# THE EXISTENCE OF GLOBAL ATTRACTOR FOR A SIXTH ORDER PARABOLIC EQUATION

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*This paper is concerned with a sixth-order parabolic equation which arises naturally as a continuum model for the formation of quantum dots and their faceting. Based on the regularity estimates for the semigroups, iteration technique and the classical existence theorem of global attractors, we prove that the sixth order parabolic equation possesses a global attractor in the $H^k$ $(k \geq 0)$ space, which attracts any bounded subset of $H^k(\Omega)$ in the $H^k$-norm.*

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## 1. Introduction

In this paper, we investigate the sixth order nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = kD^6u + D^4A(u) + \nu uDu, \quad (x,t) \in \Omega \times (0,T), \tag{1}$$

where $\Omega = (0,1)$, $k > 0$, $D = \frac{\partial}{\partial x}$. From the physical consideration, we prefer to consider a typical case of the potential $F(u)$, that is $F'(u) = -A(u) = u - u^3$, in the following form [7]

$$(H1) \qquad F(u) = \frac{1}{4}(u^2-1)^2,$$

namely, the well-known double well potential.

The equation (1) is supplemented by the boundary conditions

$$u|_{x=0,1} = D^2u|_{x=0,1} = D^4u|_{x=0,1} = 0, \tag{2}$$

and the initial value condition

$$u(x,0) = u_0(x). \tag{3}$$

The equation (1) arises naturally as a continuum model for the formation of quantum dots and their faceting, see [12]. Here $u(x,t)$ denotes the surface slope, and $\nu$ is proportional to the deposition rate. The high order derivatives are the result of the additional regularization energy which is required to form an edge between two plane surfaces with different orientations.

During the past years, many authors have paid much attention to the other sixth order thin film equation, such as the existence, uniqueness and regularity of

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the solutions [5, 6, 9]. However, as far as we know, there are few investigations concerned with the equation (1). Korzec, Evans, Münch and Wagner [7] studied the equation (1). New types of stationary solutions of a one-dimensional driven sixth-order Cahn-Hilliard type equation (1) are derived by an extension of the method of matched asymptotic expansions that retains exponentially small terms.

The dynamic properties of the equation (1), such as the global asymptotical behaviors of solutions and existence of global attractors are important. During the past years, many authors have paid much attention to the attractors of Cahn-Hilliard equation or thin-film equation (see [1, 2, 3, 8, 17]). In this paper, we are interested in the existence of global attractors for the problem (1)-(3). The main difficulties for treating the problem (1)-(3) are caused by the nonlinearity of both the fourth order diffusive and the convective factors. The method used for treating Cahn-Hilliard equation seems not applicable to the present situation. We shall use the regularity estimates for the linear semigroups, combining with the iteration technique and the classical existence theorem of global attractors, to prove that the problem (1)-(3) possesses a global attractor in the $H^k$ $(k \geq 0)$ space.

This paper is organized as follows. In section 2, we give some preparations for our consideration. In section 3, we prove that problem (1)-(3) possesses global attractors on some affined subspace of $H^3$. Based on this result, we prove the existence of global attractors for problem (1)-(3) in the $H^k$ $(k \geq 0)$ space.

Throughout this paper we denote $L^2$, $L^p$ and $H^k$ norm in $\Omega$ simply by $\|.\|$, $\|.\|_p$ and $\|.\|_{H^k}$. The symbols $C$ and $C_i$ with $i = 0, 1, 2, \cdots$ will denote positive constants that may change from line to line even if in the same inequality.

We note the Gagliardo-Nirenberg inequality ([11])

$$\|u\|_q \leq C\|D^m u\|_p^b \|u\|_r^{1-b},$$

where

$$\frac{1}{q} = b\left(\frac{1}{p} - \frac{m}{n}\right) + (1-b)\frac{1}{r}.$$

## 2. **Preliminary**

Similar to the proof in [4], we have the following results on global existence and uniqueness of solution to problem (1)-(3).

**Lemma 2.1.** *Assume $u_0 \in H^3(\Omega)$. Then the problem (1)-(3) admits a unique solution $u$ such that*

$$u \in C([0,T]; L^2(\Omega)) \cap L^\infty([0,T]; H^3(\Omega)). \tag{4}$$

By Lemma 2.1, we can define the operator semigroup $\{S(t)\}_{t\geq 0}$ as

$$S(t)u_0 = u(t), \quad \forall u_0 \in H^3(\Omega), \;\; t \geq 0, \tag{5}$$

where $u(t)$ is the solution of (1)-(3) corresponding to initial value $u_0$. It's clearly that the operator semigroup $\{S(t)\}_{t\geq 0}$ is continuous.

The following Lemma 2.2 is the classical existence theorem of global attractor by R. Temam [16].

**Lemma 2.2.** *Assume that $S(t)$ is the semigroup generated by Eq.(1), and the following conditions hold:*

*(i) For any bounded set $A \subset L^2(\Omega)$, there exists a time $t_A \geq 0$ such that $S(t)u_0 \in B$, $\forall u_0 \in A$ and $t > t_A$;*

*(ii) For any bounded set $U \subset L^2(\Omega)$ and some $T > 0$ sufficiently large, the set $\overline{\bigcup_{t\geq T} S(t)u}$ is compact in $X$.*

*Then the $\omega$-limit set $\mathcal{A} = \omega(B)$ of $B$ is a global attractor of Eq.(1), and $\mathcal{A}$ is connected providing $B$ is connected.*

We give the following theorem

**Theorem 2.1.** *Assume $u_0 \in H^3(\Omega)$ and $k \geq \max\{4, \frac{|\nu|^2}{18\pi^2}\}$. Then the semiflow associated with the solution $u$ of the problem (1)-(3) possesses a global attractor $\mathscr{A}$ in the space $H^3(\Omega)$ which attracts all the bounded set in the space $H^3(\Omega)$.*

In order to consider the global attractor for Eq.(1) in the $H^k$ space, we introduce the define as follows

$$\begin{cases} H = \{u \in L^2(\Omega)\ \ u|_{\partial\Omega} = 0\}, \\ H_{\frac{1}{2}} = \{u \in H^3(\Omega) \cap H,\ \ u|_{\partial\Omega} = D^2u|_{\partial\Omega} = 0\}, \\ H_1 = \{u \in H^6(\Omega) \cap H,\ \ u|_{\partial\Omega} = D^2u|_{\partial\Omega} = D^4u|_{\partial\Omega} = 0\}. \end{cases} \tag{6}$$

In this paper, we let $g(u) = D^3(u - u^3) + \frac{\nu}{2}u^2$ be a nonlinear function and assume that the linear operator $L = kD^6 : H_1 \to H$ in (6) is a sectorial operator, which generates an analytic semigroup $e^{tL}$, and $L$ induces the fractional power operators and fractional order spaces as follows

$$\mathscr{L}^\alpha = (-L)^\alpha : H_\alpha \to H,\ \alpha \in \mathbb{R},$$

where $H_\alpha = D(\mathscr{L}^\alpha)$ is the domain of $\mathscr{L}^\alpha$. By the semigroup theory of linear operators, $H_\beta \subset H_\alpha$ is a compact inclusion for any $\beta > \alpha$. For details of the space $H_\alpha$ see [10].

The space $H_{\frac{1}{6}}$ is given by $H_{\frac{1}{6}}$ = the closure of $H_{\frac{1}{2}}$ in $H^1(\Omega)$ and $H_k = H^{6k} \cap H_1$ for $k \geq 1$.

Then, we have the following lemma on the existence of global attractor which is equivalent to Lemma 2.2 and can be found in [13, 14, 15].

**Lemma 2.3.** *Assume that $u(t, u_0) = S(t)u_0$ $(u_0 \in H, t \geq 0)$ is a solution of (1) and $S(t)$ is the semigroup generated by (1). Assume further that $H_\alpha$ is the fractional order space generated by $L$ and*

*(i) For some $\alpha \geq 0$, there is a bounded set $B \subset H_\alpha$, such that for any $u_0 \in H_\alpha$, there exists $t_{u_0} \geq 0$ such that*

$$u(t, u_0) \in B,\ \forall\, t > t_{u_0};$$

*(ii) There is a $\beta > \alpha$, such that for any bounded set $U \subset H_\beta$, there are $T > 0$ and $C > 0$, such that*

$$\|u(t, u_0)\|_{H_\beta} \leq C,\ \ \forall t > T \ \text{ and } \ u_0 \in U.$$

*Then (1) has a global attractor $\mathscr{A} \subset H_\alpha$ which attracts any bounded set of $H_\alpha$ in the $H_\alpha-$norm.*

For sectorial operators, we also have the following lemma which is important for this paper and can be founded in [13, 14, 15].

**Lemma 2.4.** *Assume that $L$ is a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues $\lambda$ of $L$ satisfy $Re\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathscr{L}^\alpha$ ($\mathscr{L} = -L$) we have*

*(i) $T(t) : H \to H_\alpha$ is bounded for all $\alpha \in \mathbb{R}$ and $t > 0$;*

*(ii) $T(t)\mathscr{L}^\alpha x = \mathscr{L}T(t)x$, $\forall x \in H_\alpha$;*

*(iii) For each $t > 0$, $\mathscr{L}^\alpha T(t) : H \to H$ is bounded, and*

$$\|\mathscr{L}^\alpha T(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t},$$

*where some $\delta > 0$ and $C_\alpha > 0$ is a constant depending only on $\alpha$;*

*(iv) The $H_\alpha$−norm can be defined by $\|x\|_{H_\alpha} = \|\mathscr{L}^\alpha x\|_H$.*

The main result of this paper is given by the following theorem, which provides the existence of global attractors of Eq.(1) in any $k$th space $H^k$.

**Theorem 2.2.** *Assume $u_0 \in H^3(\Omega)$ and $k \geq \max\{4, \frac{|\nu|^2}{18\pi^2}\}$. Then the semiflow associated with the solution $u$ of the problem (1)-(3) possesses a global attractor $\mathscr{A}$ in the space $H^k$ which attracts all the bounded set of $H^k$ in the $H^k$-norm.*

## 3. Proofs of main results

In this section, we prove Theorem 2.1 and Theorem 2.2.

In order to prove Theorem 2.1, we establish some a priori estimates for the solution $u$ of problem (1)-(3). We always assume that $\{S(t)\}_{t\geq 0}$ is the semigroup generated by the weak solutions of equation (1) with initial data $u_0 \in H^3(\Omega)$. Then, the following lemma can be obtained.

**Lemma 3.1.** *There exists a bounded set $\mathscr{B}$ whose size depends only on $\Omega$, such that for all the orbits staring from any bounded set $B$ in $H^3(\Omega)$, $\exists t_0 = t_0(B) \geq 0$ s.t. $\forall t \geq t_0$ all the orbits will stay in $\mathscr{B}$.*

*Proof.* It suffices to prove that there is a positive constant $C$ such that for large $t$, there holds

$$\|u(t)\|_{H^3} \leq C.$$

We prove the lemma in the following steps.

Step 1. Let $z = kD^2u + A(u)$. Multiplying both sides of the equation (1) by $z$ and then integrating the resulting relation with respect to $x$ over $\Omega$, we have

$$\int_0^1 \frac{\partial u}{\partial t}(kD^2u + A(u))dx - \int_0^1 D^4 z z dx - \int_0^1 \frac{\nu}{2} Du^2 z dx = 0.$$

After integrating by parts, and using the boundary value conditions,

$$\frac{d}{dt}\int_0^1 \left(\frac{k}{2}(Du)^2 + F(u)\right) dx + \int_0^1 |D^2 z|^2 dx - \frac{\nu}{2}\int_0^1 u^2 Dz dx = 0,$$

using Hölder's inequality, we have

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \left(\frac{k}{2}(Du)^2 + F(u)\right) dx + \int_0^1 |D^2 z|^2 dx \\ \leq\ & \frac{|\nu|^2}{8}\int_0^1 u^4 dx + \frac{1}{2}\int_0^1 (Dz)^2 dx. \end{aligned}$$

Applying Poincaré's inequality and Friedrichs' inequality [4], we conclude

$$\int_0^1 |z|^2 dx \le \frac{1}{\pi}\int_0^1 |Dz|^2 dx \le \frac{1}{2\pi}\int_0^1 |D^2 z|^2 dx.$$

On the other hand, we have

$$\begin{aligned}
&\int_0^1 |z|^2 dx \\
= &\int_0^1 k^2 (D^2u)^2 dx + \int_0^1 (u-u^3)^2 dx + 2k\int_0^1 D^2u(u-u^3)dx \\
= &\int_0^1 k^2 (D^2u)^2 dx + \int_0^1 (u-u^3)^2 dx - 2k\int_0^1 Du(Du-3u^2Du)dx \\
= &\int_0^1 k^2 (D^2u)^2 dx + \int_0^1 (u-u^3)^2 dx - 2k\int_0^1 (Du)^2 dx + 6k\int_0^1 u^2(Du)^2 dx \\
\ge &\int_0^1 k^2 (D^2u)^2 dx + \int_0^1 (u-u^3)^2 dx - 2k\int_0^1 (D^2u)^2 dx + 6k\int_0^1 u^2(Du)^2 dx,
\end{aligned}$$

hence as $k \ge 4$, we have

$$\int_0^1 |z|^2 dx \ge \frac{k^2}{2}\int_0^1 (D^2u)^2 dx + \int_0^1 (u-u^3)^2 dx + 6k\int_0^1 u^2(Du)^2 dx.$$

Again by Poincaré's inequality, we obtain

$$\int_0^1 u^4 dx \le \frac{1}{\pi}\int_0^1 (Du^2)^2 dx = \frac{4}{\pi}\int_0^1 u^2(Du)^2 dx.$$

Owning to the above inequality, we finally arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \left(\frac{k}{2}(Du)^2 + F(u)\right) dx + \frac{3\pi k^2}{4}\int_0^1 (D^2u)^2 dx \\
&\quad + \frac{3\pi}{2}\int_0^1 (u-u^3)^2 dx + \left(9\pi k - \frac{|\nu|^2}{2\pi}\right)\int_0^1 u^2(Du)^2 dx \le 0.
\end{aligned}$$

Hence, as $k \ge \max\{4, \frac{|\nu|^2}{18\pi^2}\}$, we get

$$\frac{d}{dt}\int_0^1 \left(\frac{k}{2}(Du)^2 + F(u)\right) dx + C_1\int_0^1 \left(\frac{k}{2}(Du)^2 + F(u)\right) dx \le C_2.$$

Therefore,

$$\int_0^1 \left(\frac{k}{2}(Du)^2 + F(u)\right) dx \le e^{-C_1 t}\int_0^1 \left(\frac{k}{2}(Du_0)^2 + F(u_0)\right) dx + \frac{C_2}{C_1}. \tag{7}$$

Thus, for initial data in any bounded set $B \subset H^3(\Omega)$, there is a uniform time $t_1(B)$ depending on $B$ such that for $t \ge t_1(B) \ge 0$,

$$\|u(x,t)\|^2 \le \frac{2C_2}{kC_1}, \tag{8}$$

$$\|Du\|^2 \le \frac{2C_2}{kC_1}. \tag{9}$$

By the Sobolev imbedding theorem,

$$\|u\|_\infty \le C. \tag{10}$$

Step 2. Multiplying (1) with $D^4u$, and integrating it over $\Omega$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|D^2u\|^2 + k\|D^5u\|^2 = \int_0^1 (D^4u)^2dx - \int_0^1 D^4u^3D^4udx + \frac{\nu}{2}\int_0^1 u^2D^5udx.$$

Hence,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|D^2u\|^2 + k\|D^5u\|^2 \\ = &\int_0^1 (D^4u)^2dx + \int_0^1 (1-3u^2)D^3uD^5udx - 18\int_0^1 uDuD^2uD^5udx \\ &-6\int_0^1 (Du)^3D^5udx + \frac{\nu}{2}\int_0^1 u^2D^5udx. \end{aligned}$$

Using (10) and the Hölder inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|D^2u\|^2 + k\|D^5u\|^2 \\ \le &\frac{k}{2}\int_0^1 (D^5u)^2dx + C\int_0^1 (D^3u)^2dx + C\int_0^1 |DuD^2u|^2dx \\ &+C\int_0^1 (Du)^6dx + C \\ \le &\frac{k}{2}\int_0^1 (D^5u)^2dx + C\int_0^1 (D^3u)^2dx + C\int_0^1 |Du|^4dx \\ &+C\int_0^1 |D^2u|^4dx + C\int_0^1 (Du)^6dx + C. \end{aligned}$$

By (9) and the Hölder inequality, we see that

$$\int_0^1 (D^3u)^2dx = \int_0^1 DuD^5udx \le C\|D^5u\|.$$

On the other hand, using the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \int_0^1 (Du)^4dx &\le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{1}{8}}\left(\int_0^1 (Du)^2dx\right)^{\frac{15}{8}} \le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{1}{8}}, \\ \int_0^1 (Du)^6dx &\le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{1}{4}}\left(\int_0^1 (Du)^2dx\right)^{\frac{11}{4}} \le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{1}{4}}, \\ \int_0^1 (D^2u)^4dx &\le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{5}{8}}\left(\int_0^1 (Du)^2dx\right)^{\frac{11}{8}} \le C\left(\int_0^1 (D^5u)^2dx\right)^{\frac{5}{8}}. \end{aligned}$$

Hence, we obtain

$$\frac{d}{dt}\|D^2u\|^2 + C\|D^5u\|^2 \le C. \tag{11}$$

Applying Poincaré's inequality, we obtain

$$\frac{d}{dt}\|D^2u\|^2 + C_3\|D^2u\|^2 \le C_4, \tag{12}$$

which gives

$$\|D^2u\|^2 \le e^{-C_3t}\|D^2u_0\|^2 + \frac{C_4}{C_3}. \tag{13}$$

Thus, for initial data in any bounded set $B \subset \mathscr{U}_\kappa$, there is a uniform time $t_2(B)$ depending on $B$ such that for $t \ge t_2(B)$,

$$\|D^2u(x,t)\|^2 \le 2\frac{C_4}{C_3}. \tag{14}$$

Step 3. Multiplying (1) with $D^6u$, and integrating it over $\Omega$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|D^3u\|^2 + k\|D^6u\|^2 = \int_0^1 (3u^2-1)D^4uD^6udx + 24\int_0^1 uDuD^3uD^6u$$
$$+36\int_0^1 (Du)^2D^2uD^6udx + 18\int_0^1 u(D^2u)^2D^6udx - \nu\int_0^1 uDuD^6udx.$$

Using (10), (14) and the Hölder inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|D^3u\|^2 + k\|D^6u\|^2\\ \le\ &\frac{k}{2}\int_0^1 (D^6u)^2dx + C\int_0^1 (D^4u)^2dx + C\int_0^1 (D^3u)^2dx\\ &+C\int_0^1 (D^2u)^4dx + C.\end{aligned}$$

By (14) and the Hölder inequality, we see that

$$\int_0^1 (D^4u)^2dx = \int_0^1 D^2uD^6udx \le C\|D^6u\|.$$

On the other hand, using the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\int_0^1 (D^2u)^4dx &\le C\left(\int_0^1 (D^6u)^2dx\right)^{\frac{5}{8}}\left(\int_0^1 (D^2u)^2dx\right)^{\frac{11}{8}}\\ &\le C\left(\int_0^1 (D^6u)^2dx\right)^{\frac{5}{8}}.\end{aligned}$$

Hence, we obtain

$$\frac{d}{dt}\|D^3u\|^2 + C\|D^6u\|^2 \le C.$$

Applying Poincaré's inequality, we obtain

$$\frac{d}{dt}\|D^3u\|^2 + C_5\|D^3u\|^2 \le C_6, \tag{15}$$

which gives

$$\|D^3u\|^2 \le e^{-C_5t}\|D^3u_0\|^2 + \frac{C_6}{C_5}, \tag{16}$$

for $t \ge t_3(B)$.

Adding (8), (9), (14) and (16) together, we obtain

$$\|u(t)\|_{H^3} \le C.$$

Let $t_0(B) = \max\{t_1(B), t_2(B), t_3(B)\}$, then the lemma is proved. □

The above lemma implies that $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $H^3(\Omega)$. In what follows we prove the precompactness of the orbit in $H^3(\Omega)$.

**Lemma 3.2.** *For any initial data $u_0$ in any bounded set $B \subset H^3(\Omega)$, there is a $T(B) > 0$ such that*

$$\|u(t)\|_{H^4} \leq C, \quad \forall t \geq T > 0,$$

*which turns out that $\bigcup_{t\geq T} u(t)$ is relatively compact in $H^3(\Omega)$.*

*Proof.* The uniform boundedness of $H^3(\Omega)$ norm of $u(t)$ has been obtained in Lemma 3.1. In what follows we derive the estimate on $H^4$-norm.

From the equation (1) and boundary condition (2), it follows

$$kD^6u + D^4(u - u^3) + \nu u Du\Big|_{x=0,1} = 0.$$

It can be replaced by

$$D^6u\Big|_{x=0,1} = 0.$$

Multiplying (1) by $D^8u$ and integrating on $\Omega$, using the boundary conditions, we obtain

$$\frac{1}{2}\frac{d}{dt}\|D^4u\|^2 + k\|D^7u\|^2 = -\int_0^1 D^5(u-u^3)D^7u dx - \frac{\nu}{2}\int_0^1 D^2u^2D^7u dx.$$

Using (8), (9), (14), (15) and the Hölder inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|D^4u\|^2 + k\|D^7u\|^2 \\ \leq\ & \frac{k}{2}\int_0^1 (D^7u)^2dx + C\int_0^1 (D^5u)^2dx + C\int_0^1 |D^4u|^2dx \\ &+C\int_0^1 (D^3u)^2dx + C\int_0^1 (D^2u)^4dx + C. \end{aligned}$$

Similar to above, using the Gagliardo-Nirenberg inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|D^4u\|^2 + C_7\|D^7u\|^2 \leq C_8. \tag{17}$$

On the other hand, integrating (11) between $t$ and $t+1$, using (14), we have

$$\int_t^{t+1} \|D^5u\|^2d\tau \leq \|D^2u(t)\|^2 + C \leq C.$$

Hence, by Poincaré's inequality, we obtain

$$\int_t^{t+1} \|D^4u\|^2d\tau \leq \int_t^{t+1} \|D^5u\|^2d\tau \leq C. \tag{18}$$

Owning to (17), (18) and the uniform Gronwall inequality in [16], we get that

$$\|D^4u\|^2 \leq C,\ t \geq 1.$$

The lemma is proved. □

*Proof of Theorem 2.1.* By Lemma 3.1, Lemma 3.2 and Theorem I.1.1 in [16], we immediately conclude that $\mathscr{A}_\kappa = \omega(\mathscr{B})$, the $\omega$-limit set of absorbing set $\mathscr{B}$ is a global attractor in $H^3(\Omega)$. By lemma 3.2, this global attractor is a bounded set in $H^3(\Omega)$. Thus the proof of Theorem 2.1 is complete. □

By the a priori estimates of $u$, we obtain the following corollary:

**Corollary 3.1.** *Assume* $u_0 \in H^3(\Omega)$. *Then we have*

$$\|u(t)\|_\infty \le C, \quad \|Du(t)\|_\infty \le C, \quad \|D^2u(t)\|_\infty \le C. \tag{19}$$

Now, we will give the proof of the main result.

Based on [10], it's well known that the solution $u(t, u_0)$ of the problem (1)-(3) can be written as

$$u(t, u_0) = e^{tL}u_0 + \int_0^t e^{(t-\tau)L}G(u)d\tau, \tag{20}$$

where $L = kD^6$ and $G(u) = Dg(u) = D^4(u - u^3) + \nu uDu$. Then, (20) means

$$\begin{aligned} u(t, u_0) &= e^{tL}u_0 + \int_0^t e^{(t-\tau)L}Dg(u)d\tau \\ &= e^{tL}u_0 + \int_0^t (-L)^{\frac{1}{6}}e^{(t-\tau)L}g(u)d\tau. \end{aligned} \tag{21}$$

By Lemma 2.3, in order to prove Theorem 2.2, we first prove the following lemma.

**Lemma 3.3.** *Assume* $k \ge \max\{4, \frac{|\nu|^2}{18\pi^2}\}$. *Then for any bounded set* $U \subset H_\alpha$, *there exists* $C > 0$ *such that*

$$\|u(t, u_0)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ u_0 \in U \subset H_\alpha,\ \alpha \ge 0. \tag{22}$$

*Proof.* For $\alpha = \frac{1}{2}$, this follows from Theorem 2.1, i.e. for any bounded set $U \subset H_{\frac{1}{2}}$ there is a constant $C > 0$ such that

$$\|u(t, u_0)\|_{H_{\frac{1}{2}}} \le C, \quad \forall t \ge 0,\ u_0 \in U \subset H_{\frac{1}{2}}. \tag{23}$$

Then, we only need to prove (22) for any $\alpha \ge \frac{1}{2}$. there are four steps for us to prove it.

Step 1. We prove that for any bounded set $U \subset H_\alpha$ ($\frac{1}{2} \le \alpha < \frac{5}{6}$), there exists a constant $C > 0$ such that

$$\|u(t, u_0)\|_{H_\alpha} \le C, \quad \forall t \ge 0, u_0 \in U, \frac{1}{2} \le \alpha < \frac{5}{6}. \tag{24}$$

In fact, by Lemma 3.2 and (21), we obtain

$$\begin{aligned} &\|u(t, u_0)\|_{H_\alpha} \\ = \ &\|e^{tL}u_0 + \int_0^t (-L)^{\frac{1}{6}}e^{(t-\tau)L}g(u)d\tau\|_{H_\alpha} \\ \le \ &C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\frac{1}{6}+\alpha}e^{(t-\tau)L}g(u)\|_H d\tau \\ \le \ &C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\frac{1}{6}+\alpha}e^{(t-\tau)L}\| \cdot \|g(u)\|_H d\tau. \end{aligned} \tag{25}$$

We claim that $g : H_{\frac{1}{2}} \to H$ is bounded. Based on Corollary 3.1 and the embedding theorem, we obtain

$$\begin{aligned} \|g(u)\|_H^2 = \int_\Omega |g(u)|^2 dx = \int_\Omega \left(D^3(u-u^3) + \frac{\nu}{2}u^2\right)^2 dx \\ \leq \ C\|u\|_{H_{\frac{1}{2}}}^2, \end{aligned} \tag{26}$$

which means that $g : H_{\frac{1}{2}} \to H$ is bounded.

Hence, it follows from (23), (25) and (26) that

$$\begin{aligned} \|u(t,u_0)\|_{H_\alpha} &\leq C\|u_0\|_{H_\alpha} + C\int_0^t \tau^{-\beta} e^{-\delta\tau} d\tau \\ &\leq C, \quad \forall t \geq 0, \ u_0 \in U \subset H_\alpha, \end{aligned} \tag{27}$$

where $\beta = \frac{1}{6} + \alpha$, $(0 < \beta < 1)$. Then (24) is proved.

Step 2. We prove that for any bounded set $U \subset H_\alpha$ $(\frac{5}{6} \leq \alpha < 1)$, there exists a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\alpha} \leq C, \quad \forall t \geq 0, u_0 \in U, \ \frac{5}{6} \leq \alpha < 1. \tag{28}$$

In fact, by Lemma 3.2 and (21), we obtain

$$\begin{aligned} &\|u(t,u_0)\|_{H_\alpha} \\ = \ &\|e^{tL}u_0 + \int_0^t (-L)^{\frac{1}{6}} e^{(t-\tau)L} g(u) d\tau\|_{H_\alpha} \\ \leq \ &C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\frac{1}{6}+\alpha} e^{(t-\tau)L} g(u)\|_H d\tau \\ \leq \ &C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\alpha} e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{\frac{1}{6}}} d\tau. \end{aligned} \tag{29}$$

We claim that $g : H_\alpha \to H_{\frac{1}{6}}$ is bounded. By the embedding theorem, we have

$$H_\alpha \hookrightarrow W^{4,2}, \ H_\alpha \hookrightarrow W^{2,4},$$

where $\frac{1}{2} \leq \alpha < \frac{5}{6}$.

Then, we obtain

$$\begin{aligned} \|g(u)\|_{H_{\frac{1}{6}}}^2 = \int_\Omega |Dg(u)|^2 dx = \int_\Omega \left(D^4(u-u^3) + \nu u D u)\right)^2 dx \\ \leq \ C\int_\Omega (D^4u)^2 dx + C\int_\Omega (D^2u)^4 dx \\ \leq \ C(\|u\|_{H_\alpha}^2 + \|u\|_{H^\alpha}^4), \end{aligned} \tag{30}$$

which means that $g : H_\alpha \to H_{\frac{1}{6}}$ is bounded.

Hence, it follows from (23), (29) and (30) that

$$\begin{aligned} \|u(t,u_0)\|_{H_\alpha} &\leq C\|u_0\|_{H_\alpha} + C\int_0^t \tau^{-\alpha} e^{-\delta\tau} d\tau \\ &\leq C, \quad \forall t \geq 0, \ u_0 \in U \subset H_\alpha. \end{aligned} \tag{31}$$

Then, (28) is proved.

Step 3. We prove that for any bounded set $U \subset H_\alpha$ $(1 \leq \alpha < \frac{7}{6})$, there exists a constant $C > 0$, such that

$$\|u(t,u_0)\|_{H_\alpha} \leq C, \quad \forall t \geq 0,\ u_0 \in U \subset H_\alpha,\ 1 \leq \alpha < \frac{7}{6}. \tag{32}$$

In fact, by Lemma 3.2 and (20), we obtain

$$\begin{aligned}
&\|u(t,u_0)\|_{H_\alpha} \\
= \ & \|e^{tL}u_0 + \int_0^t (-L)^{\frac{1}{6}} e^{(t-\tau)L} g(u) d\tau\|_{H_\alpha} \\
\leq \ & C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\frac{1}{6}+\alpha} e^{(t-\tau)L} g(u)\|_H d\tau \\
\leq \ & C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\alpha-\frac{1}{6}} e^{(t-\tau)L}\| \cdot \|g(u)\|_{H_{\frac{1}{3}}} d\tau.
\end{aligned} \tag{33}$$

We claim that $g : H_\alpha \to H_{\frac{1}{3}}$ is bounded for $1 \leq \alpha < \frac{7}{6}$. Based on the embedding theorem, we have

$$H_\alpha \hookrightarrow W^{5,2}(\Omega), \quad H_\alpha \hookrightarrow W^{3,4}(\Omega),$$

where $\frac{5}{6} \leq \alpha < 1$.

By Corollary 3.1, we obtain

$$\begin{aligned}
&\|g(u)\|^2_{H_{\frac{1}{3}}} \\
= \ & \int_\Omega |D\left(D^4(u-u^3) + \nu u D u\right)|^2 dx \\
\leq \ & C\int_\Omega \left(|1-3u^2||D^5u| + |u||Du||D^4u| + |u||D^2u||D^3u| + |Du||D^2u|^2\right)^2 dx \\
\leq \ & C(\|D^5u\|^2 + \|D^3u\|_4^4) \\
\leq \ & C(\|u\|^4_{H_\alpha} + \|u\|^2_{H_\alpha}),
\end{aligned} \tag{34}$$

which means that $g : H_\alpha \to H_{\frac{1}{3}}$ is bounded.

Hence, it follows from (33) and (34) that

$$\begin{aligned}
\|u(t,u_0)\|_{H_\alpha} &\leq C\|u_0\|_{H_\alpha} + C\int_0^t \tau^{-\beta} e^{-\delta\tau} d\tau \\
&\leq C, \quad \forall t \geq 0,\ u_0 \in U \subset H_\alpha,
\end{aligned} \tag{35}$$

where $\beta = \alpha - \frac{1}{6}$, $(0 < \beta < 1)$. Then (32) is proved.

Step 4. We prove that for any bounded set $U \subset H_\alpha$ $(\frac{7}{6} \leq \alpha < \frac{4}{3})$, there exists a constant $C > 0$, such that

$$\|u(t,u_0)\|_{H_\alpha} \leq C, \quad \forall t \geq 0,\ u_0 \in U \subset H_\alpha,\ \frac{7}{6} \leq \alpha < \frac{4}{3}. \tag{36}$$

In fact, by Lemma 3.2 and (20), we obtain

$$\begin{aligned}
&\|u(t,u_0)\|_{H_\alpha}\\
=\ & \|e^{tL}u_0+\int_0^t(-L)^{\frac{1}{6}}e^{(t-\tau)L}g(u)d\tau\|_{H_\alpha}\\
\le\ & C\|u_0\|_{H_\alpha}+\int_0^t\|(-L)^{\frac{1}{6}+\alpha}e^{(t-\tau)L}g(u)\|_H d\tau\\
\le\ & C\|u_0\|_{H_\alpha}+\int_0^t\|(-L)^{\alpha-\frac{1}{3}}e^{(t-\tau)L}\|\cdot\|g(u)\|_{H_{\frac{1}{2}}}d\tau. \qquad (37)
\end{aligned}$$

We claim that $g: H_\alpha \to H_{\frac{1}{2}}$ is bounded for $1 \le \alpha < \frac{7}{6}$. Based on the embedding theorem, we have

$$H_\alpha \hookrightarrow W^{6,2}(\Omega), \quad H_\alpha \hookrightarrow W^{3,4}(\Omega), \quad H_\alpha \hookrightarrow W^{2,6}(\Omega),$$

where $1 \le \alpha < \frac{7}{6}$.

By Lemma 3.2 and Corollary 3.1, we obtain

$$\begin{aligned}
&\|g(u)\|^2_{H_{\frac{1}{2}}}\\
=\ & \int_\Omega |D^2[D^4(u-u^3)+\nu uDu)]|^2dx\\
\le\ & C\int_\Omega(|D^6u|+|u||Du||D^5u|+|u||D^2u||D^4u|+|u||D^3u|^2\\
&+|Du|^2D^4u+|Du||D^2u||D^3u|+|D^2u|^3)^2dx\\
\le\ & C\int_\Omega(|D^6u|^2+|D^3u|^4+|D^2u|^6)dx\\
\le\ & C(\|u\|^2_{H_\alpha}+\|u\|^4_{H_\alpha}+\|u\|^6_{H_\alpha}), \qquad (38)
\end{aligned}$$

which means that $g: H_\alpha \to H_{\frac{3}{4}}$ is bounded.

Hence, it follows from (37) and (38) that

$$\begin{aligned}
\|u(t,u_0)\|_{H_\alpha} &\le C\|u_0\|_{H_\alpha}+C\int_0^t\tau^{-\beta}e^{-\delta\tau}d\tau\\
&\le C, \quad \forall t\ge 0,\ u_0\in U\subset H_\alpha, \qquad (39)
\end{aligned}$$

where $\beta=\alpha-\frac{1}{3}$, $(0<\beta<1)$. Then (36) is proved.

In the same method as in the proof of (36), by iteration we can prove that for any bounded set $U \subset H_\alpha(\alpha > 0)$ there exists a constant $C > 0$ such that (22) holds, i.e. for all $\alpha \ge 0$ the semigroup $S(t)$ generated by problem (1)-(3) is uniformly compact in $H_\alpha$. The lemma is proved. □

**Lemma 3.4.** *Assume $k \ge \max\{4, \frac{|\nu|^2}{18\pi^2}\}$. Then for any bounded set $U \subset H_\alpha$ $(\alpha \ge 0)$ there exists $T > 0$ and a constant $C > 0$ independent of $u_0$, such that*

$$\|u(t,u_0)\|_{H_\alpha} \le C, \quad \forall t \ge T, u_0 \in U \subset H_\alpha. \qquad (40)$$

*Proof.* For $\alpha = \frac{1}{2}$, this follows from Theorem 2.1. Then, we prove (40) for any $\alpha > \frac{1}{2}$. We prove the lemma in the following steps:

Step 1. We prove that for any $\frac{1}{2} \le \alpha < \frac{5}{6}$, the problem (1)-(3) has a bounded absorbing set in $H_\alpha$.

By (21), we have

$$u(t,u_0)=e^{(t-T)L}u(T,u_0)+\int_T^t(-L)^{\frac{1}{6}}e^{(t-T)L}g(u)d\tau. \tag{41}$$

Assume $B$ is the bounded absorbing set of the problem (1)-(3) and $B$ satisfies $B\subset H_{\frac{1}{2}}$. In addition, we also assume the time $t_0>0$ such that

$$u(t,u_0)\in B,\quad \forall t>t_0, u_0\in U\subset H_\alpha, \alpha\geq\frac{1}{2}.$$

Note that

$$\|e^{tL}\|\leq Ce^{-\lambda_1^3 t},$$

where $\lambda_1>0$ is the first eigenvalue of the equation

$$\begin{cases} -\Delta u &= \lambda u,\\ \frac{\partial u}{\partial n} &= 0. \end{cases} \tag{42}$$

Then for any given $T>0$ and $u_0\in U\subset H_\alpha(\alpha\geq\frac{1}{2})$, we can obtain

$$\lim_{t\to\infty}\|e^{(t-T)L}u(T,u_0)\|_{H_\alpha}=0. \tag{43}$$

Adding (30) and (41) together, by Lemma 3.2, we get

$$\begin{aligned}
&\|u(t,u_0)\|_{H_\alpha}\\
\leq\ &\|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha}+\int_{t_0}^t\|(-L)^{\frac{1}{6}+\alpha}e^{(t-T)L}\|\cdot\|g(u)\|_H d\tau\\
\leq\ &\|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha}+C\int_{t_0}^t\|(-L)^{\frac{1}{6}+\alpha}e^{(t-T)L}\|d\tau\\
\leq\ &\|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha}+C\int_0^{T-t_0}\tau^{-\frac{1}{6}-\alpha}e^{-\delta\tau}d\tau\\
\leq\ &\|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha}+C,
\end{aligned} \tag{44}$$

where $C>0$ is a constant independent of $u_0$. Then by (43) and (44), we have that (40) holds for all $\frac{1}{2}\leq\alpha<\frac{5}{6}$.

Step 2. We can use the same method as the above step to prove that for any $\frac{5}{6}<\alpha<1$ and for any $1<\alpha<\frac{7}{6}$, the problem (1)-(3) has a bounded absorbing set in $H_\alpha$. By the iteration method, we can obtain that (40) holds for all $\alpha\geq\frac{1}{2}$. □

Now, we give the proof of Theorem 2.2.

*Proof of Theorem 2.2.* Combining Lemma 3.3 with Lemma 3.4, we have completed the proof of Theorem 2.2. □

## 4. **Conclusions**

The dynamic properties of the higher order equation, such as the global asymptotical behaviors of solutions and existence of global attractors are important. In this paper, we investigate the sixth order nonlinear parabolic equation which arises naturally as a continuum model for the formation of quantum dots and their faceting. The main difficulties for treating the problem are caused by the nonlinearity of both the fourth order diffusive and the convective factors. The method used for treating Cahn-Hilliard equation seems not applicable to the present situation. Based on the regularity estimates for the semigroups, iteration technique and the classical existence theorem of global attractors, we prove that the sixth order parabolic equation possesses a global attractor in the $H^k$ $(k \geq 0)$ space.

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