NUMERICAL STUDIES ON TWO-DIMENSIONAL SCHRÖDINGER EQUATION BY CHEBYSHEV SPECTRAL COLLOCATION METHOD

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A Chebyshev spectral collocation method for computing highly accurate numerical solutions of the two-dimensional Schrödinger equation is proposed. In this method, the equation is first discretized with respect to the spatial variables, transforming the original problem into a set of ordinary differential equations, and then the resulting system of ordinary differential equations are integrated in time by the classical fourth order Runge–Kutta method. Spatial discretization is done by using the Chebyshev spectral collocation method. The comparison between the numerical solution and the exact solution for the test cases shows the good accuracy of the present method.

Keywords: Two-dimensional Schrödinger equation, Chebyshev spectral collocation method.

MSC2000: 65M70, 65L60, 35Q55

1. Introduction

Consider the two-dimensional Schrödinger equation

$$i\frac{\partial\psi}{\partial t}(x,y,t) = \frac{\partial^2\psi}{\partial x^2}(x,y,t) + \frac{\partial^2\psi}{\partial y^2}(x,y,t) + w(x,y)\psi(x,y,t), \tag{1}$$

$$(x,y) \in \Omega \subset \mathbb{R}^2, \quad t \in [0,T],$$

with initial condition

$$\psi(x,y,0) = \psi_0(x,y), \qquad (x,y) \in \Omega, \tag{2}$$

and the boundary conditions

$$\psi(x,a,t) = \psi_1(x,t), \qquad \psi(x,b,t) = \psi_2(x,t), \qquad t \geq 0, \tag{3}$$

$$\psi(a,y,t) = \psi_3(y,t), \qquad \psi(b,y,t) = \psi_4(y,t), \qquad t \geq 0, \tag{4}$$

where $\Omega = [a,b] \times [a,b]$, $w(x,y)$ is an arbitrary potential function and $i = \sqrt{-1}$. This Schrödinger equation is of fundamental importance in quantum dynamical calculations [1],[3] and has received a great deal attention recently because of its usefulness and applicable by as a model that describe several important physical and chemical phenomena. According to Dehghan and Shokri [12] the Schrödinger equation appears in electromagnetic wave propagation [5] , in underwater acoustics

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[6] or also in optic [8] and design of certain optoelectronic devices [10] as it models an electromagnetic wave equation in a two-dimensional weakly guiding structure.

This equation is of interest from the numerical point of view, because in general, analytical solutions are not available. Dehghan and Shokri developed a numerical scheme for (1) using collocation and radial basis functions [12]. A compact finite difference method was used for solving (1) by Akbar and Dehghan [14] and mesh less local boundary integral equation method was developed in [15]. Finite difference schemes based on the second order discretization of spatial derivatives and first or second order discretization of time derivatives have been investigated in [16], [17].

The numerical solution of the problems in ordinary differential equations is a topic of research that has provided a challenge of lasting interest in numerical analysis and resulted in a large number of methods, see e.g. [2], [4], [7]. The purpose of this paper is to propose a method for solving the two-dimensional Schrödinger equation based on the Chebyshev spectral collocation method. This choice has an excellent reputation amongst numerical analysis practitioners is due to its high accuracy and relatively low computational cost [9], [11], [13]. For this reason, the numerical solutions of (1)-(4) by using a spectral collocation method should be highly accurate as well. The paper is organized as follows: In Section 2 we propose the Chebyshev spectral collocation method. The solution of the Schrödinger equation is described in Section 3. Numerical results that illustrate the efficiency of the proposed method are reported in Section 4. Section 5 contains concluding remarks.

2. **Chebyshev Spectral Collocation Method**

The Chebyshev spectral collocation method can be describe in the following way. An approximation $u_N(x)$ to $u(x)$ and $v_N(x)$ to $v(x)$ are presented for some collocation point x_i. After setting $u_N(x)$ and $v_N(x)$ in the differential equation, we have to use derivative(s) of these functions at the Chebyshev collocation points. The Chebyshev spectral collocation method would involve the use of Chebyshev differentiation matrices to compute derivatives at the collocation points. To obtain optimal accuracy this matrix must be computed carefully. For details see [13], [18].

2.1. **Chebyshev Polynomials**

The Chebyshev polynomial of the first kind $T_N(x)$ is polynomial of degree N defined for the interval $[-1, 1]$ by

$$T_N(x) = \cos(n\cos^{-1}x), \quad n = 0, 1, 2, ..., N.$$

The trigonometric relation $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$ gives the recurrence relation

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \quad n \geq 1,$$

with $T_0(x) = 1$ and $T_1(x) = x$. The recurrence relation on the derivatives

$$\frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} = 2T_n(x), \quad n > 1.$$

Let $u(x)$ be functions on $[-1, 1]$. We interpolate $u(x)$ by the polynomial $u_N(x)$ of degree at most N of the form

$$u_N(x) = \sum_{j=0}^{N} h_j(x)u(x_j),$$
(5)

with $u_N(x_j) = u(x_j)$, and $h_j(x)$ is a polynomial of degree N defined by

$$h_j(x) = \frac{(-1)^{j+1}(1-x^2)T_N'(x)}{c_j N^2 (x - x_j)}, \qquad j = 0, 1, 2, ...N,$$
(6)

where $x_j = \cos(j\pi/N)$ are Chebyshev-Gauss-Labatto (C-G-L) points and $c_0 = c_N = 2$ and $c_j = 1$, $h_j(x_k) = \delta_{jk}$, $j, k = 0, 1, 2, ..., N$ (see [19]).

2.2. One-dimensional differentiation formulation

Consider one dimensional domain: $a \leq x \leq b$. The said domain is discretized using the C-G-L points defined as

$$x_i = \frac{b-a}{2}\cos\left(\frac{i\pi}{N}\right) + \frac{b+a}{2}, \ i = 0, 1, ..., N,$$

The values of derivatives of $\dfrac{d^k u_N}{dx^k}$ and $\dfrac{d^k v_N}{dx^k}$, with $k = 1, 2, \ldots, p$ at the C-G-L points can be computed by [19]

$$\widehat{\frac{du_N}{dx}} = D^{(1)}\widehat{u}_N = D\widehat{u}_N, \qquad \widehat{\frac{dv_N}{dx}} = D^{(1)}\widehat{v}_N = D\widehat{v}_N,$$
(7)

$$\widehat{\frac{d^2 u_N}{dx^2}} = D^{(2)}\widehat{u}_N = D^2\widehat{u}_N, \qquad \widehat{\frac{d^2 v_N}{dx^2}} = D^{(2)}\widehat{v}_N = D^2\widehat{v}_N,$$
(8)

....

$$\widehat{\frac{d^p u_N}{dx^p}} = D^{(p)}\widehat{u}_N = D^p\widehat{u}_N, \qquad \widehat{\frac{d^p v_N}{dx^p}} = D^{(p)}\widehat{v}_N = D^p\widehat{v}_N,$$
(9)

where $\widehat{\ }$ labels vector, e.g., $\widehat{u}_N = (u_N(x_0), u_N(x_1), ..., u_N(x_N))^T$, $\widehat{v}_N = (v_N(x_0), v_N(x_1), ..., v_N(x_N))^T$ and $D^{(.)}$ are the differentiation matrices. The first order Chebyshev differentiation matrix $D^{(1)} = D = d_{kj}$ is given by [19]:

$$d_{kj} = \begin{cases} \frac{c_k}{c_j}\frac{(-1)^{j+k}}{x_k - x_j}, & j \neq k, j, k = 1, ..., N-1, \\ \frac{-x_k}{2(1-x_k^2)}, & k = 1, ..., N-1, \\ \frac{2N^2+1}{6}, & k = j = 0, \\ -\frac{2N^2+1}{6}, & k = j = N, \end{cases} \qquad \text{where} \quad c_k = \begin{cases} 2, & k = 0, N, \\ 1, & \text{otherwise.} \end{cases}$$
(10)

In order to minimize the round off errors for the calculation of first derivatives, the correction technique of Bayliss et al. [20] can be used to compute the diagonal entries D_{ii} by

$$D_{ii} = -\sum_{\substack{j=0 \\ j\neq i}}^{N} D_{ij}.$$
(11)

The use of (11) can lead to a significantly improved accuracy in the computation of second or higher derivatives for a wide range of functions.

To calculate the second order differentiation matrix D^2, we first calculate a provisional differentiation matrix $\widetilde{D}^2$ as the square product of the matrix D defined in (10) and (11) and then repeat the application of the correction technique of Bayliss et al. [20].

2.3. Two-dimensional differentiation formulation

Consider two-dimensional domain, $a \leq x, y \leq b$. Let N_x and N_y be nonnegative integers. The C-G-L points in the x- and y-directions are defined as follows

$$x_i = \frac{b-a}{2}\cos\left(\frac{i\pi}{N_x}\right) + \frac{b+a}{2}, \ i = 0, 1, ..., N_x,$$

$$y_j = \frac{b-a}{2}\cos\left(\frac{j\pi}{N_y}\right) + \frac{b+a}{2}, \ j = 0, 1, ..., N_y.$$

The Chebyshev differentiation over $2D$ grids can be found using the tensor product theory [13]. The derivatives with respect to x at the grid points can be computed by

$$\widehat{\frac{\partial u_N}{\partial x}} = (D_x^{(1)} \otimes \mathbf{I})\widehat{u}_N, \qquad \widehat{\frac{\partial^2 u_N}{\partial x^2}} = (D_x^{(2)} \otimes \mathbf{I})\widehat{u}_N, \tag{12}$$

$$\widehat{\frac{\partial v_N}{\partial x}} = (D_x^{(1)} \otimes \mathbf{I})\widehat{v}_N, \qquad \widehat{\frac{\partial^2 v_N}{\partial x^2}} = (D_x^{(2)} \otimes \mathbf{I})\widehat{v}_N, \tag{13}$$

where $D_x^{(.)}$ are the differentiation matrices of dimension $(N_x+1)\times(N_y+1)$ obtained from one dimensional case, $\mathbf{I}$ is the identity matrix of dimension $(N_x-1)\times(N_y-1)$ and $\otimes$ denotes the kronecker tensor product. By applying the boundary conditions of horizontal lines, equations (12)-(13) can be written as

$$\widehat{\frac{\partial u_N}{\partial x}} = (\widetilde{D}_x^{(1)})\widehat{u}_N + \widehat{k}_{u_N}^{(1x)}, \qquad \widehat{\frac{\partial^2 u_N}{\partial x^2}} = (\widetilde{D}_x^{(2)})\widehat{u}_N + \widehat{k}_{u_N}^{(2x)}, \tag{14}$$

$$\widehat{\frac{\partial v_N}{\partial x}} = (\widetilde{D}_x^{(1)})\widehat{v}_N + \widehat{k}_{v_N}^{(1x)}, \qquad \widehat{\frac{\partial^2 v_N}{\partial x^2}} = (\widetilde{D}_x^{(2)})\widehat{v}_N + \widehat{k}_{v_N}^{(2x)}, \tag{15}$$

where $\widetilde{D}^{(.)}$ are known matrices and $\widehat{k}_{u_N}^{(x)}, \widehat{k}_{v_N}^{(x)}$ are known vectors. For the homogenous boundary conditions (14)-(15) can be reduced to

$$\widehat{\frac{\partial u_N}{\partial x}} = (\widetilde{D}_x^{(1)})\widehat{u}_N, \qquad \widehat{\frac{\partial^2 u_N}{\partial x^2}} = (\widetilde{D}_x^{(2)})\widehat{u}_N, \tag{16}$$

$$\widehat{\frac{\partial v_N}{\partial x}} = (\widetilde{D}_x^{(1)})\widehat{v}_N, \qquad \widehat{\frac{\partial^2 v_N}{\partial x^2}} = (\widetilde{D}_x^{(2)})\widehat{v}_N. \tag{17}$$

Similarly in vertical block, the values of relevant derivatives with respect to y at the grid points can be computed as

$$\widehat{\frac{\partial u_N}{\partial y}} = (\widetilde{D}_y^{(1)})\widehat{u}_N + \widehat{k}_{u_N}^{(1y)}, \qquad \widehat{\frac{\partial^2 u_N}{\partial y^2}} = (\widetilde{D}_y^{(2)})\widehat{u}_N + \widehat{k}_{u_N}^{(2y)}, \tag{18}$$

$$\widehat{\frac{\partial v_N}{\partial y}} = (\widetilde{D}_y^{(1)})\widehat{v}_N + \widehat{k}_{v_N}^{(1y)}, \qquad \widehat{\frac{\partial^2 v_N}{\partial y^2}} = (\widetilde{D}_y^{(2)})\widehat{v}_N + \widehat{k}_{v_N}^{(2y)}, \tag{19}$$

where $\widehat{k}_{u_N}^{(y)}$, and $\widehat{k}_{v_N}^{(y)}$ are the boundary conditions along vertical lines.

3. Solution of the two-dimensional Schrödinger equation

In this section, we now apply the Chebyshev spectral collocation method to the two-dimensional Schrödinger equation (1)-(4). Let

$$\psi(x,y,t) = u_N(x,y,t) + iv_N(x,y,t),$$

The system of equations (1)-(2) converts to the following system of partial differential equations

$$\frac{\partial u_N}{\partial t}(x,y,t) + \frac{\partial^2 v_N}{\partial x^2}(x,y,t) + \frac{\partial^2 v_N}{\partial y^2}(x,y,t) + w(x,y)v_N(x,y,t) = 0, \tag{20}$$

$$\frac{\partial v_N}{\partial t}(x,y,t) - \frac{\partial^2 u_N}{\partial x^2}(x,y,t) - \frac{\partial^2 u_N}{\partial y^2}(x,y,t) + w(x,y)u_N(x,y,t) = 0, \tag{21}$$

with the following initial conditions

$$u_N(x,y,0) = \text{Re}(\psi(x,y,0)), \qquad v_N(x,y,0) = \text{Im}(\psi(x,y,0)), \tag{22}$$

and the following boundary conditions

$$u_N(x,a,t) = \text{Re}(\psi_1(x,t)), \qquad u_N(x,b,t) = \text{Re}(\psi_2(x,t)), \tag{23}$$

$$v_N(x,a,t) = \text{Im}(\psi_1(x,t)), \qquad v_N(x,b,t) = \text{Im}(\psi_2(x,t)), \tag{24}$$

$$u_N(a,y,t) = \text{Re}(\psi_3(y,t)), \qquad u_N(b,y,t) = \text{Re}(\psi_4(y,t)), \tag{25}$$

$$v_N(a,y,t) = \text{Im}(\psi_3(y,t)), \qquad v_N(b,y,t) = \text{Im}(\psi_4(y,t)), \tag{26}$$

where Re and Im denotes the real part and imaginary part respectively. The approximate solution is found in the polynomial $u_N(x,y,t)$ and $v_N(x,y,t)$ of degree at at most N_x and N_y in the x- and y-directions respectively. We Substitute (14)-(15) and (18)-(19) into (20)-(21) and take into account the boundary conditions (23)-(26). The resulting equations are collected at the $(N_x - 3) \times (N_y - 3)$ interior points $(x_i, y_j), i = 2, 3, \ldots N_x - 2, \quad j = 2, 3, \ldots N_y - 2$. Along the two horizontal lines there are four boundary points for the variables v, and along the two vertical lines four boundary conditions are imposed for the variable u. This leads to a set of $(N_x + 1) \times (N_y + 1)$ equations in $(N_x + 1) \times (N_y + 1)$ unknowns.

$$\frac{du}{dt}(x_i, y_j, t) + (\widetilde{D}_x^{(2)})\widehat{v}_N + (\widetilde{D}_y^{(2)})\widehat{v}_N + w(x_i, y_j)v(x_i, y_j, t)$$
$$+ \widehat{k}_{v_a}^{(2x)} + \widehat{k}_{v_b}^{(2x)} + \widehat{k}_{v_a}^{(2y)} + \widehat{k}_{v_b}^{(2y)} = 0, \tag{27}$$
$$i = 0, \ldots, N_x, \quad j = 0, \ldots, N_y,$$

$$\frac{dv}{dt}(x_i, y_j, t) - (\widetilde{D}_x^{(2)})\widehat{u}_N - (\widetilde{D}_y^{(2)})\widehat{u}_N + w(x_i, y_j)v_N(x_i, y_j, t)$$
$$-\widehat{k}_{u_a}^{(2x)} - \widehat{k}_{u_b}^{(2x)} - \widehat{k}_{u_a}^{(2y)} - \widehat{k}_{u_b}^{(2y)} = 0, \tag{28}$$
$$i = 0, \ldots, N_x, \quad j = 0, \ldots, N_y.$$

with the following initial conditions

$$u_N(x_i, y_j, 0) = \text{Re}(\psi(x_i, y_j, 0)), \quad v_N(x_i, y_j, 0) = \text{Im}(\psi(x_i, y_j, 0)), \tag{29}$$
$$i = 0, \ldots, N_x, \quad j = 0, \ldots, N_y.$$

Equations (27)-(28) form a system of ordinary differential equations (ODE) in time with initial conditions (29). Therefore to advance the solution in time, we use an ODE solver such as the classical Runge-Kutta method of order four for two variables.

4. Numerical Results

In this section, we present some numerical results of our scheme (27)-(28) for the the two-dimensional Schrödinger equation. All computations were carried out in Matlab 6.5 on a personal computer. For describing the error, we define maximum error for u as follows:

$$\|E(u)\|_\infty = \max_{0 \leq i,j \leq N} |u(x_i, y_j t) - u_N(x_i, y_j, t)|,$$

where $u_N(x_i, y_j, t)$ is the solution of numerical scheme (27)-(28), whereas $u(x_i, y_j, t)$ is the real part of exact solution of (1)–(4). Similarly we can define the maximum error for the variable v.

Problem (a): To examine the performance of the Chebyshev spectral collocation method for solving a two-dimensional Schrödinger equation, we set the region $0 \leq x, y \leq 1$ with potential function

$$w(x, y) = 3 - 2\tanh^2(x) - 2\tanh^2(y). \tag{30}$$

The exact solution of the equation is given in [14]:

$$\psi(x, y, t) = \frac{i\exp(it)}{\cosh(x)\cosh(y)}, \tag{31}$$

The initial conditions can be found from the exact solution as

$$\psi(x, y, 0) = \frac{i}{\cosh(x)\cosh(y)}, \tag{32}$$

and the boundary conditions are

$$\psi(x, 0, t) = \frac{i\exp(it)}{\cosh(x)}, \qquad \psi(x, 1, t) = \frac{i\exp(it)}{\cosh(x)\cosh(1)}, \tag{33}$$

$$\psi(0, y, t) = \frac{i\exp(it)}{\cosh(y)}, \qquad \psi(1, y, t) = \frac{i\exp(it)}{\cosh(1)\cosh(y)}. \tag{34}$$

Table 1 presents the maximum absolute error for the real part u and imaginary part v of the equation (1) of the present method at various time levels t. The time levels were chosen to compare the results of our Chebyshev spectral collocation method with those reported in Dehghan and Shokri [12] using multi-quadratic radial basis functions. We set the parameters $N_x = 16$, $N_y = 16$, and $\triangle t = 0.001$. We conclude that the present method provides better results than the results obtained

from Dehghan and Shokri [12]. Hence our method is efficient and reliable. Numerical solutions of the real part and imaginary part at time level $t = 1$ are displayed in Figure 1.

Table 1: Maximum error for real part and imaginary part in Problem (a)

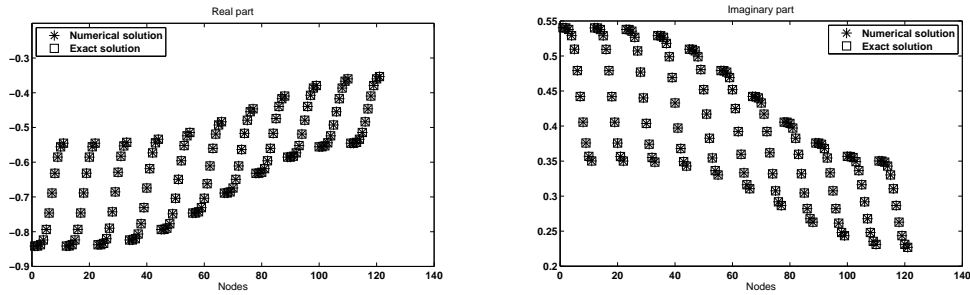| Time	Present Method		Dehghan [12]	
	Real part	Imaginary part	Real part	Imaginary part
0.1	3.5518×10^{-6}	3.0085×10^{-6}	2.4407×10^{-5}	2.9974×10^{-5}
0.3	3.0577×10^{-6}	3.4972×10^{-6}	2.9466×10^{-5}	2.3861×10^{-5}
0.5	3.8579×10^{-6}	4.5155×10^{-6}	2.7468×10^{-5}	3.4044×10^{-5}
0.7	3.6506×10^{-6}	2.9705×10^{-6}	2.5495×10^{-5}	1.8694×10^{-5}
1.0	3.0555×10^{-6}	3.5333×10^{-6}	2.9444×10^{-5}	2.4222×10^{-5}

FIGURE 1. The graph of real and imaginary part of numerical and exact solutions at $t = 1$

Problem (b): We consider the equation (1)–(4) with the region $0 \leq x, y \leq 1$ and potential function

$$w(x, y) = 1 - \frac{2}{x^2} - \frac{2}{y^2}, \tag{35}$$

with the initial conditions

$$\psi(x, y, 0) = x^2 y^2, \tag{36}$$

and the boundary conditions are

$$\psi(x, 0, t) = 0, \qquad \psi(x, 1, t) = x^2 \exp(it), \tag{37}$$

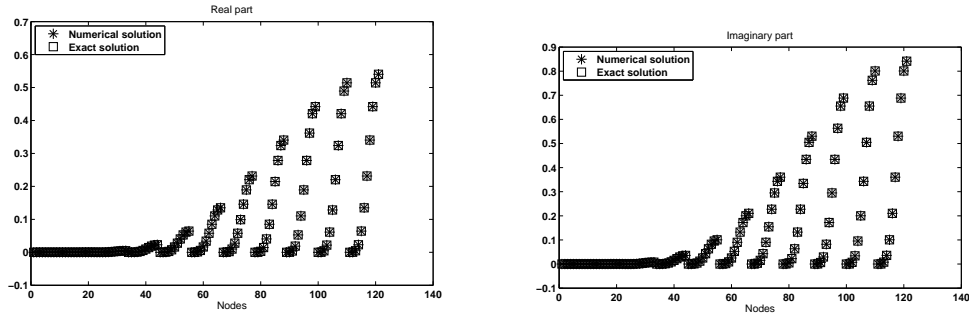$$\psi(0, y, t) = 0, \qquad \psi(1, y, t) = y^2 \exp(it). \tag{38}$$

The exact solution of the equation is given in [16]:

$$\psi(x, y, t) = x^2 y^2 \exp(it). \tag{39}$$

In Table 2, we present a comparison of the numerical solutions of this problem using Chebyshev spectral collocation method (present method) and those obtained by collocation and multi-quadratic radial basis functions method taken from Dehghan and Shokri [12]. In Table 2, we list maximum error for various values of t, and set $N_x = 16$, $N_y = 16$, $\triangle t = 0.001$. As can be seen from Table 2, the present method is considerably more accurate than the Dehghan and Shokri [12]. The graphs of the real part and imaginary part of the numerical and exact solutions at time $t = 1$ are given in Figure 2.

Table 2: Maximum error for real part and imaginary part in Problem (b)

Time	Present Method		Dehghan [12]	
	Real part	Imaginary part	Real part	Imaginary part
0.1	5.1521×10^{-5}	4.6833×10^{-5}	4.0410×10^{-4}	3.5722×10^{-4}
0.3	6.2302×10^{-5}	4.1610×10^{-5}	5.1291×10^{-4}	8.0509×10^{-4}
0.5	5.7407×10^{-5}	4.0631×10^{-5}	4.6396×10^{-4}	3.9520×10^{-4}
0.7	4.9000×10^{-5}	5.2757×10^{-5}	3.8999×10^{-4}	4.1646×10^{-4}
1.0	4.8310×10^{-5}	5.2378×10^{-5}	3.7209×10^{-4}	4.1267×10^{-4}

FIGURE 2. The graph of real and imaginary part of numerical and exact solutions at $t = 1$

Problem (c): We consider the equation (1)–(4) with the region $-2.5 \leq x, y \leq 2.5$ and potential function

$$w(x, y) = 0, \tag{40}$$

with the initial conditions

$$\psi(x, y, 0) = e^{-(x^2+y^2)-ik_0 x}, \tag{41}$$

and the boundary conditions are

$$\psi(x, 0, t) = \frac{i}{i - 4t} e^{-i(x^2+ik_0 x+ik_0^2 t)/(i-4t)}, \quad \psi(x, 1, t) = \frac{i}{i - 4t} e^{-i(x^2+1+ik_0 x+ik_0^2 t)/(i-4t)},$$

$$\psi(0, y, t) = \frac{i}{i - 4t} e^{-i(y^2+ik_0^2 t)/(i-4t)}, \quad \psi(1, y, t) = \frac{i}{i - 4t} e^{-i(1+y^2+ik_0+ik_0^2 t)/(i-4t)}.$$
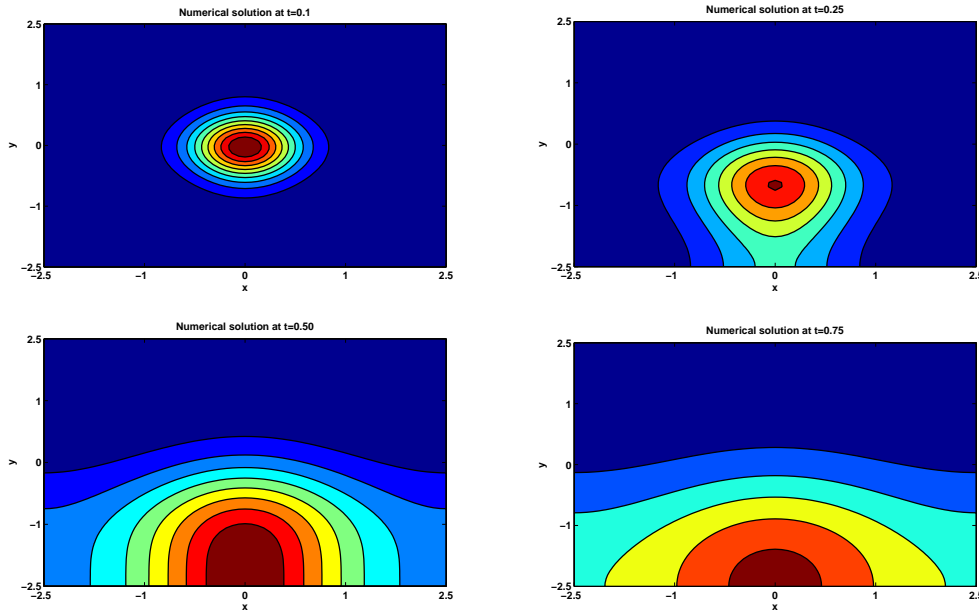
The exact solution of the equation is given in [12]:

$$\psi(x, y, t) = \frac{i}{i - 4t} e^{-i(x^2+y^2+ik_0 x+ik_0^2 t)/(i-4t)}. \tag{42}$$

In this problem, numerical results are obtained using $N_x = 32$, $N_y = 32$, $\triangle t = 0.001$. Table 3 presents the numerical and exact solutions of some selected points of t and Table 3 reports the maximum absolute error for the real and imaginary parts of the solution. As can be seen from the table, the numerical results are in good agreement with the exact solution. According to the results presented in Table 3, we can say the present method provides better results than the results obtained from Dehghan and Shokri [12] using multi-quadratic radial basis functions. In Figure 3 we can see that the contour graph of the numerical solution is moving along the negative y direction with the progress of time.

Table 3: Maximum error for real part and imaginary part in Problem (c)

Time	Present Method		Dehghan [12]	
	Real part	Imaginary part	Real part	Imaginary part
0.10	1.6924×10^{-4}	2.4816×10^{-4}	9.5813×10^{-5}	1.3705×10^{-4}
0.25	4.1169×10^{-4}	3.8990×10^{-4}	3.0058×10^{-3}	2.7889×10^{-3}
0.50	4.7014×10^{-4}	4.7181×10^{-4}	3.6903×10^{-3}	3.6072×10^{-3}
0.75	5.8125×10^{-4}	5.4532×10^{-4}	3.6905×10^{-3}	4.3421×10^{-3}

FIGURE 3. The graph of the modulus of the numerical solution at $t = 0.1, 0.25, 0.5, 0.75$

5. **Conclusion**

A numerical method based on Chebyshev spectral collocation is proposed for solving the two-dimensional Schrödinger equation. To check the numerical method, it is applied to solve different test problems from [12] with known exact solutions. The numerical solutions agree well with the exact ones. The numerical results confirm the validity of the numerical method and suggest that it is an interesting, viable and reasonable alternative to existing numerical methods for solving the two-dimensional Schrödinger equation problems under consideration.

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