ON IDEMPOTENTS IN GENERALIZED RINGS

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Studiem idempotenţii şi proprietăţile descompunerilor Peirce, ideale în aproape inele cu DCC, ifra- aproape- inele şi ringoizi.Pentru ultima structură, reamintim unele rezultate obţinute în 1964 de S.K.Sehgal. În ultima secţiune sunt prezentate aplicaţii în geometrie.

We study idempotents and properties of Peirce decompositions and ideals in near-rings with DCC, infra-near-rings and ringoids. For the last structure, we recall some results obtained in 1964 by S.K.Sehgal Applications to geometry are given in the last section.

Keywords: near-rings, infra-near-rins, ringoids, almost affine K-spaces, idempotents

1. Introduction

In the paper, we present some properties of the idempotents in generalized rings as near-rings, infra-near-rings, ringoids, and an application for almost affine geometries concerning the decomposition of the almost affine space into a sum of two component, one of them being the linear part of the space. We recall here some definitions related to near-rings. For further definitions and properties we refer to Pilz[1], Ştefănescu[2], Miron[3].

Definition 1.1. *A* ***0-symmetric right near-ring*** *is a triple $(N,+,.)$ where :*

(i) $(N, +)$ *is a group ;*

(ii) $(N, .)$ *is a semigroup ;*

(iii) $(x + y).z = x.z + y.z$, *for all* $x,y,z \in N$ *;*

(iv) $X.0 = 0$, *for all* $x \in N$.

Definition 1.2. *A* ***left ideal*** *of a right near-ring is a normal subgroupA in* $(N, +)$ *such that* $x.(y + a) - x.y \in A$, *for all* $x, y \in N$, $a \in A$. *An* ***ideal*** *of N is a left ideal A such that* $a.x \in A$, *for all* $a \in A$, $x \in N$.

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Definition 1.3. *An **N-group** is a group $(\Gamma, +)$, endowed with an external composition $\cdot : N\times\Gamma \to \Gamma$, such that, for all $\gamma \in \Gamma$ and $n, n' \in N$, $(n + n')\gamma = n\gamma + n'\gamma$ and $(nn')\gamma = n(n'\gamma)$. In N has $1 \neq 0$, then the N-group is unital if $1\gamma = \gamma$, for all $\gamma \in \Gamma$.*

Morphisms of near-rings and N-group are defined in the usual way.

2. Some remarks on idempotents in near-rings

Let N be a zero-symetric right near-ring and *I(N)* be the set of its idempotents, $I(N) = \{e \in N \mid e^2 = e\}$. Oviously, $0 \in I(N)$. We recall some properties of idempotents in special near-rings. First we consider near-rings satisfying DCC on N-subgroups.

Proposition 2.1. (Scott [4]) *If N satisfies DCC on N-subgroups and M is a right N-subgroup of N such that mM = M, then :*

(i) M contains an idempotent which is a left identity of it, e.

(ii) $Ann_M(m) = \{x \in M \mid mx=0\} = \{0\}$.

(iii) e may be chosen so that $m \cdot e = m$.

As a corollary of the above proposition, we get that if N satisfies DCC on N-subgroups and M is minimal nonnilpotent right N-subgroup of N, then M contains a left identity.

H. Lausch [5] defines a nonzero idempotent e of N to be the *primitive*, if there does not exist $f \in I(N), f \neq 0$, such that $ef = f$ and $fe \neq e$.

Proposition 2.2. (Peterson [6]) *Let N be a near-ring satisfying DCC on N-subgroups and* e $\in I(N)$. *The following statements are equivalent :*

(i) e is a primitive idempotent of N.

(ii) eN is a minimal nonnilpotent right N-subgroup of N.

(iii) eN is a minimal self-monogenic right N-subgroup of N.

Consider the following examples :

Exemple 2.3 *Consider $N = \mathbb{Z}^5$ and the operations :*

$$\begin{cases} x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + x_2 y_3 + x_3 y_1 + y_5) \\ \qquad\qquad x \cdot y = (x_1 y_1, x_2 y_2, 0,0,0). \end{cases}$$

$(N,+,\cdot)$ is a distributive near-ring and its central idempotents are : 0, $e_1 = (1,0,0,0,0)$, $e_2 = (0,1,0,0,0)$, $e = (1,1,0,0,0)$, and $e = e_1 + e_2$.

Exemple 2.4 *N*= $\mathbb{Z}^6$*,together with the operations:*

$$\begin{cases} x+y=(x_1+y_1, x_2+x_1y_4+y_2, x_3+x_1y_5+x_2y_6+y_3, x_4+y_4, x_5+x_4y_6+y_5, x_6+y_6) \\ x\cdot y=(x_1y_1,0,0,0,0,x_6y_6). \end{cases}$$

is a distributive near-ring with the central idempotents 0, $e_1=(1,0,0,0,0,0)$, $e_2=(0,0,0,0,0,1)$, $e=(1,0,0,0,0,1)$.

There are distributive near-rings with a finite number of central idempotents. For such near-rings we may prove properties which are very similar with those for rings, in spite of the noncommutativity of the addition.

Proposition 2.5. *Let N be a distributive near-ring with a finite set of nonzero central idempotents, none of them of additive order 2, let us denote their set by* $I=\{e_i \mid i=1,2,\dots n, e_i\neq 0\}\cup\{0\}, n\geq 2.$ *Then :*

(i) $e_i+e_j\in I$ *if and only if* $e_i\cdot e_j=0$, *and then* $e_i+e_j\neq 0$.

(ii) If $e_i\cdot e_j=e_j$, *then there exists* $e_k\in I, e_k\neq 0$, *such that* $e_i=e_j+e_k$, $e_ie_k=e_k$, $e_je_k=0$.

(iii) $\prod_{i=1}^{n} e_i=0$.

(iv) If e_i *and* e_j*,* $i\neq j$*, are not decomposable in sum of two different elements of I, then* $e_i\cdot e_j=0$.

(v) Each element of I is either indecomposable or can be written uniquely up to the order of the terms as a sum of indecomposable elements of I.

(vi) If the set of indecomposable elements of I has the cardinal m, then $|I|=2^n$.

Proof. (i) If $e_i+e_j\in I$, we have $(e_i+e_j)^2=e_i+e_j$, hence $2e_ie_j=0$ and $e_ie_j=0$. But then $e_i+e_j\neq 0$, otherwise $e_j=-e_i\in I$ and $(-e_i)^2=-e_i$, false. The converse is straightforward.

(ii) $e_k=e_i-e_j$.

(iii) If $\prod_{i=1}^{n} e_i=e_1\neq 0$ (by changing indices, if necessary), we denote $\prod_{i=2}^{n} e_i=e_2$. Then $e_1\cdot e_2=e_1$, and, by (ii), $e_2=e_1+e_3$, $e_1\cdot e_3=0$. Hence $\prod_{i=1}^{n} e_i=0$ (contradiction). Therefore $\prod_{i=1}^{n} e_i=0$.

(iv) If $e_i e_j = e_k \neq 0$, then $e_i e_k = e_k$ and $e_i = e_k + e_h$, so e_i is decomposable (contradiction).

(v) Let $e \in I$, then e is not decomposable or $e = e_1 + e_2$, with at least e_1 indecomposable. Apply the argument to e_2 and so on, till we get, say, $e = e_1 + e_2 + ... + e_k$, with different terms, otherwise I has elements of order 2. Let us note that $e_i + e_j = e_j + e_i$, since $e_i = e_i^2$ and $e_j = e_j^2$.

(vi) It is easily seen.

As it has been proved since 1966 by J.C. Beidleman, for a central idempotent e of N, the Peirce decomposition is valid:
$N = K_e + N_e$ (semidirect sum of groups),
where $N_e = \{e \cdot x | x \in N\}$ and $K_e = \{x - e \cdot x | x \in N\}$, N_e being an N-subgroup of N. Here N_e is a ring with e as its identity.

It is immediate the following proposition:

Proposition 2.6. *If $I_1 = \{e_1, ..., e_m\} \subseteq I$ is the set of all indecomposable central idempotents of a distributive near-ring N, then:*

$$N = \left(\bigcap_{i=1}^{m} K_{e_i} \right) + \left(\bigoplus_{i=1}^{m} N_{e_i} \right).$$

Here the first term is an ideal, not containing central idempotents.

This structure theorem suggests us a construction procedure for distributive near-rings with a finite number of central idempotents.

Construction procedure 2.7. *Take R_i a ring without idempotents different from 0 and e_i, with the identity e_i, char(R_i) ≠ 2, i = 1, 2,...,n, and K a distributive near-ring without non-zero central idempotents. Then $N = K \times R_1 \times ... \times R_n$ can be structured with componentwise addition and multiplication as a distributive near-ring with a finite number of central idempotents.*

An easy consequence of Proposition 2.6 is:

Corollary 2.8. *If (N,+,·) is a nontrivial near-ring with (N, +) a simple group, then either N has no central idempotents different from 0, or N is a ring.*

We present now an unexpected result on N-groups with DCC and ACC on N-subgroups, connected to idempotents of N.

Let N be a zero-symmetric right near-ring with $1 \neq 0$, and $(\Gamma, +)$ be a unital N-group. Γ is called ***tame*** if for all $\alpha, \beta \in \Gamma$ and $a \in N$, there is an element $b \in N$, such that

$$a \cdot (\alpha + \beta) - a \cdot \alpha = b \cdot \beta .$$

Each N-subgroup of a tame N-group Γ is an ideal of Γ, hence we may consider the sum of two N-subgroups and their intersection.

An irreducible ideal A of Γ is an ideal for which the following implication holds:

$$A=B+C \Rightarrow A = B \text{ or } A = C.$$

(Here B, C are ideals of Γ.)

The ideals of Γ form a lattice with respect to join $A \vee B := A + B$ and intersection $A \wedge B := A \cap B$. In case both chain conditions are satisfied for ideals of Γ, we may consider the ***subcover*** of an ideal A, denoted by A^-.

E. Aichinger [7] proved the following result of characterizing those ideals of a tame N-group Γ which are the range of an idempotent of N.

Proposition 2.9. (Aichinger [7]) *Let N be a zero-symmetric near-ring with $1 \neq 0$, Γ be a tame N-group and H be an N-subgroup of Γ. If N satisfies DCC on left ideals and Γ satisfies both ACC and DCC on ideals ($\equiv$ N-subgroups), then the following are equivalent:*

(i) There exists $e \in I(N)$, with $e \cdot h = h$ for all $h \in H$ and $e \cdot \Gamma \subseteq H$:

(ii) If A and B are irreducible (in sums) ideals of Γ, $A \subseteq H$ and $A=A/A^-$ and B/B^- are N-isomorphic, then $B \subseteq H$.

G. Peterson [6] proved that, for near-rings satisfying DCC on N-subgroups, it is possible to lift idempotents from N/I, with I an ideal of N, to N. Namely, we have:

Proposition 2.10. (Peterson [6]) *(i) If N satisfies DCC on N-subgroups, I is an ideal of N and ε is an idempotent in N/I, then there is $e \in Id(N)$, such that $\varepsilon = e + I$.*

(ii) If ε is primitive (in the sense of Lausch), then e is primitive. Conversely, if e is a primitive idempotent of N, then $e + I$ is a primitive idempotent of N/I.

(iii) If e and f are primitive idempotents in N, I is a nilpotent ideal of N, then $eN \cong fN$ if and only if $\overline{e}\overline{N} \cong \overline{f}\overline{N}$, where $\overline{N} = N/I$, $\overline{e} = e + I$, $\overline{f} = f + I$.

These lifted idempotents satisfy the following theorem.

Proposition 2.11. *If N satisfies DCC on N-subgoups, I is a nilpotent ideal and* $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_k\}$ *are orthogonal primitive idempotents of* $\overline{N}$*, then there are orthogonal primitive idempotents of N,* $\{e_1, e_2, ..., e_k\}$*, such that* $\overline{e_i} = \varepsilon_i$*, i = 1, 2,..., k.*

Proof. We use the induction on k, since for $k = 1$ the statement is true (by the above proposition). Consider $\{f_1, f_2, ..., f_{k-1}\}$ primitive idempotents with $\overline{f_i} = \varepsilon_i$, $i = 1,..., k$, $\overline{f_i}\,\overline{f_j} = 0$, $i \neq j$, and $f \in Id(N)$, primitive, such that $\overline{f} = \varepsilon_k$.

Then $g = f - f_1 f - ... - f_{n-1} f$ has the property: $f_i g = 0$, for $i = 1, 2, ..., k - 1$, $\overline{g} = \varepsilon_k$. Because of DCC on N-subgroups, for g there is $t \in \iddots$, such that M = $g^t N = g^{t+j} N$, for all $j \in \iddots$. There is a left identity of M, e_k, such that $x^{t+1} e_k = x^{k+1}$. We see that $\overline{e_k} = \varepsilon_k, e_k \in gN$ and $f_i e_k = 0$, for all i.

By considering $y = (f_1 - e_k f_1) f_1$, we get $\overline{y} = \varepsilon_1$, $e_n y = 0$ and $f_i y = 0$, $i > 1$. Since N satisfies DCC, we get $M = y^t M = y^{t+1} M$, hence, since $y^{t+1} e_1 f_i = y^{t+1} f_i = 0$, $i > 1$ and $y^{t+1} e_1 e_k = y^{t+1} e_k = 0$, we get $e_1 f_i$, $e_1 e_k \in Ann_M (y^{t+1}) = 0$. Hence $e_1 f_i = 0$, $e_1 e_k = 0$.
So we repeat the argument obtaining the desired set of orthogonal idem- potents. They are also primitive, from Peterson's theorem (Proposition 2.10).

Just in order to see which are the idempotents in polynomial near-rings, consider $(\Gamma, +)$ a group and $\Gamma[X]$ be the set of elements of the form:

$$\{\sum_{i=1}^{r} (\gamma_{i-1} + n_i X + \gamma_i) \,\big|\, r \in \mathbb{N}^*, n_i \in \mathbb{Z}^*, \gamma_i \in G, i \in \{1, 2, ..., r\}, \gamma_r \neq 0\}.$$

This set together with concatenation of sums (and reduction) and substitution as multiplication is a right near-ring of polynomials.

Meldrum, Pilz, So [8] proved the following theorem:

Proposition 2.12. (Meldrum, Pilz, So [8]) *The only idempotents in* $\Gamma[X]$ *are X and the constant polynomials* $\gamma \in \Gamma$*. If e is a central idempotent, then* $e \in N_0, N_0 := \{y \in \Gamma[X] \,|\, y \cdot 0 = 0\}$ *1 - e is also a central idempotent.*

3. Idempotents in affine infra-near-rings

We take some definitions and properties of the defined concepts from Ştefăanescu [2], [9].

Definition 3.1. ***A left infra-near-ring*** *is a triple $(N,+,\cdot)$, where $(N,+;,0)$ is a group and $(N,\cdot)$ is a semigroup, satisfying the condition, which is a „weak" left distributivity law:*

$$a \cdot (b + c) = a \cdot b - a \cdot 0 + a \cdot c, \ \forall a, b, c \in N,$$

and the condition

$$0 \cdot 0 = 0.$$

An element $a \in N$ is **distributive***, if $a \cdot 0 = 0$.*

Then we have the Peirce decomposition with respect to a distributive idempotent $e \in N$.

Proposition 3.2. *Let N be a left infra-near-ring and e be a distributive idempotent of N. Then:*

(i) $\varphi_e : N \to N, \varphi_e(x) = e \cdot x, x \in N$, is an idempotent endomorphism of the right N-infra-module N;

(ii) $Im\,\varphi_e = \{e\dot{x} | x \in I\}$ is a right N-infra-module with a left identity, e, on which φ_e is the identity map;

(iii) $Ker\,\varphi_e$ is a right ideal in N and $Ker\,\varphi_e \cap \operatorname{Im}\varphi_e = \{0\}$;

(iv) $N = Ker\,\varphi_e + Im\,\varphi_e$ (semidirect sum of groups).

Proof. Straightforward verifications of required properties.

In the following, N will be always a left infra-near-ring.

Proposition 3.3. *If $e \in N$ is a central distributive idempotent, then φ_e is an endomorphism of N, $Ker\,\varphi_e$ is an ideal of N and $Im\,\varphi_e$ is a sub-infra-near-ring of N, with identity, $e \in N$.*

Proof. Verifications.

Moreover, if we write $x = x_1 + x_2$, $y = y_1 + y_2$, with $x_1, y_1 \in Ker\,\varphi_e$ and x_2, $y_2 \in Im\,\varphi_e$, then:

(1) $x_2 \cdot y = x_2 \cdot y_2, \ \forall x_2 \in Im\,\varphi_e, \ \forall y = y_1 + y_2 \in N.$

(2) $x_1 \cdot y = x_1 \cdot y_1 + 0 \cdot x_2, \ \forall x_1 \in Ker\,\varphi_e, \ \forall y = y_1 + y_2 \in N.$

(3) $x \cdot y_2 = x_2 \cdot y_2, \ \forall x = x_1 + x_2 \in N, y_2 \in Im\,\varphi_e.$

Definition 3.4. *If N is a left infra-near-ring with $0 \cdot x = 0$, for all $x \in N$, and it satisfies:*

(i) $(x + y) \cdot 0 = x \cdot 0 + y \cdot 0, \ \forall x, y \in N$;

(ii) $x - x \cdot 0 = -x \cdot 0 + x,\ \forall x \in N$;
(iii) $(s + d) \cdot x = s \cdot x + d \cdot x,\ \forall x \in N$ *and* $d \in D := \{y \in N \mid y \cdot 0 = 0\}$,
$s \in S := \{y \in N \mid y \cdot 0 = 0\}$, *then N is called an* ***affine left infra-near-ring***.

We know that for an affine left infra-near-ring, D is a zero-symmetric near-ring, S is a left D-near module and $N = D + S$, $(D, +)$ being a normal subgroup of $(N, +)$. If $x = d_1+s_1$, $y = d_2+s_2$, then $x \cdot y = d_1 \cdot d_2+(s_1 + d_1 \cdot s_2)$.

Therefore, in such an affine left infra-near-ring, all the elements of S are nondistributive idempotents and all distridutive idempotents belong to D.

Proposition 3.5. *If* $(N,+,\cdot)$ *is a left infra-near-ring with identity* $1 \neq 0$, *and a right near-ring, then we have:*
(i) $(N,+)$ *is Abelian.*
(ii) If $N = I_1 \oplus I_2$, *with* I_1, I_2 *right ideals in N, then there are orthogonal idempotents in N,* e_1, e_2, *such that* $1 = e_1 + e_2$ *and* e_i *is a left identity in* I_i, $i = 1, 2$.
(iii) A right ideal I of N is a direct summand of N if and only if there exists $e_1 \in Id(N)$, *such that* $I = e_1 \cdot N$.

Proof is not very easy, but it can be done by using the same arguments as for rings.

It is amazing how many semigroupal properties of rings remain valid by using weaker condition of distributivity.

4. Idempotents in ringoids

This structure has been introduced by S.K. Sehgal [10] in 1964, in his thesis supervised by H. Zassenhaus.

We point out that some properties of idempotents remain valid also in this case.

Definition 4.1. *A nonempty set R is said to be a* ***ringoid****, if operations + and* $\cdot$ *are partially defined, such that the following conditions are satisfied:*

(i) $R = \bigcup_{i \in I} A_i, (A_i, +)$ *being Abelian groups with* $|A_i| \geq 2$.

(ii) For all a, b, c $\in$ *R, the following hold if either side is defined i.e. if one side is defined, then the other side is defined also and the two sides are equal:*

$$a(bc) = (ab)c;$$
$$a(b + c) = ab + ac;$$
$$(b + c)a = ba + ca:$$

(iii) Defining, for $a \in R$, $L(a) := \{x \in R \mid x \cdot a$ *is defined*$\}$ *and* $R(a) := \{x \in R \mid ax$ *is defined*$\}$*, R satisfies conditions:*

(iii)$_1$ *For all* $a \in R$, $L(a) \neq$ f , $R(a) \neq$ f .

(iii)$_2$ *If* $L(a) \cap L(b) \neq$ f *and* $R(a) \cap R(b) \neq$ f *, then* $a + b$ *is defined.*

Since $|A_i| \geq 2$, for all $a \in R$, there is $b \in R$, $b \neq a$, such that $a + b$ is defined.

In Sehgal [10], the Peirce decomposition for an idempotent $e \in R$ is given:
$R = eR + R'$, with R' a right ideal.

Indeed, $R' = \{r - er \mid r \in R\} \cap \{r \mid r - er$ is not defined$\}$.

A right ideal I of R is a nonempty set such that

$$I - I = \{r - s \mid r, s \in I, r - s \text{ defined}\} \subseteq I.$$

We summarized some results in Sehgal's paper [10]:

Proposition 4.2. *If R is a ringoid satisfying the conditions:*

(i) R does not have nilpotent right ideals I, $I \neq \{0\}$*;*

(ii) R satisfies the DCC on right ideals,

then R is a direct sum of a finite number of minimal right idals e_iR*, with* $e_i^2 = e_i$.

We must note that by giving up the associativity law (which is replaced by a weaker condition), we have obtained, in [11], Peirce decomposition for some nonassociative algebras.

5. Some geometry on almost affine K-module

Let K be a field or a unitary ring.

Definition 5.1. *An almost affine space (module) over the field (ring) K is an ordered system* $(M, \rho, TM, *)$ *such that M is a nonempty set of* ***points*** *denoted A, B, C, ... ,* ρ *is an equivalence relation on* $M \times M$ *whose classes called* ***vectors*** *and denoted by x, y, ... , with* $x = \overrightarrow{AB} := \{\overline{CD} \in M \times M \mid \overline{CD} \rho \overline{AB}\}$ *form the set* $TM := M \times M / \rho$, $* : K \times TM \to TM$ *is a mapping,* $(\alpha, x) \to \alpha * x$, *and the following axioms are fulfilled:*

(i) For all $A \in M$ *and* $x \in TM$*, there exists a point* $B \in M$ *such that* $x = \overrightarrow{AB}$.

(ii) For all A, B, $A' \in M$*, if* $\overrightarrow{AB} = \overrightarrow{A'B}$*, then* $A = A'$

(iii) (Triangle axiom) For all A, B, C, A', B', $C' \in M$ *if* $\overrightarrow{AB} = \overrightarrow{A'B'}$ *and* $\overrightarrow{BC} = \overrightarrow{B'C'}$*, then* $\overrightarrow{AC} = \overrightarrow{A'C'}$.

(iv) For all $\alpha \in K$, $x, y \in TM$, $\alpha * (x+y) = \alpha * x + \alpha * y$.

(v) For all $\alpha, \beta \in K$, $x \in TM$, $(\alpha\beta) * x = \alpha * (\beta * x)$.

Proposition 4.2. *Let M be an almost affine K-module. Then:*

(i)TM decomposes as a semidirect sum TM =$L_0 + L_1$*, where* $L_0 = \{ x \in TM \mid 1 * x = 0 \}$ *is a normal almost subspace of Tm and* $L_1 = \{ x \in TM \mid 1 * x = x \}$ *is a linear K–subspace of TM. The mapping* $\varphi_1 : TM \to TM$, $\varphi_1(x) = 1 * x$, *is an idempotent endomorphism of TM.*

(ii) If K is a ring and $\varepsilon \in K$ *is a central idempotent of K, then we get a Peirce decomposition for TM, TM =* $L_\varepsilon + N_\varepsilon$, *where* $L_\varepsilon = \{x \in TM \mid \varepsilon * x = x\}$ *and* $N_\varepsilon = \{x \in TM \mid \varepsilon * x = 0\}$

Proof The proof consists in straightforward verification of the axioms.

6. Conclusions

In the above developments, we have seen that Peirce decompositions in generalized rings are based upon the semigroupal properties of idempotents, no matter we gave up the commutativity of the addition and one of the distributivity lows of multiplication with respect to addition.

Future results in this topics are expected for lifting idempotents in generalized rings.

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