SOME SUFFICIENT OPTIMALITY CONDITIONS IN NONDIFERENTIABLE MULTIOBJECTIVE PROGRAMMING

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This study focuses on the nondiferentiable multiobjective programming problem. We start from the invexity proposed by H. Slimani and M.S. Radjef. They consider the invexity for a differentiable vector function when each component of function is invex with respect to its own function η_i. *We extend this concept to a general invexity class of* ρ*-invexity and, moreover for this case, we consider that the vector function is nondifferentiable. In this framework, we investigate the optimality conditions and give some new theorems that state the sufficient conditions for a feasible point to be efficient.*

Keywords: multiobjective programming, nonsmooth functions.

1. Introduction

The study of the multiobjective programming problems is one of the great significance in optimization theory (see [1]). A certain situation is when the objective function is a nondiferentiable vector function. To treat this case we consider a constrained multiobjective programming problem.

The class of invex functions was introduced by Hanson (see [2]). This class of functions is designed to relax the assumptions of the convexity which are imposed on functions when we want to state the sufficient optimality conditions. In the specialized literature exists some concepts concerning the class of invex functions (see [3], [4]). One of these approaches is that proposed by H. Slimani and M.S. Radjef (see [4]). In their study the invexity for a differentiable vector function is solved by searching the invexity for each component. On the other hand, for studying the invexity of a nondifferentiable vector function with respect to the same function η exist an other approach (see [5], [6],) which use the Elster and Thierfeld'concept (see [7]) of a K cone approximations for finding the K-directional derivative. Our paper starts from the work done in other personal research (see [8]) where we generalized the concept of invexity by stating the ρ-invexity definition for nonsmooth functions refered of different η_i. In this paper

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we complet the research by stating of three new theorems which treat the optimality of the multiobjective programming.

In the following Sections we use K-directional derivative of a function f at x to define $\rho-$invex and (weakly) $\rho-$pseudo-invex vector functions with respect to different $(\eta_i)_{i=1,p}$. Sections 3 and 4 contain some new theorems that establish sufficient conditions for a feasible point to be weakly efficient or properly efficient. The last Section refers to the further directions that derive from our paper.

2. Preliminaries

In this section we deal with nondifferentiable functions and with their $\rho-$invexity and $\rho-$weakly pseudo-invexity when the invexity is treated with respect to each different $(\eta_i)_{i=1,p}$.

Let I a nonempty open set of R^n and a nondifferentiable function $f: I \to R^p$. The local cone approximation K represents the locally approximation of the set epigraf of f at the point $(x, f(x))$, denoted by $epi\ f$ (see [4]), where the epigraf of f is the set:

$$epi \quad f = \{(x,a) \in I \times R \quad with \quad f(x) \le a\}.$$

For a certainly local con approximation K we can uniquely determine the K-directional derivative of function f at x.

Definition 1 (see [5]). Let $f: I \to R$, $x \in I$ and K the local cone approximation. The K- directional derivative of f at x is the positively homogeneous function denoted with $f^K(x,.)$ where $f^K(x,.): R^n \to [-\infty,\infty]$ and $f^K(x,d) = \inf\{\xi \in R \quad with (d,\xi) \in K(epif,(x, f(x)))\}$.

Definition 2 (see [8]). Let $f: I \to R^p$ and K a local cone approximation. We say that the function f is strictly K-ρ-invex at $x_0 \in I$ with respect to $(\eta_i)_{i=1,p}$, if there exist N vector function $\eta_i : I \times I \to R^n$, $i = 1, p$ and $\rho = (\rho_1,...,\rho_p)$ a vector with real components where $\rho_i \in R$, $i = 1, p$ such that for each $x \in I$ follows:

$$f_i(x) - f_i(x_0) > f_i^K(x_0, \eta_i(x, x_0)) + \rho_i d(x, x_0) \text{ for all } i = 1, p.$$

The function f is said to be strictly K-ρ-invex on I with respect to $(\eta_i)_{i=1,p}$, if is strictly K-ρ-invex at each $x_0 \in I$ with respect to $(\eta_i)_{i=1,p}$.

Definition 3 (see [8]). Let $f : I \to R^p$ and K be a local cone approximation. The function f is said to be strictly K-ρ-weakly pseudo-invex at $x_0 \in I$ with respect to $(\eta_i)_{i=\overline{1,p}}$, if there exist p vector function $\eta_i : I \times I \to R^n$, $i = \overline{1,p}$ and $\rho = (\rho_1,...,\rho_p)$ a vector with real components such that for a $x \in I$ with $f(x) - f(x_0) < 0$ exists $\overline{x} \in I$ such that:

$$f_i^K\left(x_0, \eta_i\left(\overline{x}, x_0\right)\right) + \rho_i d\left(\overline{x}, x_0\right) < 0 \text{ for all } i = \overline{1,p}.$$

If $\overline{x} = x$ in the previous relation, than f is said to be K-ρ-weakly pseudo-invex at $x_0 \in I$ with respect to $(\eta_i)_{i=\overline{1,p}}$. We say that f is (weakly) K-ρ-weakly pseudo-invex on I with respect to $(\eta_i)_{i=\overline{1,p}}$, if f is (weakly) K-ρ-pseudo-invex at each $x_0 \in I$ with respect to the same $(\eta_i)_{i=\overline{1,p}}$.

Definition 4 A function $f : I \subseteq R^n \to R$ is said to be K-σ -quasi-invex in x_0 with respect to a vector function $\theta : I \times I \to R^n$ if $\sigma \in R, \quad \sigma \geq 0$ and:

$$h(x) - h(x_0) \leq 0 \Rightarrow h_j^K\left(x_0, \theta_j(x, x_0)\right) + \sigma d(x, x_0) \leq 0 \quad \forall x \in I.$$

3. The sufficient conditions for a feasible point to be weakly efficient

In the following we state a sufficient conditions theorem for a weakly efficient solution of a constrained multiobjective programming problem.

We consider a multiobjective programming problem (MP): min $f(x)$ with the constraints $g(x) \leq 0$, where f is the objective function $f : I \to R^p$ and $g : I \to R^m$ represents the constraints. For a MP $x \in I$ for whigh $g(x) \leq 0$, is named a feasible solution. We denote the feasible solutions set with X and with *J*(*x*) the set $J(x) = \{j = \overline{1,m} \quad with \quad g_j(x) = 0\}$.

Definition 5 $x_0 \in X$ is said to be an efficient (respectively weakly efficient) solution of MP if there is no $x \in X$ such that $f(x) \leq f(x_0)$, (respectively $f(x) < f(x_0)$).

Definition 6 An efficient solution of MP $x_0 \in X$ is said to be a properly efficient solution if there exists a constant a, $a > 0$ such that exists at least one *j*=1,..,*p* such that: $f_j(x_0) < f_j(x)$ and $f_i(x_0) - f_i(x) \leq a\left(f_j(x) - f_j(x_0)\right)$, for each $x \in X$ and for each *i*=1,..,p satisfying $f_i(x) < f_i(x_0)$.

Theorem 1 Let $x_0 \in X$. If:

i_1) f is K- ρ -weakly pseudo-invex at $x_0 \in I$ with respect to $(\eta_i)_{i=1,p}$;

i_2) If exist vector $\mu \in R^p, \mu \geq 0$ and $\lambda, \sigma \in R^{cardJ(x_0)}, \lambda \geq 0$ such that the scalar function $\sum_{j \in J(x_0)} \lambda_j g_j$ is K- σ quasi-invex at x_0 with respect to $\theta : X \times X \to R^n$;

i_3) $\sum_{i=1}^{p} \mu_i f_i^K(x_0, \eta_i(x, x_0)) + \sum_{j \in J(x_0)} \lambda_j g_j^K(x_0, \theta_j(x, x_0)) \geq 0 \quad \forall x \in X$

i_4) $\sum_{i=1,..,p} \rho_i \mu_i + \sigma \geq 0$

then x_0 is a weakly efficient solution for MP.

Proof. We consider that x_0 is not a weakly efficient solution for MP. From the hypothesis i_1 we have:

$$\exists \quad \bar{x} \in X \quad with \quad f_i^K\left(x_0, \eta_i\left(\bar{x}, x_0\right)\right) < -\rho_i d\left(\bar{x}, x_0\right)$$

But $\mu \geq 0$ then:

$$\sum_{i=1}^{p} \mu_i f_i^K\left(x_0, \eta_i\left(\bar{x}, x_0\right)\right) < -\sum_{i=1}^{p} \rho_i \mu_i d\left(\bar{x}, x_0\right) \tag{1}$$

Then the hypothesis i_2 were $x = \bar{x}$ implies:

$$\sum_{j \in J(x_0)} \lambda_j g_j^K\left(x_0, \theta\left(\bar{x}, x_0\right)\right) \geq -\sigma d\left(\bar{x}, x_0\right) \tag{2}$$

Using the relation (1) and (2) and hypothesis i_4 we have:

$$\sum_{i=1}^{p} \mu_i f_i^K\left(x_0, \eta_i\left(\bar{x}, x_0\right)\right) + \sum_{j \in J(x_0)} \lambda_j g_j^K\left(x_0, \theta_j\left(\bar{x}, x_0\right)\right) <$$

$$< -\left(\sum_{i=1}^{p} \rho_i \mu_i + \sigma\right) d\left(\bar{x}, x_0\right) < 0$$

By hypothesis i_3 , x_0 is a weakly efficient solution for MP.

4. The sufficient conditions for a feasible point to be properly efficient

This Section refers to the sufficient conditions for a feasible point to be properly efficient for MP, the following theorems describe this situation.

Theorem 2 Let $x_0 \in X$ and suppose that:

i_1) f is K-ρ-invex at $x_0 \in I$ with respect to $(\eta_i)_{i=\overline{1,p}}$;

i_2) If for any $j \in J(x_0)$ there exists the function $\theta_j : X \times X \to R^n$ such that $g_j^K(x_0, \theta_j(x, x_0)) \le -\sigma_j d(x, x_0),\ \forall x \in X$, and for a vector $\sigma = (\sigma_j)_{j \in J(x_0)}$;

i_3) There exist vectors $\mu \in R^p, \mu \ge 0$ and $\lambda \in R^{cardJ(x_0)}, \lambda \ge 0$ such that $(x_0, \mu, \lambda, (\eta_i)_i, (\theta_j)_j)$ satisfies the following condition:

$$\sum_{i=1}^{p} \mu_i f_i^K(x_0, \eta_i(x, x_0)) + \sum_{j \in J(x_0)} \lambda_j g_j^K(x_0, \theta_j(x, x_0)) \ge 0 \quad \forall x \in X$$

i_4) $\sum_{i=1,..,p} \rho_i \mu_i + \sum_{j \in J(x_0)} \lambda_j \sigma_j \ge 0$

then x_0 is a properly efficient solution for MP.

Proof. The hypothesis i_1 of theorem and definition 2 imply:
For each $x \in X$ we have:

$$f_i(x) - f_i(x_0) > f_i^K(x_0, \eta_i(x, x_0)) + \rho_i d(x, x_0) \text{ for all } i = \overline{1, p}.$$

Then:

$$\sum_{i=1}^{p} \mu_i f_i(x) - \sum_{i=1}^{p} \mu_i f_i(x_0) \ge \sum_{i=1}^{p} \eta_i f_i^K(x_0, \eta_i(x, x_0)) + \sum_{i=1}^{p} \rho_i \mu_i d(x, x_0)$$

Using the relation from hypothesis i_3, we obtain:

$$\sum_{i=1}^{p} \mu_i f_i(x) - \sum_{i=1}^{p} \mu_i f_i(x_0) \ge -\sum_{j \in J(x_0)} \lambda_j g_j^K(x_0, \theta_j(x, x_0)) + \sum_{i=1}^{p} \rho_i \mu_i d(x, x_0) \tag{3}$$

Since $\lambda_j \ge 0$ for all $j \in J(x_0)$ and

$$g_j^K(x_0, \theta_j(x, x_0)) \le -\sigma_j d(x, x_0) \text{ for all } x \in X$$

Then the right member of relation (3) satisfies:

$$-\sum_{j \in J(x_0)} \lambda_j g_j^K(x_0, \theta_j(x, x_0)) + \sum_{i=1}^{p} \rho_i \mu_i d(x, x_0) \ge$$

$$\ge \left(\sum_{i=1}^{p} \rho_i \mu_i + \sum_{j \in J(x_0)} \lambda_j \sigma_j \right) d(x, x_0) \ge 0 \quad \forall x \in X$$

Finally, the hypothesis i_3 involves that:

$$\sum_{i=1}^{p} \mu_i f_i(x) \geq \sum_{i=1}^{p} \mu_i f_i(x_0) \quad \forall x \in X$$

Take into account that $\mu \geq 0$ and by Geoffrion (see [9]), we obtain that x_0 is a properly efficient solution for MP.

Theorem 3 Let $x_0 \in X$ and suppose that:

i_1) If exists $\mu \in R^p$, $\mu \geq 0$ such that the scalar function $\sum_{i=1}^{p} \mu_i f_i$ is K-ρ_0 weakly pseudo-invex at x_0 with respect to $\eta : X \times X \to R^n$;

i_2) If there exists a function $\theta_j : X \times X \to R^n$ such that

$$g_j^K\left(x_0, \theta_j(x, x_0)\right) \leq -\sigma_j d(x, x_0), \ \forall x \in X \ ;$$

i_3) If there exist two vectors $\sigma, \lambda \in R^{cardJ(x_0)}, \lambda \geq 0$ such that :

$$\sum_{i=1}^{p} \mu_i f_i^K\left(x_0, \eta(x, x_0)\right) + \sum_{j \in J(x_0)} \lambda_j g_j^K\left(x_0, \theta_j(x, x_0)\right) \geq 0 \quad \forall x \in X$$

i_4) If $\rho_0 + \sum_{j \in J(x_0)} \lambda_j \sigma_j \geq 0$

then x_0 is a properly efficient solution for MP.

Proof. If x_0 is not a properly efficient solution for MP, by Geoffrion (see [9]) there exists a feasible point x such that:

$$\sum_{i=1}^{p} \mu_i f_i(x) - \sum_{i=1}^{p} \mu_i f_i(x_0) < 0$$

From the hypothesis i_1 we have $\sum_{i=1}^{p} \mu_i f_i$ is K-ρ_0 weakly pseudo-invex at x_0 with respect to η, then:

$$\exists \bar{x} \in X \quad \sum_{i=1}^{p} \mu_i f_i^K\left(x_0, \eta_i(\bar{x}, x_0)\right) < -\rho_0 d(\bar{x}, x_0) \tag{4}$$

But $\lambda_j \geq 0 \ \forall j \in J(x_0)$ and hypothesis i_2 , then:

$$\sum_{j \in J(x_0)} \lambda_j g_j^K\left(x_0, \theta_j(\bar{x}, x_0)\right) \leq -\sum_{j \in J(x_0)} \lambda_j \sigma_j d(\bar{x}, x_0) \tag{5}$$

Summing the relations (4) and (5), it follows that:

$$\sum_{i=1}^{p}\mu_i f_i^K\left(x_0,\eta\left(\bar{x},x_0\right)\right)+\sum_{j\in J(x_0)}\lambda_j g_j^K\left(x_0,\theta_j\left(\bar{x},x_0\right)\right) <$$

$$< -\left(\rho_0+\sum_{j\in J(x_0)}\lambda_j\sigma_j\right)d\left(\bar{x},x_0\right) \tag{6}$$

Using the hypothesis i_4 we have:

$$\left(\rho_0+\sum_{j\in J(x_0)}\lambda_j\sigma_j\right)d\left(\bar{x},x_0\right)>0$$

This means that the relation (6) becomes:

$$\sum_{i=1}^{p}\mu_i f_i^K\left(x_0,\eta\left(\bar{x},x_0\right)\right)+\sum_{j\in J(x_0)}\lambda_j g_j^K\left(x_0,\theta_j\left(\bar{x},x_0\right)\right)<0$$

The previous relation is a contradiction for the hypothesis i_3, so x_0 is a properly efficient solution of MP.

5. Conclusions

Our research study intends to extend the invexity with respect to different η_i to the nondifferentiable vector functions. By the local cone approximation concept, we renounce to the differentiability functions assumption and to the use of the *K*-directional derivative of functions. Our theorems (1-4) provide the sufficient conditions which show that a feasible point can be less efficient or sufficiently efficient.

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