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FIDUCIAL INFERENCE: AN APPROACH BASED ON BOOTSTRAP TECHNIQUES

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În prima parte a acestei lucrări sunt prezentate conceptele de bază ale inferenței fiduciale. La aplicarea acestui principiu inferențial apar dificultăți când distribuția variabilei pivotale folosite nu este cunoscută. În partea a doua este propusă o soluție pentru această problemă constând în folosirea de metode bootstrap.

The basic concepts of fiducial inference are presented in the first part of this paper. Problems arise if the form of the distribution of the pivot used by the fiducial argument is not known. Our main result consists in showing how bootstrap techniques can be used to handle this kind of situation.

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1. Introduction

Commenting on Fisher's work, [3] stated at the beginning of his paper:

> 'The history of fiducial probability dates back over thirty years and so is long by statistical standards; however, thirty years have not proved long enough for agreement to be reached among statisticians as to the derivation, manipulation and interpretation of fiducial probability'.

Forty years later, the situation is very much the same. What is actually the fiducial theory? The fiducial model designed by Fisher leads us to objective probability statements about values of unknown parameters solely on the basis of given data, without resorting to any prior distribution. The idea that the initially assumed randomness may be preserved was one of Fisher's great contributions to the theory of statistical estimation; the lack of acceptance arose from the fact that

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Fisher failed to formulate clearly and completely the framework in which it can be applied.

The main contribution of this paper is to show how computer-intensive resampling methods can circumvent some of the difficulties associated with fiducial inference, aiming to extend the applicability of the fiducial method in practical situations.

This paper is organized as follows. First, fiducial inference is presented with the help of two illustrative examples; some of its properties and characteristics are summarized. Section 3 provides a proposal for enlarging the area of applications of the fiducial argument, based on bootstrap techniques. The final section concludes.

2. Fiducial inference

Methods of measuring uncertainty of parameters in statistical models include use of Bayes theorem, likelihood-inference as well as confidence and significance intervals. Although statisticians have a variety of procedures, none of these covers all the possible situations and that is why fiducial inference deserves a place among them.

It is generally agreed that statistical inference should rely on the Bayes theorem, if the statistical model includes a valid prior distribution. But a prior distribution is not always obtainable and the use of uninformative priors can lead to many inconsistencies and contradictions. If there is a prior, it may be empirical, as in technical applications, or of subjective nature, as possibly in economic problems. Nevertheless, this subjectivity can cause some loss of accuracy in the results.

If the prior distribution of the Bayesian methodology is not available, statistical inference can use the likelihood function, which expresses, for example, the plausibility of parameter θ_1 against parameter θ_2 provided by the sample. Because the likelihood approach is a relatively weak method of expressing uncertainty, it is used only when other methods of inference are not available.

The confidence interval is known to be a random interval, which covers the fixed but unknown parameter at a confidence level $1-\alpha$ with a probability of $(1-\alpha)\cdot 100\%$ in repeated sampling. It is universally applicable, but has the disadvantage that the probability statement rests on repeated sampling, while actually only one sample is available. Intervals constructed by this method include the true value in the mean, but possibly not in an actual case, based on a given sample. Consequently, the probability statement is associated to a procedure concerning the whole sample space and not to a random variable, as it may be desired.

With the introduction of fiducial probability, Fisher paved the way for objective probability statements related to the unknown real-valued parameter θ without relying on prior distributions, but based exclusively on sample data and conditional on the assumed statistical model. Since parameters are viewed as constants, fiducial probability is not a frequentistic probability, but a probability statement that mirrors the uncertainty of the parameter due to the randomness of the sample.

The one-dimensional case, in which the fiducial distribution is known to be unique, is obviously the simplest situation to which the fiducial method can be applied, see [3] and his sources.

Example 1 (adapted from [9]). *Let $X_1, X_2, ..., X_K$ be a random sample from the population X, $X \sim N\left(m, \sigma^2\right)$, with m und σ unknown. Then, the two-sided fiducial interval can be derived in the following way.*
First, the pivot T is considered:

$$T = \frac{\overline{X} - m}{S}\sqrt{K} \sim t_{K-1},$$

so that:

$$P\left(t_{K-1}^{\frac{\alpha}{2}} \le \frac{\overline{X} - m}{\frac{S}{\sqrt{K}}} \le t_{K-1}^{1-\frac{\alpha}{2}}\right) = 1 - \alpha.$$

The fiducial argument states that, given an observed sample (here, with mean $\overline{x}$ and standard deviation s), this probability statement can be transferred onto the parameter m itself. $\overline{X}$ is a sufficient statistic for the unknown parameter m, and, based on the distribution of $\overline{X}$, a proper fiducial interval for the parameter m can be obtained as follows:

$$P\left(\overline{x} - t_{K-1}^{1-\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{K}} \le M \le \overline{x} - t_{K-1}^{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{K}}\right) = 1 - \alpha.$$

Here, M is a random variable defined on a suitable probability space which captures the uncertainty associated with the parameter m in light of the given sample. A similar result occurs for a one-sided interval:

$$P\left(M \le \overline{x} - t_{K-1}^{\alpha} \cdot \frac{s}{\sqrt{K}}\right) = 1 - \alpha.$$

Hence, the fiducial distribution of m is Student's t distribution with $K-1$ degrees of freedom, scaled by $s/\sqrt{K}$ and shifted by $\overline{x}$.

Note that, initially, m is assumed to be a fixed, unknown constant. There is no prior distribution for m and the Bayes theorem is not used in the above inversion.

More than that, the existence of a prior distribution for m would invalidate the fiducial distribution, because $\overline{X}$ would then not contain all available information about m; the additional knowledge will change the resulting distribution yet in the beginning of the reasoning.

In this way, the informational content of the sample is fully used to express the uncertainty of a parameter, uncertainty generated by the sample itself. This transformation of a probability statement for an observable variable into a probability statement for a parameter by means of a pivotal variable on the basis of a given sample will be explained in a more general setting as follows:

1. Let Z be a statistic/estimator for a real-valued parameter θ and $F(z,\theta)$ its cumulative distribution function.
2. Given a suitable pivot $T(Z,\theta)$, the cumulative distribution function $F_T(t)$ of this pivot can be derived, so that it should not depend on the true value of θ.
3. Considering a fixed value z (provided by the sample) of Z, one obtains the fiducial distribution of θ; in the continuous case, under additional regularity conditions, its density is well-defined and expressed as follows:

$$f(\theta \mid z) = \left| \frac{\partial F(z,\theta)}{\partial \theta} \right|,$$

turning out not to depend on the pivot used. Note that the fiducial distribution is not necessarily unique, since there may be more than one statistic for the considered parameter.

Pivots, although not explicitly appearing in this density, are important in deriving the fiducial probability and interpreting it. A pivot could be obtained by considering the pivotal transformation $T(Z,\theta)$ as an unknown function, deriving the distribution function of T from the distribution function of Z, requiring that its partial derivative with respect to θ equals zero, and solving the resulting equation. Note that pivots are by no means unique, since any function of T will also have a distribution that is independent of θ.

A number of authors - [4], [8], [16] - criticized Fisher's fiducial approach and presented inconsistent or seemingly paradoxical results of his theory; other authors defended Fisher's position, e.g. [18], or set up a theoretical framework that restricts its application, , e.g. [11], which actually seems the best way to go on. One of the points of criticism was that it involves the transformation from a probability statement of a random variable to a probability statement of a parameter which is claimed to be constant. This argument arises from a misinterpretation, because it has never been claimed that the parameter θ is a random variable defined by an actual distribution, but, to the contrary, it is said that the fiducial distribution of a parameter θ is an expression of the uncertainty of the fixed parameter due to the sampling. Especially those situations, where the dimension of the parameter space is two or more, have frequently generated criticism about ambiguities of the fiducial argument. Many of the difficulties have been resolved in the meantime or shown to be a consequence of restrictive axiomatic conclusions, see [1], [17].

In response to early criticism, Fisher found himself forced to write a list of criteria of fiducial inference that should be adhered to: the random variable involved should be continuous, the statistic should be complete and sufficient, the pivot should involve only one parameter and vary monotonically with the parameter, etc. Nevertheless, [19] proved that the distribution induced by a proper pivotal variable is not necessarily unique; the induced fiducial distribution depends on the particular set of pivotal quantities chosen. No supplementary requirements which could ensure uniqueness could be found; however, the non-uniqueness of the induced distribution must not be considered as a great disadvantage. The criterion of uniqueness is likely to have been applied more severely to fiducial probability than to other areas of inference, not because of its real necessity, but simply because Fisher's claim of uniqueness. Moreover, the required sufficiency is likely to have been an attempt to secure uniqueness.

One of the points that were also considered as a major inconsistency of fiducial theory was the fact that fiducial probability sometimes doesn't obey the Bayes theorem, see [16]. The fiducial distribution of a parameter based on a first sample cannot be always used as a prior in Bayes theorem along with a second sample in order to produce a posterior distribution for the parameter based on both the samples; the problem consists in the fact that the two samples from this mechanism are not always interchangeable. As later stated by some authors, it would be reasonable to accept that there may be some limitations to the applicability of Bayes theorem and that universal applicability is a task very difficult to achieve. [16] also provided the conditions that a fiducial distribution must fulfill to be a Bayes distribution in the one-dimensional case and [3] proved that an extension for a multidimensional situation is not always possible.

Another criticism was the fact that both fiducial argument and confidence argument often use the same pivotal variable and coincide in many situations. This led some people to think that the whole fiducial method is nothing else than the classical confidence theory. [1] and [5] explained in their papers why the two arguments are conceptually different and provide a more detailed analysis of the difference between the concepts of 'fiducial' and 'confidence' interval. In addition to that, confidence intervals actually need not be identical to the fiducial intervals, as can be seen in an example for two-sided intervals, proposed below.

Example 2. *Consider the case where a parameter θ is known to lie in an open interval $I \equiv (a,b)$, although its exact value is not known. The fiducial distribution*

of the parameter θ *can be shown to be the usual fiducial distribution, censored on the interval* I *.*
Confidence intervals can be built by inverting a two-sided test for the parameter in cause. This is done by taking all those values of the parameter for which a two-sided (usually symmetric, or equal-tailed) test at level α *would not be rejected, when using them as null hypotheses. Thus, coverage is guaranteed, whatever the true value of the parameter is.*
In the general case, one is not bound to split the significance level of the tests equally. In fact, these do not even have to be constant for all $\theta \in I$. *Imagine the case where the significance level is split depending on the value of the parameter under the null hypothesis. For instance, choose the lower critical value under the null hypothesis* $\theta = \theta_0$ *such that the probability of the test statistic being smaller equals* $\alpha \cdot g(\theta_0)$, *where* g *is a continuous and increasing mapping from* I *onto* $[0,1]$. *A linear function may be used, motivated by symmetry reasons. Then, the probability that the test statistic exceeds the upper critical value is* $\alpha \cdot (1 - g(\theta_0))$.
This is done for all possible null hypotheses in I *.*
Based on these rejection bounds, one obtains confidence intervals with exact coverage, but they no longer have the disadvantage that degenerated confidence intervals appear with large probability, should the true value of the parameter be close the boundaries. But, more importantly, it is straightforward to check that these confidence intervals are not identical to the fiducial intervals, since, given the form of the fiducial distribution when restricting the parameter space, the latter can degenerate to a point, while the former always have non-zero length.

Until recently, the pivotal inference was supposed to be rigorously applicable only to data whose distribution was taken to be continuous. Although this paper deals with fiducial inference for continuous distributions, it is useful to know that an approach for the discrete case is possible, as suggested, among others, by [13]. See also [14] for a discussion of the case of the multinomial proportions.

The general nowadays conclusion about the fiducial theory is that the general form of fiducial inference is appealing, but that many of the restrictions imposed by Fisher are awkward or ambiguous and ought to be replaced or, in some cases, removed.

3. A proposal enlarging the basis of applications of fiducial probability

A comparison of different methods of assessing the uncertainty of parameters in statistical models reveals that fiducial inference deserves its own area of application. The use of fiducial inference is appropriate when parameter uncertainty due to sample evidence has to be measured by a probability approach.

Certainly, the applicability of fiducial inference is limited by numerous safeguards to be obeyed. An important restriction is the necessity of knowing the probability distribution of the pivotal quantity to be used in the inversion. We suggest that this difficulty can be solved by bootstrapping the unknown distribution of the pivotal quantity. One could use, for instance, the following resampling procedure as in [6]:

1. Suppose a random sample $X_1, X_2, ..., X_K$ is taken from a population with an unknown cumulative distribution function $F(x,\theta)$;
2. Let $\hat{\Theta}$ be a statistic for θ, where $\hat{\Theta}$ is a function of the sample mean
$$\hat{\Theta} = h(\overline{X});$$
3. Generate (with replacement) B samples from the original sample
$$X_{11}, X_{12}, ..., X_{1K}, ..., X_{b1}, X_{b2}, ..., X_{bK}, ..., X_{B1}, X_{B2}, ..., X_{BK};$$
4. Compute the approximate pivot on each data set
$$Y_b = \frac{\hat{\Theta}_b - \overline{\hat{\Theta}}}{S_{\hat{\Theta}_b}}$$
where
$$\hat{\Theta}_b = h(\overline{X}_b),$$
$$\overline{\hat{\Theta}} = \frac{1}{B}\sum_{b=1}^{B}\hat{\Theta}_b$$
and $S_{\hat{\Theta}_b}$ is the bootstrap estimator of the standard deviation of $\hat{\Theta}_b$;
5. Produce the ordered data set e.g. with $B = 1000$
$$Y_{(1)}, ..., Y_{(1000)};$$
6. Then, the following probability statement stands:
$$P\left(y^{\frac{\alpha}{2}} \le \frac{\hat{\Theta}_b - \overline{\hat{\Theta}}}{S_{\hat{\Theta}_b}} \le y^{1-\frac{\alpha}{2}}\right) \approx 1-\alpha.$$
With e.g. $\alpha = 0.05$ it follows
$$y^{\frac{\alpha}{2}} = Y_{(26)}$$
$$y^{1-\frac{\alpha}{2}} = Y_{(975)};$$
Since this probability statement apparently doesn't depend on θ, it follows in analogy that
$$P\left(\hat{\theta} - y^{1-\frac{\alpha}{2}} \cdot s_{\hat{\Theta}_b} \le \Theta \le \hat{\theta} - y^{\frac{\alpha}{2}} \cdot s_{\hat{\Theta}_b}\right) \approx 1-\alpha,$$
and, for the one-sided case,

$$P\left(\hat{\theta} - y^{1-\alpha} \cdot s_{\hat{\Theta}_b} \leq \Theta\right) \approx 1-\alpha,$$

respectively.

The last two relations are to be interpreted as fiducial statements, consequently Θ takes the role of a random variable and, with given $\hat{\theta} = h(\overline{x})$ and $s_{\hat{\Theta}_b}$ obtained from the sample, the fiducial distribution of θ can be ascertained. One could, of course, apply some smoothing technique to obtain a proper fiducial density.

The quantity

$$Y_b = \frac{\hat{\Theta}_b - \overline{\hat{\Theta}}}{S_{\hat{\Theta}_b}}$$

is called an approximate pivot, since its distribution is only approximately independent of θ. The pivot quality and its large sample distribution are guaranteed by the following extension of the central limit theorem, see for instance [2, p. 29].

Given the *i.i.d.* random variables $X_k \quad (k = 1,...,K)$, with $E\left(X_k\right) = \mu$ and $Var\left(X_k\right) = \sigma^2$ (finite), the classical central limit theorem states that:

$$\frac{\sqrt{K}\left(\overline{X} - \mu\right)}{\sigma} \xrightarrow{d} N\left(0,1\right) \;\; for\ K \to \infty,\ i.e.$$

$$P(\overline{X} \leq x) \approx \Phi\left(\frac{\sqrt{K}(x-\mu)}{\sigma}\right).$$

The following result is known from the probability theory: suppose that a_K is a sequence of constants tending to ∞, b is a fixed number and $\tilde{X}_K$ a sequence of random variables for which holds

$$a_K(\tilde{X}_K - b) \xrightarrow{d} \tilde{X} \quad for\ K \to \infty.$$

If h is a real function of a real variable, differentiable, and with a continuous derivative h' at b, then

$$a_K(h(\tilde{X}_K) - h(b)) \xrightarrow{d} h'(b)\tilde{X} \quad for\ K \to \infty.$$

In our case, $\tilde{X}_K = \overline{X}$, $b = \mu$, $a_K = \sqrt{K}$, $\tilde{X} \sim N(0,\sigma^2)$ and, as a supplementary condition, $h'\left(\mu\right)$ is presumed to be different from zero.

As a result of the above properties, approximations to the distributions of functions $h(\overline{X})$ can be obtained under certain regularity conditions:

$$P\left((\sqrt{K}(h\left(\overline{X}\right) - h(\mu)) \leq x\right) \approx \Phi\left(\frac{x}{\sigma \cdot h'\left(\mu\right)}\right)$$

$$P\left(h\left(\overline{X}\right)\leq x\right)\approx\Phi\left(\frac{\sqrt{K}\left(x-h(\mu)\right)}{\sigma\cdot h'\left(\mu\right)}\right),\ \ i.e.\ \ h\left(\overline{X}\right)\overset{approx}{\sim}N\left(h\left(\mu\right),\frac{\sigma^2\left(h'\left(\mu\right)\right)^2}{K}\right).$$

This approximation covers among others inference concerning mean, variance, correlation coefficient and exponential model applications.

Since the pivot has a valid asymptotic distribution, fiducial intervals generated by means of this resampling approach are most likely correct of higher order than the usual t and normal intervals respectively, see e.g. [12] for regularity conditions. The bootstrap estimates of the α-level quantile of bootstrap distributions are in error by only $O(K^{-1})$ (where K is the sample size). Comparatively, the traditional normal approximation is in error by a much worse $O(K^{-0.5})$. Also, the approximations by Student's t distributions hardly improve on the normal approximation. Thus, the bootstrap has higher accuracy over traditional methods employed to approximate critical points, while the fiducial theory could allow for estimation, when no other powerful estimation methods are available.

A difficulty that can appear is that the fiducial intervals may vary erratically if the sample size is small, but this problem can be solved by variance stabilizing transformations. In fact, these kind of transformations are sometimes recommended, because the use of the untransformed bootstrap procedure can lead to intervals too wide and even outside the allowable range of the values for the estimated parameter. One can obtain different bootstrap intervals depending on what scale is used, and finding the most advantageous transformation for our goals is an important challenge. The most appropriate transformation should normalize and also stabilize the variance, but this is not always possible. Fortunately, it was shown that it is sufficient to have variance stabilizing transformation and that this can be determined with the help of the given data, see for instance the algorithm given in [7, p. 165].

Of course, there are ”better” bootstrapping algorithms, with respect to different criteria, but we used this simple one in order to illustrate the use of resampling procedures within fiducial inference.

Aside from the discussed case, other problems of fiducial inference can be solved by the proposed method, as the use of fiducial distributions in discrete cases and in multivariate applications.

4.Conclusions

As a result of criticism and caveats the range of applications of fiducial inference is limited, although the central argument is theoretically sound and fiducial probability should have its place in the foundation of statistics. Our proposal to modify the fiducial approach may help to establish fiducial inference in statistical practice as a means of evaluating the uncertainty of parameters, based exclusively on the informational content of samples through objective probability statements.

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