SUBGRADIENT AND HYBRID ALGORITHMS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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In this paper, we studied the fixed point and equilibrium problems in Hilbert spaces. We present an iterative algorithm combined with subgradient and hybrid methods for solving the fixed point problems of pseudocontractive operators and the equilibrium problems of pseudomonotone operators. We show the strong convergence of the proposed algorithm.

Keywords: Fixed point, equilibrium problem, pseudomonotone operators, pseudocontractive operators, subgradient, projection.

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1. **Introduction**

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H.

Let $f : C \times C \to \mathbb{R}$ be a bifunction. Recall that f is said to be

- strongly monotone with modulus $\mu > 0$ if

$$f(x, y) + f(y, x) \leq -\mu \|x - y\|^2, \ \forall x, y \in C.$$

- monotone if

$$f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C.$$

- pseudomonotone if

$$f(x, y) \geq 0 \text{ implies that } f(y, x) \leq 0, \ \forall x, y \in C.$$

- Lipschitz-type continuous with constants $\mu_1 > 0, \mu_2 > 0$ if

$$f(x, y) + f(y, z) \geq f(x, z) - \mu_1 \|x - y\|^2 - \mu_2 \|y - z\|^2, \ \forall x, y, z \in C.$$

In this paper, we focus on the equilibrium problem associated with the bifunction f which is to solve the following problem:

$$\text{find } x \in C \text{ such that } f(x, y) \geq 0, \ \forall y \in C. \tag{1}$$

The solution set of equilibrium problem (1) is denoted by $Sol(f, C)$.

It is well known that some important problems such as variational inequalities ([2, 6, 8, 12, 19, 20, 25, 27, 28, 32]), fixed point problems ([4, 5, 9, 23, 24, 26, 29, 30, 31, 33]), Nash equilibrium ([3, 17]), can be formulated in the form of the equilibrium problem (1). The most approaches to the equilibrium problem are relied on the resolvent of equilibrium bifunction

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([7]) in which a strongly monotone regularization problem is solved at each iterative step. Mastroeni ([15]) used the auxiliary problem principle to the equilibrium problem involving a strongly monotone bifunction and satisfying Lipschitz-type condition. Another important method for solving (1) is proximal point method which was firstly proposed by Martinet ([13]) and further studied by Rockafellar ([18]) for finding a zero of maximal monotone operators. Konnov ([11]) extended proximal point method to the equilibrium problem for monotone bifunctions.

Very recently, iterative algorithms for solving (1) and fixed point problems have been future studied in the literature, see, for instance ([1, 10, 21, 35]). Especially, Nguyen, Strodiot and Nguyen ([16]) presented a hybrid method for solving equilibrium problem (1) and a fixed point problem. Yang and Liu ([22]) suggested a subgradient extragradient method for solving the pseudomonotone equilibrium problems and fixed point of nonexpansive mappings. Yao, Li and Postolache ([24]) investigated the split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions. Zhu, Yao and Postolache ([35]) proposed a projection algorithm with linesearch technique for solving equilibrium problems and fixed point problems.

Motivated and inspired by the work in this direction, the main objective of this paper is to investigate the equilibrium problem (1) and fixed point problems in Hilbert spaces. We present an iterative algorithm combined with subgradient and hybrid methods for solving the fixed point problems of pseudocontractive operators and the equilibrium problems of pseudomonotone operators. We show the strong convergence of the proposed algorithm.

2. **Notations and Lemmas**

Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $g : C \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Then, the subdifferential ∂g of g is defined by

$$\partial g(u) := \{v^\dagger \in H : g(u) + \langle v^\dagger, u^\dagger - u\rangle \leq g(u^\dagger), \forall u^\dagger \in C\} \tag{2}$$

for each $u \in C$.

It is known that $u^\dagger$ is a solution to the optimization problem

$$\min_{u\in C} g(u)$$

if and only if

$$0 \in \partial g(u^\dagger) + N_C(u^\dagger),$$

where $N_C(u^\dagger)$ means the normal cone of C at $u^\dagger$ defined by

$$N_C(u^\dagger) = \{\omega \in H : \langle \omega, u - u^\dagger\rangle \leq 0, \forall u \in C\}.$$

Recall that a bi-function $f : C \times C \to \mathbb{R}$ is called jointly sequently weakly continuous, if there exist two sequence $\{x^k\}$ and $\{y^k\}$ in C such that $x^k \rightharpoonup x^\dagger$ and $y^k \rightharpoonup y^\dagger$, then $f(x^k, y^k) \to f(x^\dagger, y^\dagger)$.

Let $f : C \times C \to \mathbb{R}$ be a bi-function. Assume that the following conditions hold:

(f1): $f(z^\dagger, z^\dagger) = 0$ for all $z^\dagger \in C$;
(f2): f is pseudomonotone on $Sol(f, C)$;
(f3): f is jointly sequently weakly continuous on $C \times C$;
(f4): $f(z^\dagger, \cdot)$ is convex and subdifferentiable for all $z^\dagger \in C$.

Recall that an operator $S : C \to C$ is said to be pseudocontractive if

$$\|Su - Su^\dagger\|^2 \leq \|u - u^\dagger\|^2 + \|(I - S)u - (I - S)u^\dagger\|^2, \forall u, u^\dagger \in C$$

and $S : C \to C$ is called L-Lipschitz if

$$\|Su - Su^\dagger\| \le L\|u - u^\dagger\|$$

for all $u, u^\dagger \in C$.

Recall that the metric projection $P_C : H \to C$ is an orthographic projection from H onto C, which possesses the following characteristic: for given $x \in H$,

$$\langle x - P_C[x], y - P_C[x]\rangle \le 0, \forall y \in C. \tag{3}$$

The following symbols are needed in the paper.

- $p^k \rightharpoonup z^\dagger$ indicates the weak convergence of p^k to $p^\dagger$ as $k \to \infty$.
- $p^k \to p^\dagger$ implies the strong convergence of p^k to $p^\dagger$ as $k \to \infty$.
- $Fix(S)$ means the set of fixed points of S.
- $\omega_w(p^k) = \{p^\dagger : \exists\{p^{k_i}\} \subset \{p^k\}$ such that $p^{k_i} \rightharpoonup p^\dagger(i \to \infty)\}$.

Lemma 2.1 ([16])**.** *Let H be a real Hilbert space. Then, we have*

$$\|\kappa u + (1-\kappa)u^\dagger\|^2 = \kappa\|u\|^2 + (1-\kappa)\|u^\dagger\|^2 - \kappa(1-\kappa)\|u - u^\dagger\|^2,$$

$\forall u, u^\dagger \in H$ and $\forall \kappa \in [0,1]$.

Lemma 2.2 ([35])**.** *Assume that the operator $S : C \to C$ is L-Lipschitz pseudocontractive. Then, for all $\tilde{u} \in C$ and $u^\dagger \in Fix(S)$, we have*

$$\|u^\dagger - S((1-\eta)\tilde{u} + \eta S\tilde{u})\|^2 \le \|\tilde{u} - u^\dagger\|^2 + (1-\eta)\|\tilde{u} - S((1-\eta)\tilde{u} + \eta S\tilde{u})\|^2,$$

where $0 < \eta < \frac{1}{\sqrt{1+L^2}+1}$.

Lemma 2.3 ([21])**.** *Assume that the bi-function $f : C \times C \to \mathbb{R}$ satisfies assumptions (f3) and (f4). For given two points $\bar{u}, \bar{v} \in C$ and two sequences $\{u^k\} \subset C$ and $\{v^k\} \subset C$, if $u^k \rightharpoonup \bar{u}$ and $v^k \rightharpoonup \bar{v}$, respectively, then, for any $\epsilon > 0$, there exist $\eta > 0$ and $N_\epsilon \in \mathbb{N}$ verifying*

$$\partial_2 f(v^k, u^k) \subset \partial_2 f(\bar{v}, \bar{u}) + \frac{\epsilon}{\eta}B$$

for every $k \ge N_\epsilon$, where $B := \{b \in H | \|b\| \le 1\}$.

Lemma 2.4 ([34])**.** *If the operator $S : C \to C$ is continuous pseudocontractive, then S is demi-closedness, i.e., $y^k \rightharpoonup \tilde{u}$ and $Sy^k \to v^\dagger$ as $k \to \infty$ imply that $S\tilde{u} = v^\dagger$.*

Lemma 2.5 ([14])**.** *For given a sequence $\{x^k\} \subset H$ and $p \in H$, if $\omega_w(x^k) \subset C$ and $\|x^k - p\| \le \|p - P_C[p]\|$ for all $k \in \mathbb{N}$, then $x^k \to P_C[p]$.*

3. Main results

In this section, we first introduce an algorithm for solving the pseudomonotone equilibrium problem and fixed pint problem. Consequently, we show the convergence of the proposed algorithm.

Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbb{R}$ be a function which satisfies the assumptions (f1)-(f4). Let $S : C \to C$ be $L(>0)$-Lipschitz pseudocontractive operator. Assume that $Sol(f,C) \cap Fix(S) \neq \emptyset$. Let $\gamma \in (0,2)$ and $\mu \in (0,1)$ be two constants. Let $\{\lambda_k\}$, $\{\delta_k\}$, $\{\sigma_k\}$ and $\{\gamma_k\}$ be four real number sequences satisfying the following conditions:

(i) $\lambda_k \in [\rho, 1]$ with $0 < \rho \le 1$ for all $k \ge 0$;
(ii) $0 < \underline{\delta} < \delta_k < \overline{\delta} < \sigma_k < \overline{\sigma} < \frac{1}{\sqrt{1+L^2}+1}$ for all $k \ge 0$;

(iii) $0 < \liminf_{n\to\infty} \gamma_k \le \limsup_{n\to\infty} \gamma_k < 2$.
Next, we present our iterative algorithm.

Algorithm 3.1. *Step 1. Let $x^0 \in H$ be an initial value. Set $C_1 = C$ and compute $x^1 = P_{C_1}[x^0]$. Set $k = 0$.*

Step 2. For given x^k, calculate

$$v^k = (1-\delta_k)x^k + \delta_k S[(1-\sigma_k)x^k + \sigma_k S x^k]. \tag{4}$$

Step 3. Compute

$$y^k = \arg\min_{y^\dagger \in C} \left\{ f(v^k, y^\dagger) + \frac{1}{2\lambda_k}\|v^k - y^\dagger\|^2 \right\}. \tag{5}$$

Find the smallest positive integer m such that

$$f(z^{k,m}, y^k) + \frac{\gamma}{2\lambda_k}\|v^k - y^k\|^2 \le 0 \tag{6}$$

where

$$z^{k,m} = (1-\mu^m)v^k + \mu^m y^k, \tag{7}$$

and consequently, write $\mu^m = \mu_k$ and $z^{k,m} = z^k$.

Step 4. Calculate u^k by

$$u^k = \begin{cases} z^k, & \text{if } 0 \in \partial_2 f(z^k, z^k), \\ P_C\left[v^k + \dfrac{\mu_k\gamma_k f(z^k, y^k)}{(1-\mu_k)\|g^k\|^2} g^k\right], \text{ where } g^k \in \partial_2 f(z^k, z^k), & \text{if } 0 \notin \partial_2 f(z^k, z^k). \end{cases} \tag{8}$$

Step 5. Calculate x^{k+1} by the following form

$$\begin{cases} C_{k+1} = \{u^\dagger \in C_k : \|u^k - u^\dagger\| \le \|x^k - u^\dagger\|\}, \\ x^{k+1} = P_{C_{k+1}}[x^0]. \end{cases} \tag{9}$$

Step 6. Set $k := k+1$ and return to Step 2.

Proposition 3.1 ([35])**.** *For each $z^\dagger \in C$, we have*

$$f(v^k, z^\dagger) \ge f(v^k, y^k) + \frac{1}{\lambda_k}\langle v^k - y^k, z^\dagger - y^k\rangle. \tag{10}$$

Proposition 3.2. *The search rule* (6) *is well-defined, i.e., there exists m such that* (6) *holds. In this case, $f(z^k, y^k) < 0$ when $v^k \ne y^k$.*

Proof. Case 1. $v^k = y^k$. Hence, $z^k = v^k$ and $f(z^k, v^k) = f(z^k, y^k) = 0$ by (f1). Thus, (6) holds and select $m = 1$.

Case 2. $v^k \ne y^k$. If (6) is not well-defined, then m must violate the inequality (6), i.e., for every $m \in \mathbb{N}$, we have

$$f(z^{k,m}, y^k) + \frac{\gamma}{2\lambda_k}\|v^k - y^k\|^2 > 0. \tag{11}$$

In $z^k = (1-\mu^m)v^k + \mu^m y^k$, letting $m \to \infty$, we conclude that $z^k \to v^k$ as $m \to \infty$. From (f3), we deduce that $f(z^k, v^k) \to 0$ and $f(z^k, y^k) \to f(v^k, y^k)$. This together with (11) implies that

$$f(v^k, y^k) + \frac{\gamma}{2\lambda_k}\|v^k - y^k\|^2 \ge 0. \tag{12}$$

Letting $z^\dagger = v^k$ in (10) and noting that $f(v^k, v^k) = 0$, we deduce

$$0 \ge f(v^k, y^k) + \frac{\|v^k - y^k\|^2}{\lambda_k}.$$

Combine the above inequality and (12) to derive that $0 \leq (\frac{1}{\lambda_k} - \frac{\gamma}{2\lambda_k})\|v^k - y^k\|^2 \leq 0$. Hence, $v^k = y^k$, which is incompatible with the assumption. Thus, the search rule (6) is well-defined. Consequently, $f(z^k, y^k) \leq -\frac{\gamma}{2\lambda_k}\|v^k - y^k\|^2 < 0$ when $v^k \neq y^k$. □

Proposition 3.3. *The sequence* $\{x^k\}$ *generated by* (9) *is well-defined.*

Proof. Firstly, we prove by induction that $Sol(f,C) \cap Fix(S) \subset C_k$ for all $k \geq 1$. $Sol(f,C) \cap Fix(S) \subset C_1$ is obvious. Suppose that $Sol(f,C) \cap Fix(S) \subset C_k$ for some $k \in \mathbb{N}$. Pick up $p \in Sol(f,C) \cap Fix(S) \subset C_k$.

By (4) and Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
\|v^k - p\|^2 &= \|(1-\delta_k)(x^k - p) + \delta_k(S[(1-\sigma_k)x^k + \sigma_k Sx^k] - p)\|^2 \\
&= (1-\delta_k)\|x^k - p\|^2 - \delta_k(1-\delta_k)\|S[(1-\sigma_k)x^k + \sigma_k Sx^k] - x^k\|^2 \\
&\quad + \delta_k\|S[(1-\sigma_k)x^k + \sigma_k Sx^k] - p\|^2 \\
&\leq (1-\delta_k)\|x^k - p\|^2 - \delta_k(1-\delta_k)\|S[(1-\sigma_k)x^k + \sigma_k Sx^k] - x^k\|^2 \\
&\quad + \delta_k(\|x^k - p\|^2 + (1-\sigma_k)\|x^k - S[(1-\sigma_k)x^k + \sigma_k Sx^k]\|^2) \\
&= \|x^k - p\|^2 - \delta_k(\sigma_k - \delta_k)\|x^k - S[(1-\sigma_k)x^k + \sigma_k Sx^k]\|^2.
\end{aligned} \tag{13}$$

Note that $f(z^k, p) \leq 0$. Applying the subdifferential inequality, we have

$$f(z^k, p) \geq \langle g^k, p - z^k \rangle.$$

It follows that

$$\langle g^k, z^k - p \rangle \geq -f(z^k, p) \geq 0.$$

So,

$$\langle g^k, v^k - p \rangle = \langle g^k, v^k - z^k \rangle + \langle g^k, z^k - p \rangle \geq \langle g^k, v^k - z^k \rangle.$$

Note that

$$v^k - z^k = \frac{\mu_k}{1-\mu_k}(z^k - y^k)$$

and

$$f(z^k, y^k) \geq \langle g^k, y^k - z^k \rangle.$$

Therefore,

$$\langle g^k, v^k - p \rangle \geq \frac{\mu_k}{1-\mu_k}\langle g^k, z^k - y^k \rangle \geq \frac{-\mu_k}{1-\mu_k} f(z^k, y^k). \tag{14}$$

Case 1. If $0 \notin \partial_2 f(z^k, z^k)$. According to (8), we get

$$\begin{aligned}
\|u^k - p\|^2 &= \left\| P_C\left[v^k + \frac{\mu_k\gamma_k f(z^k, y^k)}{(1-\mu_k)\|g^k\|^2} g^k\right] - p \right\|^2 \\
&\leq \left\| v^k + \frac{\mu_k\gamma_k f(z^k, y^k)}{(1-\mu_k)\|g^k\|^2} g^k - p \right\|^2 \\
&= \|v^k - p\|^2 + \frac{2\mu_k\gamma_k f(z^k, y^k)}{(1-\mu_k)\|g^k\|^2}\langle g^k, v^k - p \rangle + \frac{\mu_k^2\gamma_k^2 f^2(z^k, y^k)}{(1-\mu_k)^2\|g^k\|^2} \\
&\leq \|v^k - p\|^2 - \frac{2\mu_k^2\gamma_k f^2(z^k, y^k)}{(1-\mu_k)^2\|g^k\|^2} + \frac{\mu_k^2\gamma_k^2 f^2(z^k, y^k)}{(1-\mu_k)^2\|g^k\|^2} \\
&= \|v^k - p\|^2 - \gamma_k(2-\gamma_k)\frac{\mu_k^2 f^2(z^k, y^k)}{(1-\mu_k)^2\|g^k\|^2}.
\end{aligned} \tag{15}$$

Combining (13) and (15), we obtain

$$\begin{aligned}\|u^k - p\|^2 &\le \|x^k - p\|^2 - \gamma_k(2-\gamma_k)\frac{\mu_k^2 f^2(z^k, y^k)}{(1-\mu_k)^2\|g^k\|^2} \\ &\quad - \delta_k(\sigma_k - \delta_k)\|x^k - S[(1-\sigma_k)x^k + \sigma_k S x^k]\|^2 \\ &\le \|x^k - p\|^2,\end{aligned} \tag{16}$$

and hence $p \in C_{k+1}$.

Case 2. If $0 \in \partial_2 f(z^k, z^k)$. In this case, $u^k = v^k$ and $\|u^k - p\| \le \|x^k - p\|$ is obvious. So, $Sol(f, C) \cap Fix(S) \subset C_k$ for all $k \ge 1$.

It is known that C_k is closed and convex for all $k \in \mathbb{N}$. Therefore, the sequence $\{x^k\}$ is well-defined. □

Proposition 3.4. $\lim_{k\to\infty} \frac{\mu_k f(z^k, y^k)}{(1-\mu_k)\|g^k\|} = 0$ *and* $\lim_{k\to\infty} \|x^k - Sx^k\| = 0$.

Proof. Since $x^k = P_{C_k}[x^0]$, by the definition of the projection, we have

$$\|x^k - x^0\| \le \|x^0 - u\|, \forall u \in C_k, \tag{17}$$

which by selecting $u = p$, implies that the sequence $\{x^k\}$ is bounded. Consequently, the sequences $\{v^k\}$ and $\{u^k\}$ are bounded. By the maximum theorem, $\{y^k\}$ is also bounded. Thus, $\{z^k\}$ is bounded. Together with Lemma 2.3, $\{g^k\}$ is bounded.

Since $x^{k+1} \in C_{k+1} \subset C_k$, we have from (3) that

$$\langle x^0 - x^k, x^{k+1} - x^k\rangle \le 0.$$

So,

$$\begin{aligned}\|x^{k+1} - x^k\|^2 &= 2\langle x^0 - x^k, x^{k+1} - x^k\rangle + \|x^{k+1} - x^0\|^2 - \|x^0 - x^k\|^2 \\ &\le \|x^{k+1} - x^0\|^2 - \|x^k - x^0\|^2.\end{aligned} \tag{18}$$

Choosing $u = x^{k+1}$ in (17), we deduce $\|x^k - x^0\| \le \|x^0 - x^{k+1}\|$. Thus, the limit $\lim_{k\to\infty} \|x^k - x^0\|$ exists, denoted by q. This together with (18) implies that $\|x^{k+1} - x^k\| \to 0$. Thanks to the definition of C_{k+1} and $x^{k+1} \in C_k$, we derive $\|u^k - x^{k+1}\| \le \|x^k - x^{k+1}\| \to 0$. Hence,

$$\|u^k - x^k\| \le \|u^k - x^{k+1}\| + \|x^{k+1} - x^k\| \to 0.$$

By (16), we obtain

$$\begin{aligned}0 &\le \gamma_k(2-\gamma_k)\frac{\mu_k^2 f^2(z^k, y^k)}{(1-\mu_k)^2\|g^k\|^2} + \delta_k(\sigma_k - \delta_k)\|x^k - S[(1-\sigma_k)x^k + \sigma_k S x^k]\|^2 \\ &\le \|x^k - p\|^2 - \|u^k - p\|^2 \\ &\le \|x^k - u^k\|[\|x^k - p\| + \|u^k - p\|] \\ &\to 0.\end{aligned}$$

It follows that

$$\lim_{k\to\infty} \frac{\mu_k f(z^k, y^k)}{(1-\mu_k)\|g^k\|} = 0, \tag{19}$$

and

$$\lim_{k\to\infty} \|x^k - S[(1-\sigma_k)x^k + \sigma_k S x^k]\| = 0. \tag{20}$$

Consequently,

$$\lim_{k\to\infty} \|v^k - x^k\| = 0.$$

On the other hand,

$$\begin{aligned}\|x^k - Sx^k\| &\le \|x^k - S[(1-\sigma_k)x^k + \sigma_k Sx^k]\| + \|S[(1-\sigma_k)x^k + \sigma_k Sx^k] - Sx^k\| \\ &\le \|x^k - S[(1-\sigma_k)x^k + \sigma_k Sx^k]\| + L\sigma_k\|x^k - Sx^k\|.\end{aligned}$$

It follows from (20) that

$$\|x^k - Sx^k\| \le \frac{1}{1-L\sigma_k}\|x^k - S[(1-\sigma_k)x^k + \sigma_k Sx^k]\| \to 0. \tag{21}$$

□

Proposition 3.5. $\omega_w(x^k) \subset Sol(f,C) \cap Fix(S)$.

Proof. By (6), we have

$$f(z^k, y^k) + \frac{\gamma}{2\lambda_k}\|v^k - y^k\|^2 \le 0. \tag{22}$$

We will consider two cases:

Case 1. $\limsup_{k\to\infty} \mu_k > 0$. Then there exists $\mu_0 > 0$ and N_0 such that $\mu_{k_i} \ge \mu_0$ for every $i \ge N_0$. From (19) and (22), we deduce

$$\lim_{i\to\infty} \|v^{k_i} - y^{k_i}\| = 0. \tag{23}$$

Selecting any $x^\dagger \in \omega_w(v^{k_i})$, there exists a subsequence $\{v^{k_{i_j}}\} \subset \{v^{k_i}\}$, still denoted by $\{v^{k_i}\}$, such that $v^{k_i} \rightharpoonup x^\dagger \in C$. Using (23), we also deduce that $y^{k_i} \rightharpoonup x^\dagger \in C$. By (5), we have

$$\begin{aligned}f(v^{k_i}, y^{k_i}) + \frac{1}{2\lambda_{k_i}}\|v^{k_i} - y^{k_i}\|^2 &\le f(v^{k_i}, y^\dagger) + \frac{1}{2\lambda_{k_i}}\|v^{k_i} - y^\dagger\|^2 \ (\forall y^\dagger \in C) \\ &\le f(v^{k_i}, y^\dagger) + \frac{1}{2\rho}\|v^{k_i} - y^\dagger\|^2.\end{aligned} \tag{24}$$

Letting $i \to \infty$ in (24), we obtain

$$0 \le f(x^\dagger, y^\dagger) + \frac{1}{2\rho}\|x^\dagger - y^\dagger\|^2, \ \forall y^\dagger \in C. \tag{25}$$

Therefore, $x^\dagger \in Sol(f,C)$.

Case 2. $\lim_{k\to\infty} \mu_k = 0$. Selecting any $x^\ddagger \in \omega_w(v^k)$, there exists a subsequence $\{v^{k_j}\} \subset \{v^k\}$ such that $v^{k_j} \rightharpoonup x^\ddagger \in C$. Since $\{y^{k_j}\}$ is bounded, without loss of generality, we may assume that $y^{k_j} \rightharpoonup y^\ddagger \in C$. By the definition of y^{k_j}, we have

$$f(v^{k_j}, y^{k_j}) + \frac{1}{2\lambda_{k_j}}\|v^{k_j} - y^{k_j}\|^2 \le f(v^{k_j}, y^\dagger) + \frac{1}{2\lambda_{k_j}}\|v^{k_j} - y^\dagger\|^2 \ (\forall y^\dagger \in C). \tag{26}$$

Letting $j \to \infty$ with $\lim_{j\to\infty} \lambda_{k_j} = \rho^\dagger$ in (26), we derive

$$f(x^\ddagger, y^\ddagger) + \frac{1}{2\rho^\dagger}\|x^\ddagger - y^\ddagger\|^2 \le f(x^\ddagger, y^\dagger) + \frac{1}{2\rho^\dagger}\|x^\ddagger - y^\dagger\|^2 \ (\forall y^\dagger \in C). \tag{27}$$

Choose $y^\dagger = x^\ddagger$ in (27) to deduce

$$f(x^\ddagger, y^\ddagger) + \frac{1}{2\rho}\|x^\ddagger - y^\ddagger\|^2 \le f(x^\ddagger, x^\ddagger) + \frac{1}{2\rho}\|x^\ddagger - x^\ddagger\|^2 = 0. \tag{28}$$

On the other hand, m is the smallest positive integer satisfying (6), so we have

$$f(z^{k_j, m-1}, y^{k_j}) + \frac{\gamma}{2\lambda_{k_j}}\|v^{k_j} - y^{k_j}\|^2 > 0. \tag{29}$$

Note that

$$z^{k_j,m-1} = (1-\mu^{m-1})v^{k_j} + \mu^{m-1}y^{k_j} \to x^\ddagger.$$

This together with (29) implies that

$$f(x^\ddagger, y^\ddagger) + \frac{\gamma}{2\rho}\|x^\ddagger - y^\ddagger\|^2 \geq 0. \tag{30}$$

Taking into account (28) and (30), we deduce that

$$0 \leq \frac{1-\gamma}{2\rho}\|x^\ddagger - y^\ddagger\|^2 \leq 0,$$

which implies that $x^\ddagger = y^\ddagger$. Therefore,

$$f(x^\ddagger, y^\dagger) + \frac{1}{2\rho}\|x^\ddagger - y^\dagger\|^2 \geq 0,\ \forall y^\dagger \in C. \tag{31}$$

Therefore, $x^\dagger \in Sol(f, C)$.

Next, we show $x^\dagger \in Fix(S)$. Note that $x^{k_i} \rightharpoonup x^\dagger$. This together with Lemma 2.4 and (21) implies that $x^\dagger \in Fix(S)$. So, $\omega_w(x^k) \subset Sol(f, C) \cap Fix(S)$. □

Theorem 3.1. *The iterate $\{x^k\}$ defined by Algorithm 3.1 converges to $P_{Sol(f,C)\cap Fix(S)}[x^0]$.*

Proof. First, it is obvious that $Sol(f, C) \cap Fix(S)$ is nonempty, closed and convex. Thus, $P_{Sol(f,C)\cap Fix(S)}$ is well-defined. Thanks to (17), we deduce

$$\|x^k - x^0\| \leq \|x^0 - P_{Sol(f,C)\cap Fix(S)}[x^0]\|.$$

By Proposition 3.5, we obtain $\omega_w(x^k) \subset Sol(f, C)\cap Fix(S)$. Hence, all conditions of Lemma 2.5 are fulfilled. Consequently, we conclude that $x^k \to P_{Sol(f,C)\cap Fix(S)}[x^0]$. □

4. Conclusion

Recently, the equilibrium problem and fixed point problem have attracted so much attention. In this paper, we devote to construct an iterative algorithm for solving the equilibrium problem (1) and fixed point problems in Hilbert spaces. We present an iterative algorithm combined with subgradient and hybrid methods for solving the fixed point problems of pseudocontractive operators and the equilibrium problems of pseudomonotone operators. We show the strong convergence of the proposed algorithm.

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REFERENCES

[1] P.K. Anh and D.V. Hieu, *Parallel hybrid methods for variational inequalities, equilibrium problems and common fixed point problems*, Vietnam J. Math., **44** (2016), 351–374.

[2] J.Y. Bello-Cruz and A.N. Iusem, *Convergence of direct methods for paramonotone variational inequalities*, Comput. Optim. Appl., **46** (2010), 247–263.

[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), 123–145.

[4] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Hybrid viscosity extragradient method for systems of variational inequalities, fixed Points of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory, **19** (2018), 487–502.

[5] L.C. Ceng, A. Petrusel, J.C. Yao and Y. Yao, *Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions*, Fixed Point Theory, **20** (2019), 113–134.

[6] S.Y. Cho, X. Qin, J.C. Yao and Y. Yao, *Viscosity approximation splitting methods for monotone and nonexpansive operators in Hilbert spaces*, J. Nonlinear Convex Anal., **19** (2018), 251–264.

[7] P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005), 117–136.

[8] V. Dadashi and M. Postolache, *Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators*, Arab. J. Math. **9** (2020), No. 1, 89–99.

[9] Q.L. Dong, Y. Peng and Y. Yao, *Alternated inertial projection methods for the split equality problem*, J. Nonlinear Convex Anal., in press.

[10] D.V. Hieu, L.D. Muu and P.K. Anh, *Parallel hybrid extragradientmethods for pseudomonotone equilibrium problems and nonexpansive mappings*, Numer. Algorithms, **73** (2016), 197–217.

[11] I.V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer, Berlin, 2001.

[12] Y. Malitsky, *Proximal extrapolated gradient methods for variational inequalities*, Optim. Meth. & Software, **33** (2018), 140–164.

[13] B. Martinet, *Régularisation dínéquations variationelles par approximations successives*, Rev. Fr. Autom. Inform. Rech. Opér., Anal. Numér., **4** (1970), 154–159.

[14] C. Martinez-Yanes and H.K. Xu, *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal., **64** (2006), 2400–2411.

[15] G. Mastroeni, *On auxiliary principle for equilibrium problems*, Publicatione del Dipartimento di Mathematica DellUniversita di Pisa 3 (2000), 1244–1258.

[16] T.T.V. Nguyen, J.J. Strodiot and V.H. Nguyen, *Hybrid methods for solving simultaneously an equilibrium problem and countably many fixed point problems in a Hilbert space*, J. Optim. Theory Appl., **160** (2014), 809–831.

[17] T.D. Quoc, L.D. Muu and N.V. Hien, *Extragradient algorithms extended to equilibrium problems*, Optim., **57** (2008), 749–776.

[18] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim., **14** (1976), 877–898.

[19] B.S. Thakur and M. Postolache, *Existence and approximation of solutions for generalized extended nonlinear variational inequalities*, J. Inequal. Appl., **2013** (2013), Art. No. 590.

[20] D.V. Thong and A. Gibali, *Extragradient methods for solving non-Lipschitzian pseudo-monotone variational inequalities*, J. Fixed Point Theory Appl., **21** (2019), Art. ID. UNSP 20.

[21] P.T. Vuong, J.J. Strodiot and V.H. Nguyen, *On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space*, Optim., **64** (2015), 429–451.

[22] J. Yang and H. Liu, *The subgradient extragradient method extended to pseudomonotone equilibrium problems and fixed point problems in Hilbert space*, Optim. Lett., in press.

[23] Y. Yao, L. Leng, M. Postolache and X. Zheng, *Mann-type iteration method for solving the split common fixed point problem*, J. Nonlinear Convex Anal., **18** (2017), 875–882.

[24] Y. Yao, H. Li and M. Postolache, *Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions*, Optim., in press.

[25] Y. Yao, M. Postolache and J.C. Yao, *Iterative algorithms for the generalized variational inequalities*, U.P.B. Sci. Bull., Series A, **81** (2019), 3–16.

[26] Y. Yao, M. Postolache and J.C. Yao, *An iterative algorithm for solving the generalized variational inequalities and fixed points problems*, Mathematics, **7** (2019), Art. ID. 61.

[27] Y. Yao, M. Postolache and J. C Yao, *Strong convergence of an extragradient algorithm for variational inequality and fixed point problems*, U. Politeh. Buch. Ser. A, **82** (2020), No. 1, 3–12.

[28] Y. Yao, M. Postolache and J.C. Yao, *Strong convergence of Halpern-type projection subgradient algorithms with linear search for split equilibrium problems*, J. Nonlinear Convex Anal., in press.

[29] Y. Yao, M. Postolache and Z. Zhu, *Gradient methods with selection technique for the multiple-sets split feasibility problem*, Optim., **69** (2020), 269–281.

[30] Y. Yao, X. Qin and J.C. Yao, *Projection methods for firmly type nonexpansive operators*, J. Nonlinear Convex Anal., **19** (2018), 407–415.

[31] H. Zegeye, N. Shahzad and Y. Yao, *Minimum-norm solution of variational inequality and fixed point problem in Banach spaces*, Optim., **64** (2015), 453–471.

[32] X. Zhao, J.C. Yao and Y. Yao, *A proximal algorithm for solving split monotone variational inclusions*, U.P.B. Sci. Bull., Series A, **82(3)** (2020), 43–52.

[33] X. Zhao and Y. Yao, *Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems*, Optim., **69** (2020), 1987–2002.

[34] H. Zhou, *Strong convergence of an explicit iterative algorithm for continuous pseudocontractions in Banach spaces*, Nonlinear Anal., **70** (2009), 4039–4046.

[35] L.J. Zhu, Y. Yao and M. Postolache, *Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems*, U.P.B. Sci. Bull., Series A, in press.