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APPROXIMATE AMENABILITY MODULO AN IDEAL OF BANACH ALGEBRAS

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In this paper we introduce and study the concept of approximate amenability (contractibility) modulo an ideal of Banach algebras. Based on the obtained results, we prove that the weighted semigroup algebra $\ell^1(S,\omega)$ is approximately amenable modulo an ideal if and only if S is amenable and ω_σ is diagonally bounded where ω_σ is the induced weight on S/σ for the least group congruence σ on S.

Keywords: Amenability modulo an ideal, Approximate amenability modulo an ideal, Semigroup algebra, group congruence

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1. Introduction

The notion of approximate amenability of Banach algebra was introduced by Ghahramani and Loy in [6]. The structure of approximately amenable (contractible) Banach algebras is considered in [6] through different ways. It is shown two concepts of approximately amenable and approximately contractible for Banach algebras are equivalent and the group algebra $L^1(G)$, where G is a locally compact group, is approximately amenable (contractible) if and only if G is amenable. For a discrete semigroup S, the necessary and sufficient conditions for approximately amenable (contractible) of the semigroup algebra $l^1(S)$ is quite more complicated. It has not been explored when $l^1(S)$ is approximately amenable. For discrete semigroup S, it has been shown that if $l^1(S)$ is approximately amenable, then S is regular and amenable [7], and if S is right cancellative semigroup such that $l^1(S)$ is approximately amenable then S is an amenable group and $l^1(S)$ is amenable [11]. To see some remarks of approximate amenability of Banach algebras see [2, 3, 7, 11, 9].

The concept of amenability modulo an ideal for the l^1- semigroup algebra was introduced by the first author and M. Amini in [1]. The effect is to gather up the non-group structure into a congruence and its corresponding ideal and establish some analogue of Johnson's theorem on groups and their l^1-algebras. Following up some basic properties of amenability modulo an ideal, contractibility modulo an ideal of Banach algebras and their applications for semigroup algebras were studied in [13].

This paper is organized as follows: in section 2, we give some preliminaries. In section 3, we study basic properties of approximately amenable (contractible) modulo an ideal of Banach algebras. Among other results, we give a characterization of approximately amenable (contractible) modulo an ideal of Banach algebras in terms of the existence of a bounded approximate diagonal modulo an ideal of A. Finally, in section four, using the obtained results we investigate approximately amenable modulo an ideal of weighted semigroup algebras $l^1(S,\omega)$ for a large class of semigroups. As an interesting result we show

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that $l^1(S,\omega)$ is approximately amenable modulo an ideal if and only if S is amenable and Ω_{ω_σ} is bounded where Ω_{ω_σ} is the symmetrization of the induced weight ω_σ on S/σ and σ is the least group congruence on semigroup S.

2. **Preliminaries**

For a Banach algebra A, let X be a Banach A-bimodule. A linear mapping $D : A \to X$ is said to be a derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A)$. A derivation D is said to be inner (approximately inner) if there exists $x \in X$ ($(x_\alpha)_\alpha \subseteq X$) such that $D(a) = a.x - x.a$ ($D(a) = \lim_\alpha ad_{x_\alpha}(a)$) $(a \in A)$. A Banach algebra A is said to be amenable (approximately amenable) if every continuous derivation $D : A \to X^*$ is inner (approximately inner) for all Banach A-bimodules X. We now begin by recalling some definitions and then we shall give some of the basic consequences of our definitions. Further results in this area are given in the recent papers [1, 12, 13, 14], which in particular contains interesting results on contractibility modulo an ideal, some hereditary properties of amenability modulo an ideal, together with several illuminating examples.

Definition 2.1. [13] *Let I be a closed ideal of A.*
(i) A Banach algebra A is amenable (contractible) modulo I if every bounded derivation $D : A \to X^*$ ($D : A \to X$) is inner on $A\backslash I := \{a \in A : a \notin I\}$ for all Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$.
(ii)A bounded net $\{u_\alpha\}_\alpha \subseteq A$ is called approximate identity modulo I if $\lim_\alpha u_\alpha \cdot a = \lim_\alpha a \cdot u_\alpha = a \quad (a \in A \setminus I)$.

All over this paper we fix A and I as above, unless they are otherwise specified.

3. **Approximately amenable modulo an ideal**

Definition 3.1. *Let I be a closed ideal of A. A Banach algebra A is approximately amenable modulo I if every bounded derivation $D : A \to X^*$ is approximately inner on the set theoretic difference $A\backslash I := \{a \in A : a \notin I\}$ for all Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$.*

Theorem 3.1. *Let I be a closed ideal of A. Then the following assertions holds:*
(i) if $\frac{A}{I}$ is approximately amenable and $I^2 = I$, then A is approximately amenable modulo I;
(ii) if A is approximately amenable modulo I, then $\frac{A}{I}$ is approximately amenable.

Proof. (i) Let X be a Banach A-bimodule such that $X \cdot I = I \cdot X = 0$ and $D : A \to X^*$ be a bounded derivation. Then X can be regarded as an $\frac{A}{I}$-bimodule. Set $\overline{D} : \frac{A}{I} \to X^*$ by $\overline{D}(a + I) = D(a)$ $(a \in A)$. Since $I^2 = I$, $D(x) = 0$ $(x \in I)$ and hence $\overline{D}$ is well-defined. Approximately amenability of $\frac{A}{I}$ implies that there exists $(\phi_\alpha)_\alpha \subset X^*$ such that $\overline{D} = \lim_\alpha ad_{\phi_\alpha}$. Then

$$\begin{aligned} D(a) = \overline{D}(a + I) &= \lim_\alpha[(a + I)\phi_\alpha - \phi_\alpha(a + I)] \\ &= \lim_\alpha(a\phi_\alpha - \phi_\alpha a) = \lim_\alpha ad_{\phi_\alpha}(a). \end{aligned}$$

(ii) Let X be a Banach $\frac{A}{I}$-bimodule and $D : \frac{A}{I} \to X^*$ be a bounded derivation. By defining $a.e = \pi(a).e$, $e.a = e.\pi(a)$ $(a \in A, e \in X)$ where $\pi : A \to \frac{A}{I}$ is the canonical quotient map, X can be regarded as a Banach A-module. Obviously $I \cdot X = X \cdot I = 0$ and $D \circ \pi : A \to X^*$ is a bounded derivation. Thus there exists $(\psi_\alpha)_\alpha \subset X^*$ such that $D \circ \pi = \lim_\alpha ad_{\psi_\alpha}$ on $A\backslash I$. Now $D(a + I) = D \circ \pi(a) = \lim_\alpha ad_{\psi_\alpha}(a) = \lim_\alpha(a.\psi_\alpha - \psi_\alpha.a)$ $(a \in A)$. Hence $D = \lim_\alpha ad_{\psi_\alpha}$. □

Corollary 3.1. *If A is approximately amenable modulo I and I is approximately amenable, then A is approximately amenable.*

One could modify the proof of the following Lemma (which is essentially due to Ghahramani and Loy [6]) to get the same result for approximately amenable modulo an ideal.

Lemma 3.1. *If A is an unital Banach algebra which is approximately amenable modulo I, X is an A-bimodule such that $I \cdot X = X \cdot I = 0$ and $D : A \to X^*$ is a derivation, then there are $(\phi_\alpha)_\alpha \subset e \cdot X^* \cdot e$ and $\eta \in X^*$ such that*

(1) $\|\eta\| \leq 2C_X \|D\|$;

(2) $D = \lim_\alpha ad_{\phi_\alpha} + ad_\eta$ *on* $A \setminus I$.

Proof. By [6, Lemma 2.3], there are derivation $D_1 : A \to e \cdot X^* \cdot e$ and $\eta \in X^*$ such that $\|\eta\| \leq 2C_X\|D\|$ and $D = D_1 + ad_\eta$. Now $e \cdot X^* \cdot e \cong (e \cdot X \cdot e)^*$, A is approximately amenable modulo I and $I \cdot (e \cdot X \cdot e) = (e \cdot X \cdot e) \cdot I = 0$, so there exists $(\phi_\alpha)_\alpha \subset e \cdot X^* \cdot e$ such that $D_1 = \lim_\alpha ad_{\phi_\alpha}$ on $A \setminus I$. Thus $D = \lim_\alpha ad_{\phi_\alpha} + ad_\eta$ on $A \setminus I$. □

Let A be a Banach algebra. Then $A^\sharp$ is also a Banach algebra in a standard way which gives the adjoined identity norm one.

Theorem 3.2. *A is approximately amenable modulo I if and only if $A^\sharp$ is approximately amenable modulo I.*

Proof. Let X be an $A^\sharp$-bimodule such that $I \cdot X = X \cdot I = 0$ and $D : A^\sharp \to X^*$ be a derivation. By [6, Lemma 2.3], $D = D_1 + ad_\eta$ where $\eta \in X^*, D_1 : A^\sharp \to e.X^*.e$. Now $D_1(e) = 0$ and $D_1|_A : A \to e \cdot X^* \cdot e$ is a derivation, so approximately amenable modulo I of A implies that there exists $(\psi_\alpha)_\alpha \subset e \cdot X^* \cdot e$ such that $D_1|_A = \lim_\alpha ad_{\psi_\alpha}$ on $A \setminus I$. Thus $D = \lim_\alpha ad_{\phi_{\alpha+\eta}}$ $(a \in A \setminus I)$.

Conversely, let X be an A-bimodule such that $I \cdot X = X \cdot I = 0$ and $D : A \to X^*$ be a derivation. Set $\overline{D} : A^\sharp \to X^*$ by $\overline{D}(a, \lambda) = D(a)$ $((a, \lambda) \in A^\sharp)$. Now X can be made into a Banach $A^\sharp$-bimodule by defining $(a, \lambda) \cdot x = a \cdot x + \lambda x$, $x \cdot (a, \lambda) = x \cdot a + \lambda x$ $(x \in X,\ (a, \lambda) \in A^\sharp)$. We have $e \cdot Da = Da = Da \cdot e$ $(a \in A)$ and

$$\begin{aligned}\overline{D}((a_1, \lambda_1)(a_2, \lambda_2)) &= D(a_1a_2) + \lambda_2 e \cdot D(a_1) + \lambda_1 e \cdot D(a_2)\\ &= D(a_1) \cdot a_2 + a_1 \cdot D(a_2) + \lambda_2 e \cdot D(a_1) + \lambda_1 e \cdot D(a_2)\\ &= \overline{D}(a_1, \lambda_1) \cdot (a_2, \lambda_2) + (a_1, \lambda_1) \cdot \overline{D}(a_2, \lambda_2),\end{aligned}$$

so $\overline{D}$ is a derivation $(a_1, a_2 \in A, \lambda_1, \lambda_2 \in \mathbb{C})$. Since $A^\sharp$ is approximately amenable modulo I, there exists net $(\psi_\alpha)_\alpha \subset X^*$ such that $\overline{D} = \lim_\alpha ad_{\psi_\alpha}$ on $A^\sharp \setminus I$. Thus $D = \overline{D}\,|_A$ is approximately inner on $A \setminus I$. □

We recall that an A-bimodule X is called neo-unital modulo I [13], if

$$X = (A \setminus I) \cdot X \cdot (A \setminus I) = \{\, a \cdot x \cdot b \mid a, b \in A \setminus I, x \in X\}.$$

It is known that for a Banach algebra A with bounded approximate identity modulo I, $H^1_I(A, X^*) = 0$ for each Banach A-bimodule X if and only if $H^1_I(A, X^*) = 0$ for each neo-unital modulo I Banach A-bimodule X [14, Theorem 2.5]. Similarly, if A is a Banach algebra with a bounded approximate identity modulo I, then A is approximately amenable modulo I if for every neo-unital modulo I Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$, every derivation $D : A \to X^*$ is approximately inner on $A \setminus I := \{a \in A : a \notin I\}$. We denote the identity of $\frac{A^\sharp}{I}$ by $\bar{e}$ and fix it till end of this paper.

Theorem 3.3. *The following conditions are equivalent:*

(a) A is approximately amenable modulo I;

(b) There is a net $(M_\nu) \subset (\frac{A^\sharp}{I} \hat{\otimes} A^\sharp)^{**}$ such that $a \cdot M_\nu - M_\nu \cdot a \to 0$ $(\forall a \in A^\sharp \backslash I)$ and $\pi^{**}(M_\nu) \longrightarrow \bar{e}$;

(c) There is a net $(M'_\nu) \subset (\frac{A^\sharp}{I} \hat{\otimes} A^\sharp)^{**}$ such that $a \cdot M'_\nu - M'_\nu \cdot a \longrightarrow 0$ $(\forall a \in A^\sharp \backslash I)$ and $\pi^{**}(M'_\nu) = \bar{e}$ for every ν.

Proof. $(a) \to (c)$ Let A be approximately amenable modulo I, then $A^\sharp$ is approximately amenable modulo I (by Theorem 3.2). Put $u = \widehat{(\bar{e}\otimes e)} \in (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)^{**}$ and define $\pi : (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp) \to \frac{A^\sharp}{I}$ by $\pi(\bar{a}\otimes b) = \overline{ab}$. Now $ad_u : A^\sharp \to (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)^{**}$ by $a \mapsto a.u - u.a$ is a bounded derivation and

$$\begin{aligned}\pi^{**}(ad_u(a))(f) = {} & ad_u(a)(\pi^* f)\\ & = a.u(\pi^* f) - u.a(\pi^* f)\\ & = \widehat{(\bar{e}\otimes e)}(\pi^* f.a) - \widehat{(\bar{e}\otimes e)}(a.\pi^* f)\\ & = \pi^* f(\bar{a}\otimes e) - \pi^* f(\bar{e}\otimes a) = 0\end{aligned}$$

Thus $ad_u \in Z^1(A^\sharp, ker(\pi^{**}))$. Let $X := (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)^*$ and $M := \overline{(\pi^*((\frac{A^\sharp}{I})^*))}$, then $\ker\pi^{**} = (\frac{X}{M})^*$. Since $I\cdot(\frac{X}{M}) = (\frac{X}{M})\cdot I = 0$, approximately amenability modulo I of $A^\sharp$ implies that there exists $(M_\nu)_\nu \subset \ker(\pi^{**})$ such that $ad_u(a) = st - \lim_\nu ad_{M_\nu}(a)$ on $A^\sharp\backslash I$. Let $M'_\nu := u - M_\nu$, then for each $\phi \in (\frac{A^\sharp}{I})^*, \bar{c} \in \frac{A^\sharp}{I}$;

$$\begin{aligned}(\pi^{**}(M'_\nu)\cdot\bar{c})(\phi) & = (\pi^{**}(u - M_\nu)\cdot\bar{c})(\phi)\\ & = (\pi^{**}(u)\cdot\bar{c})(\phi) - (\pi^{**}(M_\nu)\cdot\bar{c})(\phi) = (\pi^{**}(u))(\bar{c}.\phi)\\ & = u(\pi^*(\bar{c}.\phi)) = \pi^*((\bar{c}.\phi)(\bar{e}\otimes e) = (\bar{c}.\phi)(\pi(\bar{e}\otimes e))\\ & = \phi(\bar{c}),\end{aligned}$$

and for each $a \in A^\sharp\backslash I$;

$$a\cdot M'_\nu - M'_\nu\cdot a = a\cdot(u - M_\nu) - (u - M_\nu)\cdot a = ad_u(a) - ad_{M_\nu}(a) \to 0.$$

$(c) \to (a)$ Let X be a neo-unital Banach $A^\sharp-$bimodule modulo I such that $I\cdot X = X\cdot I = 0$ and $D : A^\sharp \to X^*$ be a bounded derivation. Since $I\cdot X = X\cdot I = 0$, with the module actions $\bar{a}\cdot x = a\cdot x$, $x\cdot\bar{a} = x\cdot a$, X is an $\frac{A^\sharp}{I}-$bimodule. Put $\mu_x(\bar{a}\otimes b) = (\bar{a}\cdot Db)(x)$ $(\bar{a}\otimes b \in \frac{A^\sharp}{I}\hat{\otimes}A^\sharp, x \in X)$. We have;

$$\begin{aligned}\mu_{x.a-a.x}(\bar{b}\otimes c) & = \mu_{x.\bar{a}-\bar{a}.x}(\bar{b}\otimes c)\\ & = (\bar{b}.Dc)(x.\bar{a} - \bar{a}.x)\\ & = (\bar{b}.Dc)(x.\bar{a}) - (\bar{b}.Dc)(\bar{a}.x)\\ & = (\bar{a}\bar{b}.Dc)(x) - (\bar{b}.Dc.\bar{a})(x) + (\bar{b}\bar{c}.Da)(x)\\ & = \mu_x(\bar{a}\bar{b}\otimes c) - \mu_x(\bar{b}\otimes ca) + (\bar{b}\bar{c}.Da)(x).\end{aligned}$$

Thus $\mu_{x.a-a.x}(m) = (\mu_x.a - a.\mu_x)(m) + (\pi(m).Da)(x)\,(m \in (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp))$. For each ν, define $f_\nu(x) = M_\nu(\mu_x)$. Then $\{f_\nu\} \subseteq X^*$ and $\| f_\nu \|\leq\| M_\nu \|\|\, D \,\| C_X$. We claim that $D = st - \lim_\nu ad_{f_\nu}$. Since $(M_\nu)_\nu \subset (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)^{**}$, there exists $\{m_\nu^\alpha\} \subseteq (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)$ such that $M_\nu = w^* - \lim_\alpha m_\nu^\alpha$. We have;

$$\begin{aligned}(\bar{a}.f_\nu - f_\nu.\bar{a})(x) & = f_\nu(x.\bar{a} - \bar{a}.x) = M_\nu(\mu_{x\cdot\bar{a}-\bar{a}\cdot x}) \quad (x \in X, \bar{a} \in \tfrac{A^\sharp}{I})\\ & = w^* - \lim_\alpha \widehat{m_\nu^\alpha}(\mu_{x.\bar{a}-\bar{a}.x})\\ & = w^* - \lim_\alpha(\mu_x.\bar{a} - \bar{a}.\mu_x)(m_\nu^\alpha) + (\pi(m_\nu^\alpha)Da)(x)\\ & = M_\nu(\mu_x.\bar{a} - \bar{a}.\mu_x) + w^* - \lim_\alpha(\pi(m_\nu^\alpha)Da)(x)\\ & = M_\nu(\mu_x.\bar{a}) - M_\nu(\bar{a}.\mu_x) + (\pi^{**}(M_\nu)Da)(x)\\ & = f_\nu(x.\bar{a}) - f_\nu(\bar{a}.x) + (\pi^{**}(M_\nu)Da)(x)\\ & = f_\nu(x.a) - f_\nu(a.x) + (\pi^{**}(M_\nu)Da)(x)\\ & = a.f_\nu(x) - f_\nu(x).a + (\pi^{**}(M_\nu)Da)(x)\\ & = (a\cdot M_\nu - M_\nu\cdot a)(\mu_x) + (\pi^{**}(M_\nu)Da)(x).\end{aligned}$$

Thus

$$\begin{aligned}\|(a.f_\nu - f_\nu.a)(x) - (Da)(x)\| & \leq \|(a.M_\nu - M_\nu.a)\|\cdot\|D\|.\|x\|\\ & +\|x\|\cdot\|\pi^{**}(M_\nu) - \bar{e}\|\cdot\|Da\|\end{aligned}$$

Whence $D(a) = st - \lim_\alpha ad_{f_\nu}(a)$ $(a \in A^\sharp\backslash I)$.

$(b) \to (a)$ It is clear. □

Theorem 3.4. *A is approximately amenable modulo I if and only if there are nets $(M_\nu^{''}) \subseteq (\frac{A}{I}\hat{\otimes}A)^{**}$, $(F_\nu) \subseteq (\frac{A}{I})^{**}$, $(G_\nu) \subseteq (A)^{**}$, such that for each $a \in A\backslash I$;*

i) $a \cdot M_\nu^{''} - M_\nu^{''} \cdot a + F_\nu \otimes a - a \otimes G_\nu \longrightarrow 0$
ii) $a \cdot F_\nu \longrightarrow a, G_\nu \cdot a \longrightarrow a;$ *and*
iii) $\pi^{**}(M_\nu^{''}) \cdot a - F_\nu \cdot a - G_\nu \cdot a \longrightarrow 0$

Proof. Let $(M_\nu) \subseteq (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)^{**}$ be as in Theorem 3.3. Since

$$(\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)^{**} = (\frac{A}{I}\hat{\otimes}A)^{**} \oplus ((\frac{A}{I})^{**}\hat{\otimes}e) \oplus (e\hat{\otimes}A^{**}) \oplus \mathbb{C}(e\hat{\otimes}e),$$

there exists $(M_\nu^{''}) \subseteq (\frac{A}{I}\hat{\otimes}A)^{**}$, $(F_\nu) \subseteq (\frac{A}{I})^{**}$, $(G_\nu) \subseteq (A)^{**}$ and $(c_\nu) \subseteq \mathbb{C}$ such that $M_\nu = M_\nu^{''} - F_\nu \otimes e - e \otimes G_\nu + c_\nu e \otimes e$. Let $(F_\nu^\alpha) \subseteq (\frac{A}{I})^{**}$ be w^*-accumulation point of $(\widehat{F_\nu^\alpha})$, then for each $f \in (\frac{A^\sharp}{I})^*$;

$$\begin{aligned}(\pi^{**}(F_\nu \otimes e))(f) = \quad & (F_\nu \otimes e)(\pi^* f) = w^* - \lim((\widehat{F_\nu^\alpha \otimes e})(\pi^* f) \\ & = w^* - \lim_\alpha(\pi^* f)(F_\nu^\alpha \otimes e) = w^* - \lim_\alpha f(F_\nu^\alpha) \\ & = w^* - \lim_\alpha \widehat{F_\nu^\alpha}(f) = F_\nu(f)\end{aligned}$$

Then $\pi^{**}(F_\nu \otimes e) = F_\nu$. Similarly $\pi^{**}(e \otimes G_\nu) = G_\nu$, $\pi^{**}(c_\nu e \otimes e) = c_\nu$. Using Theorem 3.3(b), proof is completed. □

Theorem 3.5. *Let A be approximately amenable modulo I, B be a Banach algebra and J be a closed ideal of B. Let $\theta : A \to B$ be a continuous surjective homomorphism such that $\theta(I) \subseteq J$. Then B is approximately amenable modulo J.*

Proof. Let X be a Banach B-bimodule such that $J \cdot X = X \cdot J = 0$ and $D : B \to X^*$ is a bounded derivation. Then X become a Banach A-bimodule where the module actions are defined $a \cdot x := \theta(a) \cdot x$, $x \cdot a := x \cdot \theta(a)\,(a \in A, x \in X)$. Clearly, $I \cdot X = X \cdot I = 0$ and $D \circ \theta : A \to X^*$ is a bounded derivation. Then there exists a net $(\phi_\nu)_\nu \subset X^*$ such that $D \circ \theta = \lim_\nu ad_{\phi_\nu}$ on $A \setminus I$. □

Definition 3.2. *Let A be Banach algebra and I be a closed ideal of A. A is approximately contractible modulo I if for every Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$, every bounded derivation D from A into X is an approximately inner derivation on $A\backslash I := \{a \in A : a \notin I\}$.*

We recall that an element $e \in A$ is an identity modulo I if $ae - a \in I$, $ea - a \in I\ (a \in A)$ and a Banach algebra A is unital modulo I if A has an identity modulo I [13].

Theorem 3.6. *The following assertions hold.*

(i) If $\frac{A}{I}$ is approximately contractible and $I^2 = I$, then A is approximately contractible modulo I.
(ii) If A is approximately contractible modulo I, then $\frac{A}{I}$ is approximately contractible.
(iii) If A is approximately contractible modulo I and I is approximately contractible, then A is approximately contractible.
(iv) If A is approximately contractible modulo I, then A is unital modulo I.

Proof. (i) Let X be a Banach A-bimodule such that $X \cdot I = I \cdot X = 0$ and $D : A \to X$ be a bounded derivation. Analogous argument of Theorem 3.1, shows that X can be regarded as $\frac{A}{I}$-bimodule and $D_1 : \frac{A}{I} \to X$ by $D_1(a+I) = D(a)\,(a \in A)$ is well-define bounded derivation. Since $\frac{A}{I}$ is approximately contractible, there exits $(\phi_\alpha)_\alpha \subset X$ such that $D_1 = \lim_\alpha ad_{\phi_\alpha}$. Then

$$D(a) = D_1(a + I) = \lim_\alpha ad_{\phi_\alpha}(a) \quad (a \in A \setminus I).$$

(ii) Let X be a Banach $\frac{A}{I}$-bimodule and $D : \frac{A}{I} \to X$ be a bounded derivation. Now X is an A-bimodule by defining $a \cdot x = \pi(a) \cdot x,\ x \cdot a = x \cdot \pi(a)\,(a \in A, x \in X)$. Clearly $D \circ \pi : A \to X$ is a bounded derivation and A is approximately contractible modulo I implies that there exists $(\phi_\alpha)_\alpha \subseteq X$ such that $D \circ \pi = \lim_\alpha ad_{\phi_\alpha}$ on $A \setminus I$. Thus $D = \lim_\alpha ad_{\phi_\alpha}$.
(iii) and (iv) are straightforward. □

The following results are the analogous of Theorems 3.3 and 3.5.

Theorem 3.7. *Let A be approximately contractible modulo I, B be a Banach algebra and J be a closed ideal of B. Let $\theta : A \to B$ be a continuous surjective homomorphism such that $\theta(I) \subseteq J$. Then B is approximately contractible modulo J.*

Theorem 3.8. *The following assertions are equivalent.*
(i) *A is approximately contractible modulo I.*
(ii) *There exists $(M_\nu) \subset (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)$ such that $a \cdot M_\nu - M_\nu \cdot a \longrightarrow 0\,(a \in A^\sharp \backslash I)$ and $\pi(M_\nu) \longrightarrow \bar{e}$.*
(iii) *There exists $(M_\nu^{'}) \subset (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)$ such that $a \cdot M_\nu^{'} - M_\nu^{'} \cdot a \longrightarrow 0\,(a \in A^\sharp \backslash I)$ and $\pi(M_\nu^{'}) = \bar{e}$.*
(iv) *There are $(M_\nu^{''}) \subseteq (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp), (F_\nu) \subseteq (\frac{A^\sharp}{I}), (G_\nu) \subseteq A^\sharp$ such that*
$a \cdot M_\nu^{''} - M_\nu^{''} \cdot a + F_\nu \otimes a - a \otimes G_\nu \longrightarrow 0, \quad a \cdot F_\nu \longrightarrow a, \quad G_\nu \cdot a \longrightarrow a$ *and*
$\pi(M_\nu^{''}) \cdot a - F_\nu \cdot a - G_\nu \cdot a \longrightarrow 0.$

4. Approximate amenability modulo an ideal of weighted semigroup algebras

We begin by introducing some basic concepts that are used throughout this section. Let S be a semigroup and $E = E(S)$ be the set (possibly empty) of idempotents of S. We recall that a semigroup S is called an *E-semigroup* if $E(S)$ forms a sub-semigroup of S, *E-inversive* if for all $x \in S$, there exists $y \in S$ such that $xy \in E(S)$, *regular* if $V(a) \neq \phi$ where $V(a) = \{x \in S :\ a = axa, x = xax\}$ is the set of inverses of $a \in S$ and S is called an *inverse semigroup* if moreover, the inverse of each element is unique. An inverse semigroup S is called *E-unitary* if for each $x \in S$ and $e \in E(S)$, $ex \in E(S)$ implies $x \in E(S)$ and S is called *semilattice* if S is a commutative and idempotent semigroup. Also, a semigroup S is called *eventually inverse* if every element of S has some power that is regular and $E(S)$ is a semilattice.
A congruence ρ on semigroup S is called group congruence if S/ρ is group. We denote the least group congruence on semigroup S by σ. Following Gigon [8], if S is an E-inversive E-semigroup with commuting idempotents or S is an eventually inverse semigroup then $\sigma = \{(a, b) \in S \times S \,|\, ea = fb \text{ for some } e, f \in E(S)\}$ is the least group congruence on S.

Theorem 4.1. [1] (i) *Let S be an eventually inverse semigroup or S be an E-inversive E-semigroup with commuting idempotents . Then S is amenable if and only if S/σ is amenable.*
(ii) *Let ρ be a group congruence on S with commuting $Ker\rho$, then S is amenable if and only if S/ρ is amenable.*

By a weight on semigroup S (group G), we mean a function $\omega : S \to (0, \infty)$ such that $\omega(st) \leq \omega(s)\omega(t)\,(s, t \in S)$. The weight ω on group G is called symmetric if $\omega(g) = \omega(g^{-1})(g \in G)$. For any weight ω, its symmetrization is the weight defined by $\Omega_\omega(g) = \omega(g)\omega(g^{-1})$. Let S be a semigroup and ω be a weight on S. The weighted semigroup algebra $l^1(S, \omega) = \{f \mid f : S \to \mathbb{C}, \sum_{s \in S} |f(s)|\omega(s) < \infty\}$ under convolution is a Banach algebra where $\|f\|_{1,\omega} = \sum_{s \in S} |f(s)|\omega(s)$. If $\omega = 1$, the weighted semigroup algebra $l^1(S, \omega)$ is called semigroup algebra and is denoted by $l^1(S)$.

Lemma 4.1. *The following statements hold:*
(i) if S is a semigroup, ρ is a congruence on S and ω is a weight on S, then $\frac{l^1(S,\omega)}{I_\rho} \simeq l^1(S/\rho, \omega_\rho)$ where $\omega_\rho([s]_\rho) = \inf\{\omega(s) : s \in [s]_\rho\}$ is the induced weight on S/ρ;
(ii) if S is an E-inversive semigroup with commuting idempotents or S is an eventually inverse semigroup, σ be the least group congruence on S and ω be a weight on S, then $l^1(S/\sigma, \omega_\sigma)) \simeq \frac{l^1(S,\omega)}{I_\sigma}$ where I_σ is a closed ideal of $l^1(S,\omega)$ and $I_\sigma^2 = I_\sigma$.

Proof. (i) Let $u, v \in S$ then;

$$\begin{aligned} \omega_\rho([uv]_\rho) &\leq \inf\{\omega(t_1) : t_1 \in [u]_\rho\}.\inf\{\omega(t_2) : t_2 \in [v]_\rho\} \\ &= \omega_\rho([u]_\rho).\omega_\rho([v]_\rho) \end{aligned}$$

Thus ω_ρ is a weight on S/ρ. Now $\widehat{\pi} : l^1(S,\omega) \to l^1(S/\rho, \omega_\rho)$ by $\widehat{\pi}(\delta_s) = \delta_{\pi(s)}$ is an epimorphism where $\pi : S \to \frac{S}{\rho}$ is the natural quotient map and $I_\rho = ker\widehat{\pi}$.

(ii) It follows from [1, Lemma 2]. □

By Lemma 4.1, $\frac{l^1(S,\omega)}{I_\rho} \simeq l^1(S/\rho, \omega_\rho)$ where $ker\widehat{\pi} = I_\rho$. Now I_ρ is an ideal in $l^1(S,\omega)$ generated by the set $\{\delta_s - \delta_t : s, t \in S \text{ with} (s,t) \in \rho\}$. Conversely, if J is an ideal of $l^1(S,\omega)$, and ρ_J is the congruence on S defined by $\rho_J = \{(s,t) : s,t \in S, \delta_s - \delta_t \in J\}$, then $I_{\rho_J} \subseteq J$.

Theorem 4.2. *Suppose that ω is a weight on semigroup S. If S is to one of the following statements:*
(i) S is an E-inversive semigroup with commuting idempotents;
(ii) S be an eventually inverse semigroup;
Then S is amenable and Ω_{ω_σ} is bounded if and only if $l^1(S,\omega)$ is approximately amenable modulo I_σ.

Proof. S is amenable if and only if S/σ is amenable (Theorem 4.1(i)), if and only if $l^1(S/\sigma, \omega_\sigma) \simeq \frac{l^1(S,\omega)}{I_\sigma}$ is approximately amenable (Lemma 4.1 and [11, Theorem 0]), if and only if $l^1(S,\omega)$ is approximately amenable modulo I_σ (Theorem 3.1 and Lemma 4.1), because $I_\sigma^2 = I_\sigma$. □

By a similar argument of Theorem 4.1, we have the following result;

Corollary 4.1. *Suppose that ω is a weight on semigroup S. If S is to one of the following statements:*
(i) S is an E-inversive semigroup with commuting idempotents;
(ii) S be an eventually inverse semigroup;
Then S is amenable and Ω_{ω_σ} is bounded if and only if $l^1(S,\omega)$ is amenable modulo I_σ.

Example 4.1. *(i) Let $S = \{p^m q^n : m, n \geq 0\}$ be the bicyclic semigroup generated by $p\,, q$Using Theorem 4.1, amenability of S implies $l^1(S)$ is approximate amenable modulo I_σ, where σ is the least group congruence on S. We note that $l^1(S)$ is not approximately amenable* [10]. *Also we note that for any weight ω, $l^1(S,\omega)$ is not amenable.*
(ii) Let $G = \mathbb{F}_2$ be a free group with two generators a, b. Then G is not amenable. Let $T = (\mathbb{N}_0, +) \times (\mathbb{N}, max)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and put $S = G \times T$. Then $E(S) = \{(1_G, e) : e \in E(T)\}$ is infinite. Also G is the maximum group homomorphism image of S under the homomorphism $\phi : (g,t) \mapsto g$. Let σ be the congruence on S such that $S/\sigma \simeq G$. Then $l^1(S)$ is not approximately amenable modulo I_σ, indeed $\frac{l^1(S)}{I_\sigma}$ is isomorphic to $l^1(G)$, which is not approximately amenable.
(iii) Suppose X is a singleton and $S = FI(X)$ be the free inverse semigroup on X. Since $S = FI(X)$ has infinitely many idempotents, $l^1(S)$ is not amenable [1]. *On the other hand,*

amenability of S implies that S/σ is amenable. Thus $l^1(S/\sigma)$ is approximately amenable so $l^1(S)$ is approximately amenable modulo I_σ (by Theorem 3.1).

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REFERENCES

[1] M. Amini, H. Rahimi, Amenability of semigroups and their algebras modulo a group congruence, Acta Mathematica Hungarica, Vol 144, Issue 2, (2014), 407-415.

[2] T. D. Blackmore, Weak amenability of discrete semigroup algebras, Semigroup Forum, Vol 55, (1997), 196-205.

[3] G. K. Dales, A. T. -M. Lau and D. Strauss, Banach Algebras on Semigroups and Their Compactifications, Memoirs American Mathematical Society, Vol 205, (2010), 966.

[4] J. Duncan, I. Namioka, Amenability of inverse semigroups and their semigroup algebras, Proc. Royal Soc. Edinburgh Sect. A, Vol 80, (1978), 309-321.

[5] J. Duncan and A. L. T. Paterson, Amenability for discrete convolution semigroup algebras, Math. Scandinavica, Vol 66, (1990), 141-146.

[6] F. Ghahramani and R. J. Loy, Generalized notions of amenability, J. Func. Anal.,Vol 208, (2004), 229-260.

[7] F. Ghahramani, R. J. Loy, and Y. Zhang, Generalized notions of amenability, II, J. Functional Analysis, Vol 254, (2008), 1776-1810.

[8] R. S. Gigon, Congruences and group congruences on a semigroup, Semigroup Forum, Vol 86, (2013), 431-450.

[9] H. M. Ghlaio and C. J. Read, Irregular commutative semigroups S with weakly amenable semigroup algebra $1^1(S)$, Semigroup Forum,Vol 82, Issue 2, (2011), 367-383.

[10] F. Gheorge and Y. Zhang, A note on the approximate amenability of semigroup algebras, Semigroup Forum, Vol 79, (2009), 349-354.

[11] N. Gronbaek, Amenability of weighted discrete convolution algebras on cancellative semi-groups, Proc. Roy. Soc. Edinburgh, Section A, Vol 110, (1988), 351-360.

[12] H. Rahimi and E. Tahmasebi, A note on amenability modulo an ideal of unitial Banach algebras, J. Mathematical Extension, In press.

[13] H. Rahimi and E. Tahmasebi, Amenability and Contractibility modulo an ideal of Banach algebras, Abstract and Applied Analysis, (2014), 514-761.

[14] H. Rahimi and E. Tahmasebi, Hereditary properties of amenability modulo an ideal of Banach algebras, J. Linear Topol. Algebra, Vol 03, No 02, (2014), 107- 114.

[15] M. Rostami and A. Pourabbas, Approximate amenability of certain inverse semigroup algebras, Acta Mathematica Scientia, Vol 33, B(2), (2013), 565-577.