U.P.B. Sci. Bull., Series A, Vol. 71, Iss. 1, 2009 ISSN 1223-7027

CLASSIFICATION OF NON STATIC SPHERICALLY SYMMETRIC SPACE-TIMES ACCORDING TO THEIR PROPER CONFORMAL VECTOR FIELDS

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Direct integration technique is used to study proper conformal vector field in non conformally flat spherically symmetric non static space-times. Using the above mentioned technique we show that there exist two possibilities when the above space-times admit proper conformal vector fields.

Keywords: Direct integration technique; proper conformal vector field; Killing vector fields**.**

1. Introduction

In this paper we investigate the existence of proper conformal vector fields in non conformally flat spherically symmetric non static space-times by using the direct integration technique. The conformal vector field which preserves the metric structure upto a conformal factor carries significant interest in Einstein's theory of general relativity. Unlike Killing, homothetic and affine symmetries, conformal symmetry is difficult to study because it lacks the linearity property. These difficulties are discussed in [1-5]. Some more general results on the Lie algebra and dimensions of conformal vector fields are given in [5]. It is therefore important to study these symmetries.

Throughout M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (-, +, +, +). The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$, the Ricci tensor components are $R_{ab} = R^c{}_{acb}$ and the Weyl tensor components are $C^a{}_{bcd}$. The usual covariant, partial and Lie

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derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. The space-time M will be assumed non conformally flat in the sense that the Weyl tensor does not vanish over any non empty open subset of M.

The covariant derivative of any vector field X on M can be decomposed as

$$X_{a;b} = \frac{1}{2} h_{ab} + F_{ab} \tag{1}$$

where $h_{ab}(= h_{ba}) = L_X g_{ab}$ and $F_{ab}(= -F_{ba})$ are symmetric and skew symmetric tensors on M, respectively. Such a vector field X is called conformal vector field if the local diffeomorphisms η_t (for appropriate t) associated with X preserves the metric structure up to a conformal factor i.e. $\eta_t^* g = \psi g$, where ψ is a nowhere zero positive function on some open subset of M and η_t^* is a pullback map on some open subset of M [3,6]. This is equivalent to the condition that

$$h_{ab} = 2\psi g_{ab},$$

or, equivalently, if

$$g_{ab,c} X^c + g_{cb} X^c_{,a} + g_{ac} X^c_{,b} = 2\psi g_{ab}, \tag{2}$$

where $\psi : U \to R$ is the smooth conformal function on some open subset of M, then X is called conformal vector field. If ψ is constant on M, X is homothetic (proper homothetic if $\psi \neq 0$) while if $\psi = 0$ then it is Killing. If the vector field X is conformal, but not homothetic, then it is called proper conformal.

2. Main Results

Consider the space-times in the usual coordinate system (t, r, θ, ϕ) with line element

$$ds^2 = -e^{A(t,r)} dt^2 + e^{B(t,r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2). \tag{3}$$

For the above space-time (3) “the Ricci tensor Segre type” is {1,1(11)} or {211} or one of its degeneracies. The above space-times (3) admit three linearly independent Killing vector fields which are

$$\cos\phi \frac{\partial}{\partial\theta} - \cot\theta \sin\phi \frac{\partial}{\partial\varphi}, \quad \sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial\phi}. \tag{4}$$

A vector field X is said to be a conformal vector field if it satisfies equation (2). One can write (2) explicitly using (3) as

$$A_t(t,r)X^0 + A_r(t,r)X^1 + 2X^0_{,0} = 2\psi, \tag{5}$$

$$e^{B(t,r)}X^1_{,0} - e^{A(t,r)}X^0_{,1} = 0, \tag{6}$$

$$r^2 X^2_{,0} - e^{A(t,r)}X^0_{,2} = 0, \tag{7}$$

$$r^2 \sin^2\theta X^3_{,0} - e^{A(t,r)}X^0_{,3} = 0, \tag{8}$$

$$B_t(t,r)X^0 + B_r(t,r)X^1 + 2X^1_{,1} = 2\psi, \tag{9}$$

$$r^2 X^2_{,1} + e^{B(t,r)}X^1_{,2} = 0, \tag{10}$$

$$r^2 \sin^2\theta X^3_{,1} + e^{B(t,r)}X^1_{,3} = 0, \tag{11}$$

$$\frac{1}{r}X^1 + X^2_{,2} = \psi, \tag{12}$$

$$\sin^2\theta X^3_{,2} + X^2_{,3} = 0, \tag{13}$$

$$\frac{1}{r}X^1 + \cot\theta X^2 + X^3_{,3} = \psi, \tag{14}$$

where $\psi = \psi(t,r,\theta,\phi)$. Considering equations (8) and (7) and differentiating with respect to ϕ and θ, respectively and subtracting them we get

$$-\left[\sin^2\theta X^3_{,0}\right]_{,2} + X^2_{,03} = 0. \tag{15}$$

Differentiating equation (13) with respect to t we get

$$\sin^2\theta X^3_{,02} + X^2_{,03} = 0. \tag{16}$$

Subtracting equation (15) from equation (16) and integrating we get

$$X^3 = \operatorname{cosec}\theta E^1(t,r,\phi) + E^2(r,\theta,\phi),$$

where $E^1(t,r,\phi)$ and $E^2(r,\theta,\phi)$ are functions of integration. Using the above information in equation (8) one has

$$X^0 = e^{-A(t,r)} r^2 \sin\theta \int E_t^1(t,r,\phi)\, d\phi + E^3(t,r,\theta),$$

where $E^3(t,r,\theta)$ is a function of integration. Substituting the value of X^0 in equations (7) and (6) we get

$$X^2 = \cos\theta \int E^1(t,r,\phi) d\phi + \frac{1}{r^2}\int e^{A(t,r)} E_\theta^3(t,r,\theta)\, dt + E^4(r,\theta,\phi),$$

$$X^1 = r^2 \sin\theta \int e^{-B(t,r)}\left[\int E_r^1(t,r,\phi) d\phi\right] dt + 2r\sin\theta \int e^{-B(t,r)}\left[\int E^1(t,r,\phi) d\phi\right] dt$$
$$- r^2 \sin\theta \int e^{-B(t,r)}\left[A_r \int E^1(t,r,\phi) dt\right] dt + \int e^{A(t,r)-B(t,r)} E_r^3(t,r,\theta) dt + E^5(r,\theta,\phi),$$

where $E^4(r,\theta,\phi)$ and $E^5(r,\theta,\phi)$ are functions of integration. Thus we have

$$X^0 = e^{-A(t,r)} r^2 \sin\theta \int E_t^1(t,r,\phi) d\phi + E^3(t,r,\theta),$$
$$X^1 = r^2 \sin\theta \int e^{-B(t,r)}\left[\int E_r^1(t,r,\phi) d\phi\right] dt + 2r\sin\theta \int e^{-B(t,r)}\left[\int E^1(t,r,\phi) d\phi\right] dt$$
$$- r^2 \sin\theta \int e^{-B(t,r)}\left[A_r \int E^1(t,r,\phi) dt\right] dt + \int e^{A(t,r)-B(t,r)} E_r^3(t,r,\theta) dt + E^5(r,\theta,\phi),$$
$$X^2 = \cos\theta \int E^1(t,r,\phi) d\phi + \frac{1}{r^2}\int e^{A(t,r)} E_\theta^3(t,r,\theta)\, dt + E^4(r,\theta,\phi),$$
$$X^3 = \mathrm{cosec}\theta \int E^1(t,r,\phi) dt + E^2(r,\theta,\phi). \qquad (17)$$

In order to determine $E^1(t,r,\phi), E^2(r,\theta,\phi), E^3(t,r,\theta), E^4(r,\theta,\phi)$ and $E^5(r,\theta,\phi)$ we need to integrate the remaining six equations. To avoid details, here we will present only results, when the above space-times (3) admit conformal vector fields. It follows after some tedious and lengthy calculations that the following possibilities exist when the above space-times (3) admit conformal vector fields, which are:

Case (1)

In this case we have $A(r) = 2\ln r + c_7$ and $B(t) = c_8 t + c_9$, where $c_7, c_8, c_9 \in R (c_8 \neq 0)$. The space-time (3) becomes

$$ds^2 = -e^{2\ln r + c_7} dt^2 + e^{c_8 t + c_9} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (18)$$

The conformal vector fields in this case are

$$X^0 = c_4,\ X^1 = r\left[c_5 \ln r + c_6\right]$$
$$X^2 = c_1 \cos\phi + c_2 \sin\phi,\ X^3 = \cot\theta\left[-c_1 \sin\phi + c_2 \cos\phi\right] + c_3, \tag{19}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in R(c_4 \neq 0, c_5 \neq 0)$ and $c_5 = -\frac{c_4 c_8}{2}$. The conformal factor is $\psi = c_5 \ln r + c_6$. One can write the above equation (19) after subtracting Killing vector fields as

$$X^0 = c_4,\ X^1 = r\left[c_5 \ln r + c_6\right],\ X^2 = X^3 = 0. \tag{20}$$

The above space-time (18) admits six independent conformal vector fields which are given in equation (19) in which three are proper conformal see equation (20) and three are independent Killing vector fields.

Case (2)

In this case we have $A(r) = 2\ln r + c_7$ and $B = N(r),$ where $c_7 \in R.$ The space-time (3) can, after a suitable rescaling of t be written in the form

$$ds^2 = -r^2\, dt^2 + e^{N(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2). \tag{21}$$

The conformal vector fields in this case are

$$X^0 = c_4,\ X^1 = c_5\, r\, e^{-\frac{1}{2}N(r)},\ X^2 = c_1 \cos\phi + c_2 \sin\phi,$$
$$X^3 = \cot\theta\left[-c_1 \sin\phi + c_2 \cos\phi\right] + c_3, \tag{22}$$

where $\psi = c_5\, e^{-\frac{1}{2}N(r)}$ and $c_1, c_2, c_3, c_4, c_5 \in R(c_5 \neq 0).$ One can write the above equation (22) after subtracting Killing vector fields as

$$X^0 = 0,\ X^1 = c_5\, r\, e^{-\frac{1}{2}N(r)},\ X^2 = X^3 = 0. \tag{23}$$

The above space-time (21) admits five independent conformal vector fields which are given in equation (22) in which one is proper conformal see equation (23) and four are independent Killing vector fields.

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