GENERALIZED DERIVATIONS AND GENERALIZED AMENABILITY OF BANACH ALGEBRAS

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Let $\mathfrak{A}$ be a Banach algebra. A generalized derivation from $\mathfrak{A}$ into itself is a linear map D such that $D(xa) = D(a)x + xd(a)$ for all $a, x \in \mathfrak{A}$, where d is a derivation from $\mathfrak{A}$ into $\mathfrak{A}$. In this paper we define dual generalized derivation form Banach algebra $\mathfrak{A}$ into dual of its $\mathfrak{A}^*$ or dual of some Banach $\mathfrak{A}$-module X and study its properties.

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1. Introduction

Amenability is a cohomological property of Banach algebras which was introduced by Johnson in [14]. Let $\mathfrak{A}$ be a Banach algebra, and suppose that X is a Banach $\mathfrak{A}-$bimodule such that the following statements hold

$$\|a \cdot x\| \leq \|a\|\|x\| \qquad \text{and} \qquad \|x \cdot a\| \leq \|a\|\|x\|$$

for each $a \in \mathfrak{A}$ and $x \in X$.

We can define the right and left actions of $\mathfrak{A}$ on dual space X^* of X via

$$\langle x, \lambda \cdot a\rangle = \langle a \cdot x, \lambda\rangle \qquad \langle x, a \cdot \lambda\rangle = \langle x \cdot a, \lambda\rangle,$$

for each $a \in \mathfrak{A}$, $x \in X$ and $\lambda \in X^*$.

Suppose that X is a Banach $\mathfrak{A}$-bimodule. A derivation $D : \mathfrak{A} \to X$ is a linear map which satisfies $D(ab) = a \cdot D(b) + D(a) \cdot b$ for each $a, b \in \mathfrak{A}$ and it is called Jordan derivation in case $D(x^2) = D(x) \cdot x + x \cdot D(x)$ for each $x \in \mathfrak{A}$. It is clear that every derivation is a Jordan derivation.

A derivation δ is said to be inner if there exists a $x \in X$ such that $\delta(a) = \delta_x(a) = a \cdot x - x \cdot a$ for each $a \in \mathfrak{A}$. We denote the linear space of bounded derivations from $\mathfrak{A}$ into X by $Z^1(\mathfrak{A}, X)$ and the linear subspace of inner derivations by $N^1(\mathfrak{A}, X)$. We consider the quotient space $H^1(\mathfrak{A}, X) = Z^1(\mathfrak{A}, X)/N^1(\mathfrak{A}, X)$, it is called the first Hochschild cohomology group of $\mathfrak{A}$ with coefficients in X. The Banach algebra $\mathfrak{A}$ is said to be amenable if $H^1(\mathfrak{A}, X^*) = \{0\}$ for each Banach $\mathfrak{A}$-bimodules X. The Banach algebra $\mathfrak{A}$ is called weakly amenable if, $H^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$ (for more details

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see [1], [4], [11] and [12]). Let $n \in \mathbb{N}$; the Banach algebra $\mathfrak{A}$ is called n-weakly amenable if, $H^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\}$ (see [5]).

The concept of generalized derivation has been introduced by Brešer in [3]. An additive mapping $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ is called generalization derivation if there exists a derivation $d : \mathfrak{A} \longrightarrow \mathfrak{A}$ such that $D(xy) = D(x)y + xd(y)$ for each pairs $x, y \in \mathfrak{A}$ and we say D is a d-derivation. It is easy to see that $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ is generalized derivation if and only if D is of the form $D = d + \varphi$, where d is a derivation from $\mathfrak{A}$ into $\mathfrak{A}$ and φ is a left module mapping.

The set of bounded $\mathfrak{A}$-module homomorphisms from $\mathfrak{A}$ into an $\mathfrak{A}$-module M is itself an $\mathfrak{A}$-module, when the module operation is given by $a \cdot \phi(x) = \phi(x \cdot a)$ or $\phi(a \cdot x) = \phi(x) \cdot a$, for each $a \in \mathfrak{A}$ and each module homomorphisms ϕ. This module is denoted by $Hom(\mathfrak{A}, M)$. A map $T \in Hom(\mathfrak{A}, \mathfrak{A})$ is called a multiplier, and we write $Hom(\mathfrak{A}, \mathfrak{A}) = M(\mathfrak{A})$. The set $M(\mathfrak{A})$ is a Banach subalgebra of $B(\mathfrak{A})$, the set of all bounded operators on $\mathfrak{A}$. The homomorphic image of $\mathfrak{A}$ in $M(\mathfrak{A})$ is given by $a \mapsto L_a$, where $L_a(x) = ax$, is called the regular representation of $\mathfrak{A}$.

The generalized derivation $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ is a inner if there exist $a, b \in \mathfrak{A}$, such that $D(x) = bx - xa$. If we consider $\mathfrak{A}$ as a right $\mathfrak{A}$-module, generalized derivation $\delta : \mathfrak{A} \longrightarrow \mathfrak{A}$ is inner if there exist $a \in \mathfrak{A}$ and $\phi \in M(\mathfrak{A})$, such that $\delta(x) = \phi(x) - xa$, that $\phi(x) = bx$.

There are some generalizations for amenability of Banach algebras such as approximate amenability [10], character amenability [15, 17], approximate character amenability [13], ideal amenability [7] and approximate ideal amenability [6]. We denote the linear space of bounded generalized derivations from $\mathfrak{A}$ into X by $GZ^1(\mathfrak{A}, X)$ and the linear subspace of generalized inner derivations by $GN^1(\mathfrak{A}, X)$, we consider the quotient space $GH^1(\mathfrak{A}, X) = GZ^1(\mathfrak{A}, X)/GN^1(\mathfrak{A}, X)$, called the first generalized Hochschild cohomology group of $\mathfrak{A}$ with coefficients in X. Similar to amenability of Banach algebra we say $\mathfrak{A}$ is a generalized amenable if $GH(\mathfrak{A}, X^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule X.

2. **Basic Properties**

In this section let $\mathfrak{A}$ be a Banach algebra and M be a Banach $\mathfrak{A}$-bimodule. We use "." for module product between M and its dual and "$\cdot$" denote the module product between M and $\mathfrak{A}$.

Definition 2.1. *A linear mapping $\delta : M \longrightarrow M^*$ is said to be dual generalized derivation on M, if there exist a derivation $d : \mathfrak{A} \longrightarrow M^*$ such that*

$$\delta(xa) = \delta(x) \cdot a + x.d(a)$$

for each $x \in M$ and for each $a \in \mathfrak{A}$.

Definition 2.2. *Let M be a Banach algebra and let $\delta : M \longrightarrow M^*$ be a dual generalized derivation. δ is said to be dual generalized inner derivation, if there exist $a, b \in M^*$ such that $\delta(x) = bx - xa$, for each $x \in M$.*

As above mentioned, it is proved that $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ is generalized derivation if and only if D is of the form $D = d + \varphi$, where d is a derivation from $\mathfrak{A}$ into $\mathfrak{A}$ and

φ is a left module mapping. In the next lemma we extend this for the case when $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$.

Lemma 2.1. *A linear mapping $\delta : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ is dual generalized derivation if and only if there exist a derivation $d : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ and module map $\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ such that $\delta = d + \varphi$.*

Proof. Let δ be a dual generalized derivation on $\mathfrak{A}$, so there exist a derivation $d : \mathfrak{A} \longrightarrow \mathfrak{A}^*$. Give $\varphi = \delta - d$. Then for each $a, x \in \mathfrak{A}$, we have

$$\varphi(xa) = \delta(xa) - d(xa) = \delta(x).a + x.d(a) - (d(x).a + x.d(a)) = \varphi(x)a.$$

Thus φ is module map and $\delta = d + \varphi$.

Conversely let d be a derivation from $\mathfrak{A}$ to $\mathfrak{A}^*$ and $\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ be a module map. Take $\delta = d + \varphi$, then clearly δ is a d-derivation. □

Proposition 2.1. *Let $\mathfrak{A}$ has a bounded approximate identity and $\delta : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ be a d-derivation. Then δ is bounded if and only if d is bounded.*

Proof. By Lemma 2.1, we can decompose δ as $\delta = d + \varphi$ and by Cohen factorization Theorem [2], φ will be bounded and boundedness of δ is only depend on boundedness of d. □

Theorem 2.1. *Let $\delta : M \longrightarrow M^*$ be a bounded linear map. Then δ is a dual generalized inner derivation if and only if there exist an inner derivation $d_a : \mathfrak{A} \longrightarrow M^*$ specified by $a \in \mathfrak{A}$, such that δ is a d_a-derivation.*

Proof. Let δ be a dual generalized derivation. Then there exist $a, b \in M^*$ such that

$$\delta(x) = b.x - x.a \qquad (x \in M).$$

Also for every $x \in M$ we have

$$\begin{aligned} \delta(x) \cdot c + x.d_a(c) &= (b.x - x.a) \cdot c + xa \cdot c - x \cdot c \cdot a \\ &= b.x \cdot c - x.a \cdot c + x.a \cdot c - x \cdot c \cdot a = b.x \cdot c - x \cdot c \cdot a \\ &= \delta(x \cdot c). \end{aligned}$$

Thus δ is a d_a-derivation.

Conversely, suppose δ is a d_a-derivation for some $a \in M^*$. Define $T : M \longrightarrow M^*$ by $T(x) = \delta(x) + x.a$. Then T is linear, bounded and for each $b \in \mathfrak{A}$

$$T(x \cdot b) = (\delta(x) + x.a) \cdot b = T(x) \cdot b.$$

Thus

$$\delta(x) = (\delta(x) + x.a) - x.a = T(x) - x.a.$$

Therefore δ is a dual generalized inner derivation. □

3. Main Results

Proposition 3.1. *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be Banach algebras such that $\mathfrak{A}$ and $\mathfrak{B}$ are Banach $\mathfrak{C}$-bimodule. Suppose that $\theta : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a homeomorphism such that θ and θ^{-1} are linear module maps and $d : \mathfrak{C} \longrightarrow \mathfrak{C}$ is a derivation. Then for every d-derivation $\delta_{\mathfrak{B}} : \mathfrak{B} \longrightarrow \mathfrak{B}$ there exists a d-derivation $\delta_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathfrak{A}$. Converse is true when $\theta^{-1}$is onto.*

Proof. Let $\delta_{\mathfrak{B}} : \mathfrak{B} \longrightarrow \mathfrak{B}$ be a $d-$derivation. So for every $y \in \mathfrak{B}$ and $c \in \mathfrak{C}$ we have $\delta_{\mathfrak{B}}(y \cdot c) = \delta_{\mathfrak{B}}(y) \cdot c + y \cdot d(c)$. Therefore, there exists a $x \in \mathfrak{A}$ such that

$$\delta_{\mathfrak{B}}(\theta(x) \cdot c) = \delta_{\mathfrak{B}}(\theta(x)) \cdot c + \theta(x) \cdot d(c).$$

Consequently

$$\delta_{\mathfrak{B}} o\theta(x \cdot c) = (\delta_{\mathfrak{B}} o\theta(x)) \cdot c + \theta(x) \cdot d(c),$$

and also we have

$$\theta^{-1} o\delta_{\mathfrak{B}} o\theta(x \cdot c) = \theta^{-1} o(\delta_{\mathfrak{B}} o\theta(x)) \cdot c + x \cdot d(c) = (\theta^{-1} o\delta_{\mathfrak{B}} o\theta(x)) \cdot c + x \cdot d(c).$$

Now, assume $\delta_{\mathfrak{A}} = \theta^{-1} o\delta_{\mathfrak{B}} o\theta$, and so proof is complete. □

Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and θ be defined as above Proposition. Then for every inner d-derivation $\delta_{\mathfrak{B}} : \mathfrak{B} \longrightarrow \mathfrak{B}$ there exists a $\delta_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathfrak{A}$ such that $\delta_{\mathfrak{A}}$ is an inner d-derivation.

Proposition 3.2. *If $\delta : \mathfrak{A} \longrightarrow \mathfrak{A}$ is a d-derivation, then $\delta^{**} : \mathfrak{A}^{**} \longrightarrow \mathfrak{A}^{**}$ is a d^{**}-derivation.*

Proof. It is clear that δ^{**} is linear. For given $a, b \in \mathfrak{A}^{**}$, there exist nets (a_α) and (b_β) in $\mathfrak{A}$ such that $a = w^* - \lim_\alpha a_\alpha = a$ and $b = w^* - \lim_\beta b_\beta = b$. Then

$$\begin{aligned} \delta^{**}(ab) &= w^* - \lim_\alpha \ w^* - \lim_\beta \delta(a_\alpha b_\beta) \\ &= w^* - \lim_\alpha \ w^* - \lim_\beta \ (\delta(a_\alpha) b_\beta + a_\alpha d(b_\beta)) \\ &= \delta^{**}(a)b + ad^{**}(b). \end{aligned}$$

□

Theorem 3.1. *Suppose that the following sequence is a short exact sequence*

$$0 \longrightarrow \mathfrak{I} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{B} \longrightarrow 0,$$

of Banach algebras, Banach $\mathfrak{A}$-bimodules and bounded algebra homomorphism ($\mathfrak{A}$ is an extension of $\mathfrak{B}$ by $\mathfrak{I}$). If $\delta_1 : \mathfrak{I} \longrightarrow \mathfrak{I}^*$ and $\delta_2 : \mathfrak{B} \longrightarrow \mathfrak{B}^*$ be dual generalized d-derivations, then there exists a linear map $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ such that D is a dual generalized d-derivation

Proof. We may assume that $\mathfrak{I}$ is a closed two sided ideal in $\mathfrak{A}$ and $\mathfrak{B}$ is the quotient space $\mathfrak{A}/\mathfrak{I}$. According to our assumption we have $\delta_1(xa) = \delta_1(x).a + x.d(a)$ and $\delta_2(ya) = \delta_2(y).a + y.d(a)$, for each $a \in \mathfrak{A}, x \in \mathfrak{I}$ and $y \in \mathfrak{A}/\mathfrak{I}$.

Now, we define $D = \delta_1 + \delta_2$. It is clear that D is linear and for each $z \in \mathfrak{A}$ we have

$$\begin{aligned} D(za) &= D((x+y)a) = D(xa + ya) \\ &= D(xa) + D(ya) = \delta_1(xa) + \delta_2(ya) \\ &= (\delta_1(x) + \delta_2(y)).a + (x+y)d(a) \\ &= D(x+y).a + (x+y).d(a) \end{aligned}$$

Thus, $D(za) = D(z).a + z.d(a)$, and proof is complete. □

Let $\mathfrak{A}_1, ..., \mathfrak{A}_n$, be Banach algebras and $D_i : \mathfrak{A}_i \longrightarrow \mathfrak{A}_i^*$ be a dual d_i-derivation for each $i = 1, ..., n$. Then $D : \prod_{i=1}^n \mathfrak{A}_i \longrightarrow \mathfrak{A}_i^*$ is a dual d_i-derivation.

Proof. We have the following short exact sequence

$$0 \longrightarrow \mathfrak{A}_i \xrightarrow{\tau_i} \prod_{i=1}^n \mathfrak{A}_i \xrightarrow{\pi_i} \mathfrak{A}_i \longrightarrow 0.$$

Accorrolaryding to above Theorem, D is a dual d_i-derivation. □

Definition 3.1. *Let $\mathfrak{A}$ be a Banach algebra. We say $\mathfrak{A}$ is generalized amenable if $GH(\mathfrak{A}, X^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule X.*

Definition 3.2. *Let $\mathfrak{A}$ be a Banach algebra. We say $\mathfrak{A}$ is generalized weakly amenable if $GH(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$.*

Theorem 3.2. *Let $\mathfrak{A}$ be a amenable Banach algebra. Then for every Banach $\mathfrak{A}$-bimodule M, we have $GH^1(M, M^*) = \{0\}$.*

Proof. Let $\delta : M \longrightarrow M^*$ be a dual generalized derivation. Then there exists a derivation $d : \mathfrak{A} \longrightarrow M^*$ such that δ is a d-derivation. Thus by Theorem 2.1, $GH(M, M^*) = \{0\}$. □

If $\mathfrak{A}$ is an amenable Banach algebra, then for every Banach algebra M, which is a Banach $\mathfrak{A}$-bimodule, we have $GH^1(M, M^{(n)}) = \{0\}$ (i.e. M is generalized-n-permanent amenable).

Theorem 3.3. *Let $\mathfrak{A}$ and M be Banach algebras and M be a right Banach $\mathfrak{A}$-module. If M is weakly amenable, then for every dual generalized d-derivation $\delta : M \longrightarrow M^*$, d is inner derivation from $\mathfrak{A}$ to M^*.*

Proof. Let $\delta : M \longrightarrow M^*$ be a dual generalized d-derivation so $\delta(x \cdot b) = \delta(x) \cdot b + x.d(b)$ for $b \in \mathfrak{A}$ and $x \in M$. Since M is weakly amenable, then δ is an inner derivation. Therefore there exists an $a \in M^*$ such that

$$\delta(x) = a.x - x.a.$$

So we have

$$\begin{aligned} \delta(x \cdot b) &= a.x \cdot b - x \cdot b \cdot a = \delta(x) \cdot b + x.d(b) \\ &= a.x \cdot b - x.a \cdot b + x.d(b). \end{aligned}$$

Then $d(b) = a \cdot b - b \cdot a$ and so d is an inner derivation. □

Theorem 3.4. *Let $\mathfrak{A}_1, ..., \mathfrak{A}_n$, be Banach algebras and M_i be a Banach $\mathfrak{A}_i$-module for each $i = 1, 2, ..., n$. Let $D_i : M_i \longrightarrow M_i^*$ be a dual generalized derivation for each $i = 1, ..., n$. Then*

$$D : \prod_{i=1}^n M_i \longrightarrow \prod_{i=1}^n M_i^*$$

is a dual generalized derivation.

Proof. Since each D_i is a dual generalized derivation, therefore there exists a derivation such as $d_i : \mathfrak{A}_i \longrightarrow M_i^*$ such that $D_i(a \cdot x) = D_i(a) \cdot x + a.d_i(x)$ for each $a \in M_i$ and $x \in \mathfrak{A}_i$. Define $D : M_1 \times M_2 \times ... \times M_n \longrightarrow M_1^* \times M_2^* \times ... \times M_n^*$ by $D = (D_1, ..., D_n) = \prod_{i=1}^n D_i$. Then for every $(a_1, a_2, ..., a_n) \in \prod_{i=1}^n M_i$ and $(x_1, x_2, ..., x_n) \in \prod_{i=1}^n A_i$, we have

$$\begin{aligned} D(a \cdot x) &= D((a_1, a_2, ..., a_n) \cdot (x_1, x_2, ..., x_n)) = D((a_1 \cdot x_1, a_2 \cdot x_2, ..., a_n \cdot x_n)) \\ &= (D_1(a_1, x_1), D_2(a_2, x_2), ..., D_n(a_n, x_n)) \\ &= (D_1(a_1) \cdot x_1 + a_1.d_1(x_1), ..., D_n(a_n) \cdot x_n + a_n.d(x_n)) \\ &= (D_1(a_1), ..., D_n(a_n)) \cdot (x_1, ..., x_n) + (a_1, ..., a_n).(d_1(x_1), ..., d_n(x_n)). \end{aligned}$$

Now, take $d = (d_1, ..., d_2)$. Since each d_i is a derivation, so d is a derivation from $\prod_{i=1}^n \mathfrak{A}_i$ into $\prod_{i=1}^n M_i^*$. Then we have

$$D(a \cdot x) = D(a) \cdot x + a.d(x),$$

for every $x \in \mathfrak{A}$ and $a \in M$. Thus, D is a d-derivation and proof is complete. □

Let $\mathfrak{A}_1, ..., \mathfrak{A}_n$, be generalized amenable Banach algebras, then $\prod_{i=1}^n \mathfrak{A}_i$ is generalized amenable.

4. **Results for Triangular Banach Algebras**

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebra and suppose that $\mathcal{M}$ is Banach $\mathcal{A}, \mathcal{B}$-module. We define triangular Banach algebra

$$T = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix},$$

with the sum and product being giving by the usual 2×2 matrix operations and internal module actions. The norm on T is

$$\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \| = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}.$$

Derivation on triangular Banach algebras have been studied by B. E. Forrest and L. W. Marcoux in [6] and amenability and weak amenability of these algebras are studied in [7] and [11]. T as a Banach space is isomorphic to the ℓ^1-direct sum of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, so we have $T^{(2m-1)} \simeq \mathcal{A}^{(2m-1)} \oplus_1 \mathcal{M}^{(2m-1)} \oplus_1 \mathcal{B}^{(2m-1)}$ and $T^{(2m)} \simeq \mathcal{A}^{(2m)} \oplus_\infty \mathcal{M}^{(2m)} \oplus_\infty \mathcal{B}^{(2m)}$ for each $m \geq 1$.

When $m = 1$, for every $\tau = \begin{bmatrix} \alpha & \mu \\ & \beta \end{bmatrix} \in T^*$ and $\omega = \begin{bmatrix} x & y \\ & z \end{bmatrix}$, the actions of ω on τ and τ on ω are given by

$$\omega \circ \tau = \begin{bmatrix} x \circ \alpha + y \circ \mu & z \circ \mu \\ & z \circ \beta \end{bmatrix} \quad \textit{and} \quad \tau \circ \omega = \begin{bmatrix} \alpha \circ x & \mu \circ x \\ & \mu \circ y + \beta \circ z \end{bmatrix}$$

By above relations and easy calculations we have the following theorem:

Theorem 4.1. *Let $D : T \longrightarrow T^*$ be a bounded dual generalized derivation. Then there exist bounded dual generalized derivations $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^*, D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^*$, and* *an element* $\gamma_D \in \mathcal{M}$ *such that*

$$D\begin{bmatrix} x & y \\ & z \end{bmatrix} = \begin{bmatrix} D_{\mathcal{A}}(x) - y \circ \gamma_D & \gamma_D \circ x - z \circ \gamma_D \\ & D_{\mathcal{B}}(z) + \gamma_D \circ y \end{bmatrix} \quad (x \in \mathcal{A}, y \in \mathcal{M}, z \in \mathcal{B}).$$

Theorem 4.2. *Let* $\delta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^*$ *be a dual generalized derivation. Then* $D_{\delta_{\mathcal{A}}} : T \longrightarrow T^*$ *defined by*

$$\begin{bmatrix} x & y \\ & z \end{bmatrix} \mapsto \begin{bmatrix} \delta_{\mathcal{A}}(x) & 0 \\ & 0 \end{bmatrix}$$

is a bounded dual generalized derivation and $\delta_{\mathcal{A}}$ *is a dual generalized inner derivation if and only if* $D_{\delta_{\mathcal{A}}}$ *is a dual generalized inner derivation.*

Similarly, for $\delta_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}^*$ *with define* $D_{\delta_{\mathcal{A}}} : T \longrightarrow T^*$ *by*

$$\begin{bmatrix} x & y \\ & z \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ & \delta_{\mathcal{B}}(z) \end{bmatrix}$$

above result is true.

Proof. Since $\delta_{\mathcal{A}}$ is a dual generalized derivation thus exist derivation $d : \mathcal{A} \longrightarrow \mathcal{A}^*$ such that $\delta_{\mathcal{A}}(xa) = \delta_{\mathcal{A}}(x).a + x.d(a)$, for each $x, a \in \mathcal{A}$. Then for every $\omega = \begin{bmatrix} x & y \\ & z \end{bmatrix}, \nu = \begin{bmatrix} a & m \\ & b \end{bmatrix} \in T$ we have

$$\begin{aligned} D_{\delta_{\mathcal{A}}}(\omega\nu) &= D_{\delta_{\mathcal{A}}}(\begin{bmatrix} xa & xm + yb \\ & zb \end{bmatrix}) = \begin{bmatrix} \delta_{\mathcal{A}}(xa) & 0 \\ & 0 \end{bmatrix} \\ &= \begin{bmatrix} \delta_{\mathcal{A}}(x).a & 0 \\ & 0 \end{bmatrix} + \begin{bmatrix} x.d(a) & 0 \\ & 0 \end{bmatrix} \\ &= D_{\delta_{\mathcal{A}}}(\omega).\nu + \omega.D_d(\nu), \end{aligned}$$

where D_d is a derivation from T into T^* corrolaryresponding to d.

Now suppose that $\delta_{\mathcal{A}}$ is a dual generalized inner derivation, therefore d is a inner and accorrolaryding to Lemma 3.3 of [7], D_d is a dual generalized inner derivation and by Theorem 2.1, $D_{\delta_{\mathcal{A}}}$ is a dual generalized inner derivation. Converse by Lemma 3.3 of [7] is clear.

We can write the similar proof for $\delta_{\mathcal{B}}$ and $D_{\delta_{\mathcal{B}}}$, and the above results hold too. □

5. **Jordan Dual Generalized Derivation**

Definition 5.1. *Let* $\mathfrak{A}$ *be a Banach algebra. An additive mapping* $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ *is generalized Jordan derivation if* $D(x^2) = D(x)x + xd(x)$ *holds for each* $x \in \mathfrak{A}$ *where* $d : \mathfrak{A} \longrightarrow \mathfrak{A}$ *is a Jordan derivation.*

Definition 5.2. *Let* $\mathfrak{A}$ *be a Banach algebra. An additive mapping* $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ *is dual Jordan derivation if* $D(x^2) = D(x).x + x.D(x)$ *holds for each* $x \in \mathfrak{A}$.

Definition 5.3. *Let* $\mathfrak{A}$ *be a Banach algebra. An additive mapping* $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ *is dual generalized Jordan derivation if* $D(x^2) = D(x).x + x.d(x)$ *holds for each* $x \in \mathfrak{A}$ *where* $d : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ *is a dual Jordan derivation.*

Theorem 5.1. *Let* $\mathfrak{A}$ *be a semisimple Banach algebra and let* $D : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ *be a dual generalized Jordan derivation. Then* D *is a dual generalized derivation.*

Proof. Since D is a dual generalized Jordan derivation, we have

$$D(x^2) = D(x).x + x.d(x) \qquad (x \in \mathfrak{A})$$

where d is a dual Jordan derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$. Since $\mathfrak{A}$ is a semisimple, then d is a derivation. Define $\varphi = D - d$, then we have

$$\begin{aligned}\varphi(x^2) &= D(x^2) - d(x^2) = D(x).x + x.d(x) - (x.d(x) + d(x).x) \\ &= D(x).x - d(x).x = (D(x) - d(x)).x = \varphi(x).x\end{aligned}$$

therefore $\varphi(x^2) = \varphi(x).x$, for each $x \in \mathfrak{A}$. By Proposition 1.4 of [14], we conclude that φ is a module map. Hence $D = \varphi + d$, where φ is a module map and d is a derivation. Then by Lemma 2.1, D is a dual generalized derivation. □

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