HOMOTOPY ANALYSIS METHOD AND COMBINATORIAL APPROACH TO STUDY OF BUCKET RECURSIVE TREES

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In this paper, we give a result related to m-decomposable trees in bucket recursive trees of size n with variable bucket capacities using the well-known powerful homotopy analysis method (HAM). We prove that m-decomposable become more numerous as m grows to infinity. Also, we study the quantities out-degree and descendants of label j and give closed formulas for the probability distributions and all factorial moments.

1. Introduction

A graph is a collection of points and lines connecting a subset of them [1]. Trees are defined as connected graphs without cycles, and their properties are basics of graph theory. Recursive trees are rooted labelled trees, where the root is labelled by 1 and the labels of all successors of any node v are larger than the label of v. They are one of the most natural combinatorial tree models with applications in several fields, e.g., it has been introduced as a model for the spread of epidemics, for pyramid schemes, for the family trees of preserved copies of ancient texts and furthermore it is related to the Bolthausen-Sznitman coalescence model. There are $(n-1)!$ different recursive trees with n nodes. It is of particular interest in applications to assume the random recursive tree model and to speak about a random recursive tree with n nodes, which means that one of the $(n-1)!$ possible recursive trees with n nodes is chosen with equal probability, i.e., the probability that a particular tree with n nodes is chosen is always $1/(n-1)!$ [9]. Kazemi [4] introduced a new version of these trees where the nodes are buckets with variable capacities labelled with integers $1, 2, ..., n$. In this model, the capacity of buckets is a random variable. The motivation for studying such models is multifold. For example, if n atoms in a branching molecular structure (such as dendrimer) are stochastically labelled with integers $1, 2, ..., n$, then atoms in different functional groups can be considered as the labels of different buckets of a bucket recursive tree with variable capacities of buckets. Also, this model of tree structures is appeared in some computer science applications. We define the tree below for the reader's convenience.

Definition 1.1. *A size-n bucket recursive tree T_n with variable bucket capacities and maximal bucket size b starts with the root labelled by 1. The tree grows by progressive attraction of increasing integer labels: when inserting label $j + 1$ into an existing bucket recursive tree T_j, except the labels in the non-leaf nodes with capacity $< b$ all labels in the tree (containing label 1) compete to attract the label $j + 1$. For the root node and nodes with capacity b, we always produce a new node $j + 1$. But for a leaf with capacity $c < b$, either the label $j + 1$ is attached to this leaf as a new bucket containing only the label $j + 1$ or is added to that leaf*

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and make a node with capacity $c + 1$. This process ends with inserting the label n (i.e., the largest label) in the tree.

By definition, a node v with capacity $c(v) < b$ has the out-degree 0 or 1. In Figure 1, we diagrammatically show the step-by-step growth of a tree of size 11 with $b = 2$. The probability p, which gives the probability that label $n+1$ is attracted by node v in the model is $\frac{c(v)}{n-|\gamma|}$, where

$$\gamma = \{v \in T; \ c(v) = k < b, \text{ and } v \text{ is a non-leaf}\}.$$

A sequence of non-negative numbers $(\alpha_k)_{k \geq 0}$ with $\alpha_0 > 0$ and a sequence of non-negative
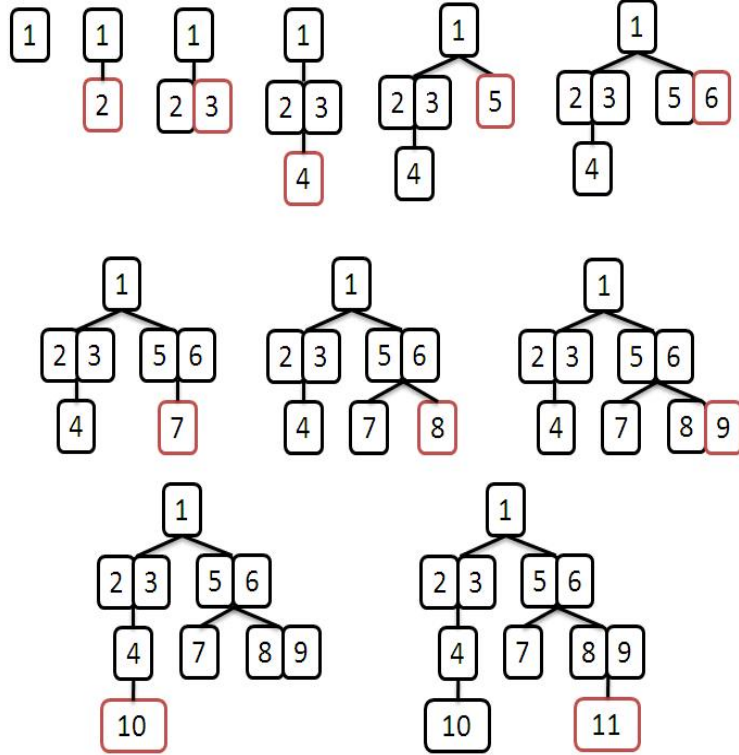

FIGURE 1. The step-by-step growth of a tree of size $n = 11$ with maximal bucket size $b = 2$ [5].

numbers $\beta_1, \beta_2, ..., \beta_{b-1}$ is used to define the weight $w(T)$ of any ordered tree T by $w(T) := \Pi_v w(v)$, where v ranges over all nodes of T. The weight $w(v)$ of a node v is given as follows:

$$w(v) := \begin{cases} \alpha_{d(v)}, & v \text{ is root or complete } (c(v) = b) \\ \beta_{c(v)}, & v \text{ is incomplete } (c(v) < b), \end{cases} \tag{1}$$

where $d(v)$ denotes the out-degree of node v. Furthermore, $\mathcal{L}(T)$ denotes the set of different increasing labelings of the tree T with distinct integers $\{1, 2, ..., |T|\}$, where $L(T) := |\mathcal{L}(T)|$ denotes its cardinality. Then the family $\mathcal{T}$ consists of all trees T together with their weights $w(T)$ and the set of increasing labelings $\mathcal{L}(T)$. For a given degree-weight sequence $(\alpha_k)_{k \geq 0}$ with a degree-weight generating function $\varphi(t) := \sum_{k \geq 0} \alpha_k t^k$ and a bucket-weight sequence $\beta_1, \beta_2, ..., \beta_{b-1}$, we define the exponential generating function

$$T_{n,b}(z) \quad := \quad \sum_{n=1}^{\infty} T_{n,b} \frac{z^n}{n!}, \tag{2}$$

where $T_{n,b} := \sum_{|T|=n} w(T) \cdot L(T)$ is the total weights. For this model,

$$
\begin{aligned}
T'_{n,b}(t) &= b!^{-\sum_{i=1}^r |\mathcal{P}_{k_i}|} \frac{(b-1)!}{1 - b!^{1-\sum_{i=1}^r |\mathcal{P}_{k_i}|} t}, \\
T_{n,b} &= \frac{(n-1)!(b!)^{n(1-\sum_{i=1}^r |\mathcal{P}_{k_i}|)}}{b}, \; n \geq 1,
\end{aligned}
\tag{3}
$$

where $|\mathcal{P}_{k_i}|$ denotes the size of the set of all trees of size $k_i(i = 1, 2, ..., r)$ [4]. In the equation (3), if ith subtree starts with a bucket with capacity $c(v) = 1$, then we set $|\mathcal{P}_{k_i}| = 0$. Kazemi studied the following random variables in this model: the depth of the largest label [4], the first Zagreb index [3], the subtree size profile [5] and the branches [6].

The paper is organized as follows. First, we give a result related to m-decomposable trees in Section 3 using the well-known powerful homotopy analysis method (HAM). Let $O_{n,j}$ be the random variable which counts the out-degree of the bucket containing label j in bucket recursive tree of size n with variable capacities of buckets. In Section 4, we first evaluate separately for probabilities $\mathbb{P}(O_{n,j} = m)$ the case $j = 1$ and $m = r$ via a bivariate generating function (Lemma 4.1). Second we introduce a trivariate generating function $T(z, u, v)$ for studying of $\mathbb{P}(O_{n,j} = m)$ if bucket v is complete. Third we compute all factorial moments for our model. In Section 5, with the same approach we obtain a closed formula for the probability distribution and all factorial moments of $Y_{n,j}$, which counts the number of descendants of label j (i.e., the size of the subtree rooted at bucket containing label j).

2. Preliminaries

In the discourse we shall use the following notations:

$$
X^{\underline{\ell}} := X(X-1)(X-2)\cdots(X-\ell+1),
$$

$$
\begin{bmatrix} n \\ k \end{bmatrix} : \text{the Stirling numbers of the first kind,}
$$

$$
H_n = \sum_{k=1}^n \frac{1}{k} : \text{the nth harmonic number,}
$$

$$
H_n^{(a)} = \sum_{k=1}^n \frac{1}{k^a} : \text{the nth harmonic number of order a.}
$$

Let $[z^n]f(z)$ denote the operation of extracting the coefficient of z^n in the formal power series $f(z) = \sum f_n z^n$. Also, we will use the following equalities:

$$
[z^n] \log\left(\frac{1}{1-z}\right)(1-z)^{-1} = H_n,
\tag{4}
$$

$$
[z^n] \log^2\left(\frac{1}{1-z}\right)(1-z)^{-1} = H_n^2 - H_n^{(2)},
\tag{5}
$$

$$
[z^n]f(qz) = q^n[z^n]f(z),
\tag{6}
$$

$$
\sum_{n \geq 0} \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{z^n}{n!} v^m = \frac{1}{(1-z)^v},
\tag{7}
$$

$$
[z^n]\frac{1}{(1-z)^\alpha} = \binom{n+\alpha-1}{n}.
\tag{8}
$$

3. Homotopy Analysis Method: The Number of m-decomposable Trees

Let us illustrate first the basic concept of homotopy analysis method (HAM) (see [2, 7, 11] for more details and applications) which we will employ it to discover the number of m-decomposable trees.

3.1. Homotopy Analysis Method (HAM)

Consider the following general nonlinear system

$$\mathcal{N}[u(r,t)] = 0,$$

where $\mathcal{N}$ is a nonlinear operator, $u(r,t)$ is an unknown function, and r and t denote spatial and temporal independent variables, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, we have the so-called zero-order deformation equation

$$(1-p)\mathcal{I}[\phi(r,t;p) - u_0(r,t)] = p\hbar\mathcal{H}(r,t)\mathcal{N}[\phi(r,t;p)], \tag{9}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $\mathcal{H}(r,t) \neq 0$ is an auxiliary function, $\mathcal{I}$ is an auxiliary linear operator, $u_0(r,t)$ is an initial guess of $u(r,t)$, $\phi(r,t;p)$ is an unknown function, respectively. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(r,t;0) = u_0(r,t), \quad \phi(r,t;1) = u(r,t),$$

respectively. Thus as p increases from 0 to 1, the solution $\phi(r,t;p)$ varies from the initial guess $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\phi(r,t;p)$ in Taylor series with respect to p, one has

$$\phi(r,t;p) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t)p^m, \tag{10}$$

where

$$u_m(r,t) = \frac{1}{m!}\frac{\partial^m \phi(r,t;p)}{\partial p^m}\Big|_{p=0}. \tag{11}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series converges at $p = 1$, so we obtain

$$u(r,t) = \phi(r,t;1) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t). \tag{12}$$

Thus the governing equation for $u_m(r,t)$ can be deduced as follow:
Define the vector

$$\vec{u}_n = \{u_0(r,t), u_1(r,t), ..., u_n(r,t)\}.$$

Differentiating m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{I}[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar\mathcal{H}(r,t)R_m(\vec{u}_{m-1}), \tag{13}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(r,t;p)]}{\partial p^{m-1}}\Big|_{p=0}, \tag{14}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

3.2. The Number of m-decomposable Trees

A tree is said to be m-decomposable if it has a spanning forest whose components are all of size m. Tree decompositions of a very general nature are of interest in the theory of networks. The special case $m = 2$ corresponds to the question whether a tree has a perfect matching or not. Let m be a fixed natural number and $A(x)$ denote the exponential generating function for the total weights of m-decomposable our models from a variety determined by the degree function $\varphi(t)$; the set of these trees is denoted by $\mathfrak{T}$. Furthermore, let $\mathfrak{T}_l$ denote the set of models from our variety with a spanning forest whose components are of size m, except for the component containing the root, which is of size l, and let $A_l(x)$ be its generating function. Note at this point that $A(x) = A_m(x)$ by definition.

We define the bivariate generating function $N(x, y) = \sum_{l \geq 1} A_l(x) y^l$ for the union $\bigcup_l \mathfrak{T}_l$, where the exponent of x gives the size of a tree and the exponent of y the size of the component containing the root in the associated spanning forest whose other components have size m.

In passing, we think of a tree T in the class $\mathfrak{T}_l$ as being decomposed into the root, followed by several subtrees $T_1, T_2, \ldots$ attached to it. Now note that the associated spanning forest F whose components (with exception of the distinguished component that contains the root) all have size m induces spanning forests of the same kind in $T_1, T_2, \ldots$: those T_i whose roots belong to the component of F that contains the root of T belong to some $\mathfrak{T}_{r_i}$, whereas the others have to belong to $\mathfrak{T}$. Conversely, a collection of rooted trees from $\mathfrak{T} \cup \bigcup_l \mathfrak{T}_l$ whose roots are joined to a new common root yields a rooted tree from $\bigcup_l \mathfrak{T}_l$ (similar to [8]). This decomposition translates directly to differential equation

$$
\begin{aligned}
\frac{\partial}{\partial x} N(x, y) &= y\varphi(A(x) + N(x, y)) \\
&= y(b-1)! \exp\Big(b(A(x) + N(x, y))\Big),
\end{aligned}
\tag{15}
$$

with initial condition $N(0, y) = 0$, since

$$
\frac{d}{dx} A_l(x) = \sum_{r_1 + \cdots = l-1} \frac{A_1(x)^{r_1} A_2(x)^{r_2} \cdots}{r_1! r_2! \cdots} \varphi^{(r_1 + r_2 + \cdots)}(A(x)).
$$

Now, using the homotopy analysis method (HAM) described in previous subsection (by setting $\hbar = -1$ and $\mathcal{H}(x, y) = 1$), the differential equation (15) has been led to the following closed-form solution

$$
N(x, y) = \sum_{m=1}^{+\infty} N_m(x, y) = \frac{1}{b} \log\left(\frac{1}{1 - yb! \int_0^x e^{bA(t)} dt} \right).
$$

Thus

$$
A_l(x) = \frac{b!^l}{bl} \left(\int_0^x \exp(bA(t)) dt \right)^l,
$$

since $[x^l] f(qx) = q^l [x^l] f(x)$. Then we arrive at the equation

$$
A_m(x) = A(x) = \frac{b!^l}{bm} \left(\int_0^x \exp(bA(t)) dt \right)^m.
$$

Simplification and differentiation with respect to x gives

$$
b^{\frac{1}{m}} m^{\frac{1}{m}-1} A'(x) A(x)^{\frac{1}{m}-1} = e^{bA(x)},
$$

which leads to the implicit solution

$$
x = b^{\frac{1}{m}} \int_0^A (mt)^{\frac{1}{m}-1} e^{-bt} dt.
\tag{16}
$$

Note that A has nonzero coefficients only for those powers x^n for which n is a multiple of m. We see that $A(x) = B(x^m)$, where B has a dominant singularity at ρ^m and ρ is given by

$$\rho = b^{\frac{1}{m}} \int_0^\infty (mt)^{\frac{1}{m}-1} e^{-bt}\, dt = m^{\frac{1}{m}-1} \Gamma\left(\tfrac{1}{m}\right).$$

A simple singularity analysis gives:

$$[x^{mn}]A(x) \sim \frac{f(b)}{n}\left(\frac{m^{1-\frac{1}{m}}}{\Gamma\left(\frac{1}{m}\right)}\right)^{mn},$$

where $f(b)$ is a function of b. Since the total weights of our models with n vertices is

$$T_{n,b} = b^{-1}(n-1)!(b!)^{n(1-\sum_{i=1}^r |\mathcal{P}_{k_i}|)},$$

we see that the ratio of m-decomposable among all trees is asymptotically

$$m\left(\frac{m^{1-\frac{1}{m}}}{\Gamma\left(\frac{1}{m}\right)}\right)^{mn} f(b).$$

Also $m^{1-1/m}\Gamma(1/m)^{-1}f(b)$ tends to $f(b)$ as $m \to \infty$. Thus m-decomposable become more numerous as m grows to infinity.

4. Out-degree

In this section, we consider the random variable $O_{n,j}$, which counts the out-degree of bucket containing label j in a random bucket recursive tree with variable capacities of buckets of size n. As our first result we have the following lemma that is immediate consequence of the stochastic growth rule of the model (similar to [9]).

Lemma 4.1. *The bivariate generating function*

$$K(z,v) = \sum_{n \geq 1} \sum_{r \geq 0} \mathbb{P}(O_{n,1} = r) T_{n,b} \frac{z^n}{n!} v^r$$

of the root-degree is given by

$$K(z,u) = b!^{-\sum_{i=1}^r |\mathcal{P}_{k_i}|} \int_0^z \varphi(v T_{n,b}(t)) dt.$$

Proof. By using the fact the total weights of the r subtrees and the root node is

$$\alpha_r b!^{-\sum_{i=1}^r |\mathcal{P}_{k_i}|} T_{k_1,b} \cdots T_{k_r,b},$$

proof is completed (see [4] for details). $\qquad\square$

For our model of size n with root-degree r and subtrees with sizes $k_1, ..., k_r$, enumerated from left to right, where the bucket containing label j lies in the leftmost subtree and is the i-th smallest bucket in this subtree, we can reduce the computation of the probabilities $\mathbb{P}(O_{n,j} = m)$ to the probabilities $\mathbb{P}(O_{k_1,i} = m)$. We get as factor the total weight of the r subtrees and the root node $\alpha_r b!^{-\sum_{i=1}^r |\mathcal{P}_{n_i}|} T_{k_1,b} \cdots T_{k_r,b}$, divided by the total weight $T_{n,b}$ of trees of size n and multiplied by the number of order preserving relabellings of the r subtrees, which are given here by

$$\binom{j-2}{i-1}\binom{n-j}{k_1-i}\binom{n-1-k_1}{k_2,k_3,...,k_r}.$$

Due to symmetry arguments we obtain a factor r, if the bucket containing label j is the i-th bucket in the second, third, ..., r-th subtree. Summing up over all choices for the rank i of

bucket containing label j in its subtree, the subtree sizes $k_1, ..., k_r$, and the out-degree r of the root node gives the following recurrence $(2 \leq j \leq n)$:

$$\mathbb{P}(O_{n,j} = m) = \sum_{r \geq 1} r\alpha_r \sum_{k_1 + \cdots + k_r = n-1} \frac{T^*_{k_1,b} \cdots T^*_{k_r,b}}{T_{n,b}}$$

$$\times \sum_{i=1}^{\min\{k_1, j-1\}} \mathbb{P}(O_{k_1,i} = m) \binom{j-2}{i-1} \binom{n-j}{k_1-i} \binom{n-1-k_1}{k_2, k_3, ..., k_r}, \qquad (17)$$

where $T^*_{n_1,b} \cdots T^*_{k_r,b} = b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} T_{k_1,b} \cdots T_{k_r,b}$. Let

$$T(z, u, v) = \sum_{k \geq 0} \sum_{j \geq 1} \sum_{m \geq 0} \mathbb{P}(O_{k+j,j} = m) T_{k+j,b} \frac{z^{j-1}}{(j-1)!} \frac{u^k}{k!} v^m.$$

Multiplying (17) with $T_{k+j,b} \frac{z^{j-2}}{(j-2)!} \frac{u^k}{k!} v^m$ and summing up over $k \geq 0, j \geq 2$ and $m \geq 0$ we obtain the following differential equation:

$$\frac{\partial}{\partial z} T(z, u, v) = b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \varphi'(T_{n,b}(z + u)) T(z, u, v). \qquad (18)$$

Equation (18) has the general solution

$$T(z, u, v) = C(u, v) \exp\left\{ b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \int_0^z \varphi'(T_{n,b}(t + u)) dt \right\}$$

with a function $C(u, v)$. From Lemma 4.1 (evaluating at $z = 0$),

$$C(u, v) = T(0, u, v) = \frac{\partial}{\partial u} K(u, v) = b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \varphi(v T_{n,b}(u)).$$

Thus

$$\begin{aligned}
T(z, u, v) &= b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \varphi(v T_{n,b}(u)) \exp\left\{ b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \int_0^z \varphi'(T_{n,b}(t + u)) dt \right\} \\
&= b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \varphi(v T_{n,b}(u)) \\
&\times \exp\left\{ b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \int_0^z \frac{\varphi'(T_{n,b}(t + u)) T'_{n,b}(t + u)}{b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \varphi(T_{n,b}(t + u))} dt \right\}.
\end{aligned}$$

Hence

$$T(z, u, v) = b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \frac{\varphi(v T_{n,b}(u)) \varphi(T_{n,b}(z + u))}{\varphi(T_{n,b}(u))}. \qquad (19)$$

Theorem 4.1. *The probabilities* $\mathbb{P}(O_{n,j} = m)$ *are given by the following formula:*

$$\mathbb{P}(O_{n,j} = m) = \frac{1}{b\binom{n-1}{j-1}} \sum_{k=m}^{n-j} \binom{n-k-2}{j-2} \frac{\left[\begin{smallmatrix} k \\ m \end{smallmatrix}\right]}{k!}. \qquad (20)$$

Proof. By (3),

$$T(z, u, v) = b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \frac{(b-1)!}{\left(1 - b!^{1-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} u\right)^v \left(1 - \frac{b!^{1-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} z}{1 - b!^{1-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} u}\right)}. \qquad (21)$$

Suppose $t = b!^{1-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|}$. Then $T_{n,b} = \frac{(n-1)!}{b} t^n$. By (6), (7) and (8),

$$
\begin{aligned}
\mathbb{P}(O_{n,j} = m) &= \frac{(j-1)!(n-j)!}{T_{n,b}} [z^{j-1} u^{n-j} v^m] T(z,u,v) \\
&= \frac{\frac{t}{b}}{\binom{n-1}{j-1} t^n} [u^{n-j} v^m] \frac{t^{j-1}}{(1-tu)^{v+j-1}} \\
&= \frac{1}{b\binom{n-1}{j-1}} \sum_{k=0}^{n-j} [u^{n-j-k}] \frac{1}{(1-tu)^{j-1}} [u^k v^m] \frac{t^{j-1}}{(1-tu)^v} \\
&= \frac{1}{b\binom{n-1}{j-1}} \sum_{k=m}^{n-j} \binom{n-k-2}{j-2} \frac{\left[\begin{smallmatrix} k \\ m \end{smallmatrix}\right]}{k!}.
\end{aligned}
$$

$\square$

Theorem 4.2. *The ℓ-th factorial moments $\mathbb{E}(O_{n,j}^{\ell})$ are for $\ell \geq 1$ given by the following formula:*

$$
\mathbb{E}(O_{n,j}^{\ell}) = \frac{\ell!}{b\binom{n-1}{j-1}} \sum_{k=0}^{n-j} \binom{n-k-1}{j-1} \frac{\left[\begin{smallmatrix} k \\ \ell \end{smallmatrix}\right]}{k!}. \tag{22}
$$

Proof. It is obvious that for $\ell \geq 1$,

$$
\mathbb{E}(O_{n,j}^{\ell}) = \frac{(j-1)!(n-j)!}{T_{n,b}} [z^{j-1} u^{n-j}] \frac{\partial^{\ell} T(z,u,v)}{\partial v^{\ell}} \Big|_{v=1}
$$

and thus proof is completed similar to proof of Theorem 4.1. $\square$

Corollary 4.1. *From Theorem 4.2, (4) and (5),*

$$
\begin{aligned}
\mathbb{E}(O_{n,j}) &= \frac{1}{b}(H_{n-1} - H_{j-1}), \\
\mathbb{E}(O_{n,j}^2) &= \frac{(H_{n-1} - H_{j-1})^2 - (H_{n-1}^{(2)} - H_{j-1}^{(2)})}{b}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbb{V}ar(O_{n,j}) &= \mathbb{E}(O_{n,j}^2) + \mathbb{E}(O_{n,j}) - \mathbb{E}^2(O_{n,j}) \\
&= \frac{1}{b}(H_{n-1} - H_{j-1})^2 \left(1 - \frac{1}{b}\right) \\
&+ \frac{1}{b}\left((H_{n-1} - H_{j-1}) - (H_{n-1}^{(2)} - H_{j-1}^{(2)})\right).
\end{aligned}
$$

5. Descendants

In this section we consider the random variable $Y_{n,j}$ which denotes the number of descendants of label j in a random bucket recursive tree of size n with variable capacities of buckets. Let

$$
Y(z,u,v) = \sum_{k \geq 0} \sum_{j \geq 1} \sum_{m \geq 0} \mathbb{P}(Y_{k+j,j} = m) T_{k+j,b} \frac{z^{j-1}}{(j-1)!} \frac{u^k}{k!} v^m.
$$

Similar to the analysis of Section 4,

$$
Y(z,u,v) = C(u,v) \exp\left\{ b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \int_0^z \varphi'(T_{n,b}(t+u)) dt \right\}
$$

with a function $C(u,v)$. Since $\mathbb{P}(Y_{k+1,1} = k+1) = 1$. Thus

$$
\begin{aligned}
C(u,v) &= Y(0,u,v) \\
&= \sum_{k \geq 0} \sum_{m \geq 0} \mathbb{P}(Y_{k+1,1} = m) T_{k+1,b} \frac{u^k}{k!} v^m \\
&= \sum_{k \geq 0} T_{k+1,b} \frac{u^k}{k!} v^{k+1} \\
&= v \sum_{n \geq 1} T_{n,b} \frac{(uv)^{n-1}}{(n-1)!} \\
&= v T'_{n,b}(uv) \\
&= v b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \varphi(T_{n,b}(uv)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
Y(z,u,v) &= v b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} \frac{\varphi(T_{n,b}(uv))\varphi(T_{n,b}(z+u))}{\varphi(T_{n,b}(u))} \\
&= v \frac{b!^{-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|}(b-1)!}{\left(1 - b!^{1-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}uv\right)\left(1 - \frac{b!^{1-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}z}{1-b!^{1-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}u}\right)}.
\end{aligned} \tag{23}
$$

Theorem 5.1. *The probabilities $\mathbb{P}(Y_{n,j} = m)$, which give the probability that the bucket with label j in a randomly chosen size-n tree has exactly m descendants, are for $m \geq 1$ given by the following formula:*

$$
\mathbb{P}(Y_{n,j} = m) = \frac{1}{b} \frac{\binom{n-m-1}{j-2}}{\binom{n-1}{j-1}}. \tag{24}
$$

Also the ℓ-th factorial moments $\mathbb{E}(Y_{n,j}^{\ell})$ are for $\ell \geq 1$ given by the following formula:

$$
\mathbb{E}(Y_{n,j}^{\ell}) = \frac{1}{b}\ell!\left(\frac{\binom{n-j}{\ell}}{\binom{j+\ell-1}{\ell}} + \frac{\binom{n-j}{\ell-1}}{\binom{j+\ell-2}{\ell-1}}\right). \tag{25}
$$

Proof. By (6) and (8),

$$
\begin{aligned}
\mathbb{P}(Y_{k+j,j} = m) &= \frac{(j-1)!k!}{T_{k+j,b}}[z^{j-1}u^k v^m]Y(z,u,v) \\
&= \frac{1}{t^{k-1}\binom{k+j-1}{j-1}}[u^k v^{m-1}]\frac{1}{(1-tuv)(1-tu)^{j-1}} \\
&= \frac{1}{t^{k-m+2}\binom{k+j-1}{j-1}}[u^k]\frac{u^{m-1}}{(1-tu)^{j-1}} \\
&= \frac{\binom{k+j-m-1}{j-2}}{b\binom{k+j-1}{j-1}}.
\end{aligned}
$$

Also

$$
\mathbb{E}(Y_{k+j,j}^{\ell}) = \frac{(j-1)!k!}{T_{k+j,b}}[z^{j-1}u^k]\frac{\partial^{\ell}Y(z,u,v)}{\partial v^{\ell}}\bigg|_{v=1}.
$$

With the same method we can obtain (25) (similar to [10]). □

Corollary 5.1. *For $\ell = 1, 2$,*

$$
\begin{aligned}
\mathbb{E}(Y_{n,j}) &= \frac{n}{bj}, \\
\mathbb{E}(Y_{n,j}^2) &= \frac{2n(n-j)}{bj(j+1)},
\end{aligned}
$$

and thus

$$
\mathbb{V}ar(Y_{n,j}) = \frac{2n(n-j)}{bj(j+1)} + \frac{n}{bj}\left(1 - \frac{n}{bj}\right).
$$

6. Conclusion

In this paper, we studied three quantities in bucket recursive trees of size n with variable bucket capacities for investigating the effect of bucketing on recursive trees. The probabilities and factorial moments of out-degree and descendants of label j are dependent on maximal bucket size b and they become more numerous as b decrees to 1. These results are quite reasonable and reduce to the ordinary random recursive trees as $b = 1$. Also

$$
\mathbb{E}(O_{n,j}) = \mathbb{V}ar(O_{n,j}) = \frac{1}{b}(\log n - \log j) + \mathcal{O}(1),
$$

where the $\mathcal{O}(1)$ bound appearing in the above equation holds independently from j. For $j = o(n)$,

$$
Z_{n,j,b} = \frac{O_{n,j} - \frac{1}{b}(\log n - \log j)}{\sqrt{\frac{1}{b}(\log n - \log j)}} \sim N(0,1).
$$

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