag{19}$$

$$\nu' X^1 + 2X^3_{,3} = C. \tag{20}$$

Equations (15), (16), (17) and (12) give

$$\begin{aligned} X^1 &= \frac{1}{2}\int (C+D)dr + A^1(t,\theta,\phi) \\ X^2 &= -A^1_{\theta}(t,\theta,\phi)\int e^{-\nu} dr + A^2(t,\theta,\phi) \\ X^3 &= -A^1_{\phi}(t,\theta,\phi)\int e^{-\nu} dr + A^3(t,\theta,\phi) \\ X^0 &= A^1_t(t,\theta,\phi)\int e^{-\nu} dr + A^4(t,\theta,\phi) \end{aligned} \tag{21}$$

where $A^1(t,\theta,\phi)$, $A^2(t,\theta,\phi)$, $A^3(t,\theta,\phi)$ and $A^4(t,\theta,\phi)$ are functions of integration. In order to determine $A^1(t,\theta,\phi)$, $A^2(t,\theta,\phi)$, $A^3(t,\theta,\phi)$ and $A^4(t,\theta,\phi)$ we need to integrate the remaining six equations. To avoid details, here we will present only the result. The solution of the equations (11) – (20) is

$$\begin{aligned} X^0 &= ta + \theta c_1 + \phi c_2 + c_3, \quad X^1 = \frac{1}{2}\int (C+D)dr + b, \\ X^2 &= \theta a + tc_1 - \phi c_4 + c_5, \quad X^3 = \phi a + tc_2 + \theta c_4 + c_6 \end{aligned} \tag{22}$$

provided that

$$\int (C+D)dx + b = \frac{1}{\nu'}(C - 2a) \qquad \nu' \neq 0,$$

where $a, b, c_1, c_2, c_3, c_4, c_5, c_6 \in R$. After subtracting Killing vector fields from (22) one has

$$X^0 = ta, \quad X^1 = \frac{1}{2}\int (C+D)dr + b, \quad X^2 = \theta a, \quad X^3 = \phi a$$

provided that

$$\int (C+D)dx + b = \frac{1}{\nu'}(C - 2a) \qquad \nu' \neq 0.$$

Suppose $X = (ta, \rho(r), \theta a, \phi a)$, where $\rho(r) = \frac{1}{2}\int (C+D)dr + b$ and $\rho(r) = \frac{1}{v'}(C - 2a)$. The vector field X is then projective if it satisfies (2). So, using the above information in (2) gives

$$v'\rho' - v'(a + \frac{1}{2}\rho v') = \frac{1}{2}(v''\rho + v'\rho'), \qquad \rho'' = v''\rho + v'\rho' \tag{23}$$

and also $\psi_a = \rho'' r_a$. A particular solution of (23) is

$$\rho = a_1 e^r - 2a, \qquad v = r + a_2 \tag{24}$$

where $a_1, a_2 \in R\ (a_1 \neq 0)$ and $C = D = a_1 e^r$. *Thus the space-time (5) admits a proper projective vector field, for the special choice of v as given in (24).*

2.2 Existence of Projective vector field in non flat conformally flat Bianchi type I space-times

Consider a Bianchi type-1 space-time in the usual coordinate system (t, x, y, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with metric [9]

$$ds^2 = -dt^2 + k(t)dx^2 + h(t)dy^2 + f(t)dz^2. \tag{25}$$

The above space-time admits three linearly independent killing vector fields, which are $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. Since we are interested in those cases when the above Space-Times (25) becomes conformally flat but non flat, It follows from [5,9] there exists only one possibility, which is:

(P2) $k(t) = h(t) = f(t)$.

Case P2

In this case the above Space-Times becomes

$$ds^2 = -dt^2 + k(t)\left(dx^2 + dy^2 + dz^2\right) \tag{26}$$

and it admits six independent Killing vector fields, which are

$$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}.$$

These six Killing vector fields are clearly tangent to the family of three dimensional timelike hypersurfaces of constant t. Consequently, these hypersurfaces are constant (zero) curvature. The Segre type of the above space-time is {1, (111)} or {(1,111)}. If the Segre is {(1,111)} then the Space-Times is of constant curvature and the projective vector fields are given in [2]. Here it is assumed that the space-times is not of constant curvature. The proper projective vector fields for the above space-time (26) are also available in [10]. The non-zero independent components of the Riemann curvature tensors are

$$R^{01}{}_{01} = R^{02}{}_{02} = R^{03}{}_{03} = \frac{1}{k}\left(\frac{\ddot{k}}{2} - \frac{\dot{k}^2}{4k}\right) \equiv A$$
$$R^{12}{}_{12} = R^{13}{}_{13} = R^{32}{}_{32} = \frac{1}{k}\left(\frac{\dot{k}^2}{4k}\right) \equiv B. \tag{27}$$

One can write the curvature tensor with the components $R^{ab}{}_{cd}$ at p as a 6×6 matrix in a well known way [7]

$$R^{ab}{}_{cd} = diag(A, A, A, B, B, B),$$

where A and B are real functions of t only and where the 6-dimensional labelling is in the order $01, 02, 03, 12, 13, 23$ with $x^0 = t$. Here, at $p \in M$ one may choose a tetrad (t, x, y, z) satisfying $-t^a t_a = x^a x_a = y^a y_a = z^a z_a = 1$ (with all other inner products zero) such that the eigenbivectors of the curvature tensor at p are all simple with blades spanned by the vector pairs $(t, x), (t, y), (t, z)$ each with eigen value $A(p)$, and -$(x, y), (x, z), (y, z)$- each with eigenvalue $B(p)$. Here we are considering the open subregion where A and B are nowhere equal (if $A = B$ then it follws from (27) that the above Space-Times (26) becomes of constant curvature, which contradicts to our assumtion; hence $A \neq B$) and $A \neq 0$. If $A = 0$ than the rank of the 6×6 Riemann matrix becomes three and it follows from [8] no proper projective vector field will exist. Hence $A \neq 0$. Thus, at p the tensor $P_{ab} = Ah_{ab} + 2\psi_{a;b}$ has eigenvectors t, x, y and z with same eigenvalue, say, γ_1 and $P_{ab} = Bh_{ab} + 2\psi_{a;b}$ has eigenvectors x, y and z with same eigenvalue, say, γ_2. First consider the equation $P_{ab} = Ah_{ab} + 2\psi_{a;b}$, where P_{ab} is a second order symmetric tensor with eigenvectors t, x, y and z with same eigenvalue γ_1. The Segre type of P_{ab} is $\{(1,111)\}$, and $P_{ab} = \gamma_1 g_{ab}$. Substituting back, we get $Ah_{ab} + 2\psi_{a;b} = \gamma_1 g_{ab}$. Now consider $P_{ab} = Bh_{ab} + 2\psi_{a;b}$, where P_{ab} is a second order symmetric tensor with eigenvectors x, y and z with same eigenvalue, say, γ_2. The Segre type of P_{ab} is $\{1,(111)\}$ and $P_{ab} = \gamma_2 g_{ab} + \gamma_3 t_a t_b$. Substituting back, we get $Bh_{ab} + 2\psi_{a;b} = \gamma_2 g_{ab} + \gamma_3 t_a t_b$. Hence on M one has

$$Ah_{ab} + 2\psi_{a;b} = \gamma_1 g_{ab}, \qquad Bh_{ab} + 2\psi_{a;b} = \gamma_2 g_{ab} + \gamma_3 t_a t_b, \tag{28}$$

where γ_1, γ_2 and γ_3 are some real functions on M. Since $A \neq B$ then it follows from equation (28) that

$$h_{ab} = \beta\, g_{ab} + \alpha\, t_a t_b, \qquad \psi_{a;b} = E\, g_{ab} + F\, t_a t_b \tag{29}$$

for some real functions α, β, E *and* F on M. Now one substitutes the first equation of (29) in (2) and contracts the resulting expression with $x^a y^b$ and then with $x^a z^b$ to get $\psi_a x^a = \psi_a y^a = \psi_a z^a = 0$ and one has $\psi_a = \xi t_a$ for some function ξ. The same expression contracted with $t^a t^b$ then infers $(\alpha - \beta)_{,a} = -4\xi t_a$ and hence $(\alpha - \beta)$ is a function of t only. Now again contract the same expression with $x^a x^b$. One finds $\beta_c = 2\xi t_c$, which implies $\beta_a x^a = \beta_a y^a = \beta_a z^a = 0 \Rightarrow \beta = \beta(t)$. Substituting back we get $\alpha_c = -2\xi t_c$, and hence $\alpha = \alpha(t)$. Now consider the second equation of (29) and use $\psi_{a;b} = \xi_b t_a + \xi t_{a;b}$ and contract this with t^a. One can easily find that $\xi = \xi(t)$. Consider the first equation of (29) and using (26) one obtains the following non zero components of h_{ab}

$$h_{00} = (\alpha - \beta), \; h_{11} = \beta k, \quad h_{22} = \beta k \quad \text{and} \quad h_{33} = \beta k, \tag{30}$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$ and $\alpha - \beta = (\alpha - \beta)(t)$. Now we are interested in finding projective vector fields by using the relation (10). Writing out equation (10) explicitly and using (26) and (30), we get

$$X^0{}_{,0} = \frac{1}{2}(\beta - \alpha) \tag{31}$$

$$kX^1{}_{,0} - X^0{}_{,1} = 0 \tag{32}$$

$$kX^2{}_{,0} - X^0{}_{,2} = 0 \tag{33}$$

$$kX^3{}_{,0} - X^0{}_{,3} = 0 \tag{34}$$

$$\frac{1}{2}\dot{k}X^0 + kX^1{}_{,1} = \frac{1}{2}\beta k \tag{35}$$

$$X^2{}_{,1} + X^1{}_{,2} = 0 \tag{36}$$

$$X^3{}_{,1} + X^1{}_{,3} = 0 \tag{37}$$

$$\frac{1}{2}\dot{k}X^0 + kX^2{}_{,2} = \frac{1}{2}\beta k \tag{38}$$

$$X^3{}_{,2} + X^2{}_{,3} = 0 \tag{39}$$

$$\frac{1}{2}\dot{k}X^0 + kX^3{}_{,3} = \frac{1}{2}\beta k. \tag{40}$$

Equations (31), (32), (33) and (34), give

$$\left.\begin{aligned} X^0 &= \frac{1}{2}\int(\beta-\alpha)dt + A^1(x,y,z) \\ X^1 &= A_x^1(x,y,z)\int\frac{1}{k}dt + A^2(x,y,z) \\ X^2 &= A_y^1(x,y,z)\int\frac{1}{k}dt + A^3(x,y,z) \\ X^3 &= A_z^1(x,y,z)\int\frac{1}{k}dt + A^4(x,y,z) \end{aligned}\right\}, \tag{41}$$

where $A^1(x,y,z), A^2(x,y,z), A^3(x,y,z)$ and $A^4(x,y,z)$ are functions of integration. In order to determine $A^1(x,y,z), A^2(x,y,z), A^3(x,y,z)$ and $A^4(x,y,z)$ we need to integrate the remaining six equations. To avoid lengthy calculations, here we will present only the result. The solution of the equations (31) – (40) is

$$\left.\begin{aligned} X^0 &= \frac{1}{2}\int(\beta-\alpha)dt + c_8 \\ X^1 &= xc_1 - yc^5 + zc^6 + c^7 \\ X^2 &= yc_1 + xc^5 - zc^8 + c^{10} \\ X^3 &= zc_1 - xc^6 + yc^8 + c^9 \end{aligned}\right\} \tag{42}$$

provided that

$$\frac{1}{2}\int(\beta-\alpha)dt + c_8 = \frac{k}{\dot{k}}(\beta - 2c_1) \qquad \dot{k} \neq 0,$$

where $c_1, c_8, c^5, c^6, c^7, c^8, c^9, c^{10} \in R$. After subtracting Killing vector fields from (42) one has

$$X^0 = \frac{1}{2}\int(\beta-\alpha)dt + c_8, X^1 = xc_1, X^2 = yc_1, X^3 = zc_1$$

provided that

$$\frac{1}{2}\int(\beta-\alpha)dt + c_8 = \frac{k}{\dot{k}}(\beta - 2c_1) \qquad \dot{k} \neq 0,$$

Suppose $X = (\eta(t), xc_1, yc_1, zc_1)$, where $\eta(t) = \frac{1}{2}\int(\beta-\alpha)\,dt + c_8$ and $\frac{k}{\dot{k}}(\beta - 2c^1) = \eta(t)$. The vector field X is said to be projective if it satisfies (2). Hence using the above information in (2) we infer

$$\frac{\ddot{\eta}}{2}=\frac{\dot{k}}{k}\dot{\eta}-\frac{\dot{k}}{k^2}\left(k\,c^1+\frac{1}{2}\dot{k}\,\eta\right)$$
$$\frac{\ddot{\eta}}{2}=\frac{1}{2k}\left(\ddot{k}\,\eta+\dot{k}\,\dot{\eta}\right)-\frac{\dot{k}^2}{2k^2}\eta \tag{43}$$

and $\psi_a=\ddot{\eta}\,t_a$. Particular solutions of (43) are

$$\eta=k=\frac{I}{F}e^{F\,t+FG}-\frac{2c_1 I}{F}\,; \tag{44}$$

$$k=Le^t\,,\ \eta=Ne^t-D, \tag{45}$$

where $F,G,I,L,N,R\in R(F\neq 0)$. *Thus the space-time (25) admits a proper projective vector field, for the special choice of* k *as given in (44) and (45).*

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