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ADJOINT VARIABLES SOLVING TRAIN CONTROL PROBLEM

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This article deals with optimizing the energy consumption of vehicles traction guided by rail such as: electric trains (including subway electric units), railcars, locomotives, and trams. The proposed optimization strategy considers the compliance time drive and aims at improving the transport system for given operation conditions. Our aim has four targets: (1) improving the optimal control techniques; (2) establish a strategy for the operating conditions of the vehicle; (3) formulate and solve additional problems of optimal movement; (4) improving automatic systems for vehicle traction to optimize energy consumption.

Keywords: optimal control involving ODEs, train optimal control, adjoint variables, speed profile.

1. Main requirements for safe railways system

For the rail transport (here we refer to all transport systems guided by rail) to be a viable alternative for travellers, it must meet certain conditions such as: low transport time, low cost, comfort, safety, accessibility, fast links etc. Perception of transport quality depends on several factors that contribute to an efficient transport both for freight transport and passenger transport. The technology enables the realization of high-performance rail vehicles. The difference between transport operators will be determined by reducing operating costs and maintenance costs. An important part of these costs are the costs for energy consumption of the vehicles. Reducing these costs is a priority target for all railway companies. It is known that between energy consumption, traffic speed and drive time there is a relationship of interdependence. It remains an open competition to optimize energy consumption in relation to the required time drive. The energy supplied to the vehicle traction as electric power or mechanical power (supplied by diesel engine) is used to drive the train. Energy balance shows that part of energy is consumed by resistances to advance (determined by the

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circulation on the railways profile) and the brake system. Remaining energy is found in the kinetic and potential energy. It is known that the energy consumption depends on drive regimes selected and of their sequence. It is confirmed that the optimal driving strategy for a train takes the form of a power-speed hold-coast-brake strategy unless the track contains steep grades.

The data used in the study of train control: T is the time allowed for the journey, x is the distance between two stations, $u(t)$ is the accelerations applied to the train, $v(t)$ is the speed of the train, and $-r(v(t))$ is the resistive acceleration due to the friction. The movement of the train is governed by the Newton law

$$\ddot{x}(t) = u(t) - r(v(t)), \tag{1}$$

where $r(v)$, $v \in [0,\infty)$ is strictly increasing and convex function and the acceleration $u(t)$ (control variable) is limited by the relation $|u(t)| \leq 1$. The theory of energy consumption involves also the positive part of $u(t)$, defined by

$$u_+(t) = \frac{1}{2}(u(t) + |u(t)|). \tag{2}$$

The increasing and convex function $r(v)$ is exemplified by the trinomial

$$r(v) = a + bv + cv^2, \; v \in [0, \infty), \tag{3}$$

where the coefficients $a > 0$, $b > 0$, $c > 0$ are known numbers usually given by rolling stock manufacturers.

2. Train control problem

The problem of finding the best way to drive to the next destination can be formulated as an optimal control problem (local energy minimization principle). That is, our aim is to find the sequence of control settings that will get the train to the next destination on time, and with minimal energy consumption.

Newton's law determines the movement equation of the train. In the following problem, x and v are state variables and u is the control variable.

Mathematical assumptions (i) $U = L^\infty([0,T])$ is the set of measurable and bounded functions on the interval $[0,T]$, endowed with the norm

$$\|u\|_\infty = sup|u(t)|, \; t \in [0,T]. \tag{4}$$

The normed space U is called the space of controls;

(ii) $V = C^{0,1}([0,T])$ is the set of piecewise C^1 functions on the interval [0, T], endowed with the norm

$$\|v\| = \|v\|_\infty + \|\dot{v}\|_\infty. \tag{5}$$

The set $F = U \times V$ is called the feasible set. A feasible pair $(u,v) \in F = U \times V$ must satisfies $\|u\|_\infty \leq 1$ and $v(0) = v(T) = 0$.

Isoperimetric Train Problem Minimize the mechanical energy consumption

$$J(u(\cdot)) = \int_0^T u_+(t)v(t)dt \tag{6}$$

subject to

(i) the ODE constraints

$$\dot{x}(t) = v(t), \dot{v}(t) = u(t) - r(v(t)), v(0) = v(T) = 0, \tag{7}$$

(ii) the isoperimetric constraint

$$\int_0^T v(t)dt = x(T) = X, \tag{8}$$

(iii) the control inequality constraint

$$|u(t)| \leq 1. \tag{9}$$

We shall look to apply the Pontryagin maximum principle. For that we use a new objective functional

$$I(u(\cdot)) = \int_0^T (-u_+(t)v(t) + p_1 v(t))dt \tag{10}$$

and the Hamiltonian

$$H(u,v) = -u_+v + p_1 v + p_2\big(u - r(v)\big), \tag{11}$$

where p_1 = ct. and $p_2 = p_2(t)$ are the Lagrange multipliers.

The Hamiltonian can be rewritten as a piecewise function of degree at most one with respect to u, namely

$$H(v,u) = \begin{cases} p_2u + p_1v - p_2r(v), \text{ for } -1 \le u < 0 \\ u(p_2 - v) + p_1v - p_2r(v), \text{ for } 0 \le u \le 1 \end{cases} \quad (12)$$

Control. If the Hamiltonian is linear in the control variables and the control variables have simple bounds, then the optimal control is a combination of bang- bang control and singular arcs.

The Hamiltonian is piecewise linear (function of degree at most one) in the control u, the control variable have simple bounds, and the switching functions are $p_2(t)$ and $p_2(t) - v(t)$, respectively. Therefore the optimal control is a combination of *bang-bang control and singular arcs.* The optimal control $u^*(t)$ is discontinuous: it jumps from a minimum to a maximum and viceversa in response to each change in the sign of switching function.

(i) The optimal control as determined by the switching function $p_2(t)$ is

$$u^*(t) = \begin{cases} 0, \text{ for } p_2(t) > 0, \text{ bang-bang control} \\ -1, \text{ for } p_2(t) < 0, \text{ bang-bang control} \\ \text{undetermined, for } p_2(t) = 0. \end{cases} \quad (13)$$

A switching time is a solution of the equation $p_2(t) = 0$. The most interesting case is those of finite number of switching times.

(ii) The optimal control as determined by the switching function $p_2(t) - v(t)$ is

$$u^*(t) = \begin{cases} 1, \text{ for } p_2(t) > v(t), \text{ bang-bang control} \\ 0, \text{ for } p_2(t) < v(t), \text{ bang-bang control} \\ \text{undetermined, for } p_2(t) = v(t). \end{cases} \quad (14)$$

A switching time is a solution of the equation $p_2(t) = v(t)$. The most interesting case is those of finite number of switching times.

State variables. The state ODE is

$$\dot{v}(t) = -(cv^2(t) + bv(t) + a - u^*), v(t) \ge 0, v(0) = v(T) = 0. \quad (15)$$

If $a - u^* \ge 0$ and $\Delta = b^2 - 4c(a - u^*) \le 0$, then $\dot{v} \le 0$ and v is a decreasing function of t.

If $a - u^* < 0$, and $\Delta = b^2 - 4c(a - u^*) > 0$, i.e., in our conditions $u^* = 1$ and $0 < a < 1$, the algebraic equation $\dot{v} = 0$ has two real roots $\alpha < 0 < \beta$. Then v is an increasing function of t while $v \in [0, \beta)$ and v is a decreasing function of t for $v > \beta$.

For $u^* = 1$ we have

$$\frac{dv}{cv^2+bv+a-1} = -dt \qquad (16)$$

and one obtains

$$t = \frac{1}{\sqrt{\Delta_1}} \ln C_1 \left|\frac{\beta(v-\alpha)}{\alpha(v-\beta)}\right|, \; C_1 > 0, \qquad (17)$$

where $\Delta_1 = b^2 - 4c(a-1)$. The condition $v(0) = 0$ gives us

$$t = \frac{1}{\sqrt{\Delta_1}} \ln \left|\frac{\beta(v-\alpha)}{\alpha(v-\beta)}\right| \qquad (18)$$

Remark In this case $v = \beta$ is a limit speed, and $v \nearrow \beta$ implies $t \to \infty$.

Analogously, we obtain $t = t(v)$ in the cases $u^* = 0$ and $u^* = -1$, without restriction on $v \geq 0$.

Finally,

$$x(t) = \int_0^t v(\tau)d\tau. \qquad (19)$$

Adjoint variable. The adjoint ODE

$$\dot{p}_2(t) = -\frac{\delta H}{\delta v} \qquad (20)$$

becomes $\dot{p}_2(t) = u_+^*(t) - p_1 + p_2 \frac{\delta r}{\delta v}(t).$ (21)

Because

$$\frac{dp_2}{dt} = \frac{dp_2}{dv}\frac{dv}{dt} = -\frac{dp_2}{dv}(cv^2 + bv + a - u^*), \qquad (22)$$

it follows the linear ODE

$$\frac{dp_2}{dv} = -\frac{2cv+b}{cv^2+bv+a-u^*}p_2 - \frac{u_+^*-p_1}{cv^2+bv+a-u^*}, \qquad (23)$$

with general solution

$$p_2(v) = \frac{(p_1 - u_+^*)v + C_2}{cv^2 + bv + a - u^*}. \quad (24)$$

3. Speed profile solving energy - efficient train movement

The book [9] suggested that an energy-efficient speed profile should contain at least three or four phases coupled by continuity: (i) *maximum acceleration, coast, and maximum brake;* (ii) *maximum acceleration, hold speed, coast and maximum brake*. All the experiments confirmed that these strategies are indeed efficient.

Accelerate-brake strategy. The feasible set F is non-empty. Indeed, the initial condition problem

$$\dot{v}(t) = 1 - r(v(t)),\ v(0) = 0 \quad (25)$$

has a unique solution $v_1(t), t \geq 0$, and the final condition problem

$$\dot{v}(t) = -1 - r(v(t)), v(T) = 0 \quad (26)$$

has a unique solution $v_2(t),\ t \leq T$. Further there exists a unique point $t = T_1$, where $v_1(T_1) = v_2(T_1)$ (the two phases are joined by continuity).

Theorem 1. The pair of functions (u(t),v(t)), where

$$u(t) = \begin{cases} 1, & \text{for } t \in (0, T_1) \\ -1, & \text{for } t \in (T_1, T) \end{cases} \quad (27)$$

and, continuous one,

$$v(t) = \begin{cases} v_1(t), & \text{for } t \in (0, T_1) \\ v_2(t), & \text{for } t \in (T_1, T) \end{cases} \quad (28)$$

satisfies the foregoing conditions and represents an accelerate – brake strategy.

Proof. Phase 1: maximum acceleration. The initial condition $v(0) = 0$ implies increasing speed. This is possible for $a \in (0,1)$ only. We must take $u^* = 1$ and $p_2(0) > 0$. Noting $a - 1 = -m^2$, we find $0 > C_2 = -k_1^2$. So, we have

$$t = \frac{1}{\sqrt{\Delta_1}}\, ln\, \frac{\beta(v-\alpha)}{\alpha(v-\beta)}, \quad (29)$$

and $p_2(v) = \frac{(p_1-1)v-k_1^2}{cv^2+bv-m^2}$. (30)

Phase 2: maximum brake. The final condition $v(T) = 0$ implies decreasing speed. We must take $u^* = -1$ and $p_2(T) < 0$. So, supposing

$$\Delta_{-1} = b^2 - 4c(a+1) < 0, \quad (31)$$

we have $T - t = \frac{2}{\sqrt{-\Delta_{-1}}} arctan \frac{v\sqrt{-\Delta_{-1}}}{bv+2a}$, (32)

and $p_2(v) = \frac{p_1 v - k_2^2}{cv^2+bv+a+1}$. (33)

Switch time. The equation

$$\frac{1}{\sqrt{\Delta_1}} ln \frac{\beta(v-\alpha)}{\alpha(v-\beta)} + \frac{2}{\sqrt{-\Delta_{-1}}} arctan \frac{v\sqrt{-\Delta_{-1}}}{bv+2a} = T \quad (34)$$

has a unique solution $v_1 \in (0,\beta)$. This v_1 is the speed at which the swich breaks in and there results the corresponding time $T_1 \in (0,T)$, necessarily.

The constants p_1, k_1^2, k_2^2 will be determinate by the conditions

$$p_1 \geq 1, \ \frac{(p_1-1)v_1-k_1^2}{cv_1^2+bv_1-m^2} = v_1, \ p_1v_1 - k_2^2 = 0. \quad (35)$$

Unfortunately, this strategy does not fulfil the isoperimetric condition since the speed v does not depend on p_1 and k_1. Taking into account the above equation, the values v_1, T and X are connected also by the relation (supplementary condition):

$$bT + 2cX = ln \frac{(a-1)(cv_1^2+bv_1+a+1)}{(a+1)(cv_1^2+bv_1+a-1)}. \quad (36)$$

We have the next alternative. Either we give the space X and calculate the time T, in which the space is covered, or vice-versa.

Accelerate-coast-brake strategy. Let us look for more feasible pairs. With the results in the above case, for a certain $T_2 \in [0,T_1]$, find the unique solution $v_3(t), t \geq T_2$ of the problem

$$\dot{v}(t) = -r(v(t)), \ v(T_2) = v_1(T_2) \quad (37)$$

In the condition $v_3(t) \geq 0$, there exists a unique point $T_3 \in [T_1, T]$ with $v_3(T_3) = v_2(T_3)$.

Theorem 2. The pair of functions (u(t),v(t)), where

$$u(t) = \begin{cases} 1, \text{ for } t \in (0, T_2) \\ 0, \text{ for } t \in [T_2, T_3] \\ -1, \text{ for } t \in (T_3, T) \end{cases} \tag{38}$$

and, continuous one,

$$v(t) = \begin{cases} v_1(t), \text{ for } t \in (0, T_2) \\ v_3(t), \text{ for } t \in (T_2, T_3) \\ v_2(t), \text{ for } t \in (T_3, T) \end{cases} \tag{39}$$

represents an accelerate – coast - brake strategy.

Proof. The optimal control $u^* = 1$ runs while $p_2(v) > v$. Let us fix, in the previous considerations, k_1^2 such that $p_2(v_2) > v_2$. This is the first positive solution of the equation

$$v(cv^2 + bv - m^2) = (p_1 - 1)v - k_1^2. \tag{40}$$

Then $T_2 = \frac{1}{\sqrt{\Delta_1}} \, ln \, \frac{\beta(v_2 - \alpha)}{\alpha(v_2 - \beta)}$. (41)

This is the first switch time.

Phase 1: maximum acceleration runs as in the previous case until $t = T_2$. Note that both v_2 and T_2 depend upon p_1 and k_1^2.

Phase 2: coast. We take

$$u^* = 0,\ \dot{v} = -(cv^2 + bv + a),\ v(T_2) = v_2,\ p_2(v) = \frac{p_1 v + C_2}{cv^2 + bv + a}. \tag{42}$$

In this situation the condition $p_2(v_2) = v_2$, i.e.,

$$\frac{p_1 v_2 + C_2}{cv_2^2 + bv_2 + a} = v_2. \tag{43}$$

From the relation (33) we find $C_2 = -k_1^2$ and the inequality $p_2(v) < v$ holds on the interval $(k_1^2/p_1, v_2)$ at least. Then T_3 corresponding to $v_3 = k_1^2/p_1$ will be the second switch time.

Phase 3: maximum brake. For $t \geq T_3$, we take $u^* = -1$ and $p_2(t) < 0$. The evolution ODE is

$$\dot{v} = -(cv_2^2 + bv_2 + a + 1),\ v(T_3) = v_3 \tag{44}$$

and $p_2(v) = \frac{p_1(v) - k_2^2}{cv_2^2 + bv_2 + a + 1}$, where $k_2^2 \geq k_1^2$.

Finally, the constants p_1, k_1^2, k_2^2 will be determined by the conditions

$$p_1 \geq 1,\ v(T) = 0,\ \int_0^T v(t)\,dt = x(T) = X. \tag{45}$$

4. Conclusions

Circumstances which make train control a pressing problem at the present time are very well known. However, automatic control cannot be done without knowledge of the mathematical theory of optimal control. That is why, in our paper we clarify the idea of cost functional, ODE constraints, isoperimetric constraint, and Pontryagin maximum principle for a train control problem in terms of adjoint variables (see, [1 - 2], [6 - 17]).

This paper is addressed to mathematicians wanting to know more about mathematical issues associated with concrete applications. The topics include modern approaches of geometric control and other mathematical notions that have gave significant enhancements in classical train problems. The experts of nonlinear control learn about applications of this discipline to nontrivial examples in transport problems.

Our aim is to review and detail the optimal control theory of train movement compared with the presentations in the papers [1, 2], [6 - 17], continuing the paper [12]. The papers [3 – 5] and [14] can be used to study stochastic perturbation of previous problem.

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