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ON A VECTOR OPTIMIZATION PROBLEM INVOLVING HIGHER ORDER DERIVATIVES

Savin Treanţă[1]

In this paper we establish necessary and sufficient efficiency conditions for a class of multiobjective fractional variational problems (MFP) subject to ODEs & ODIs constraints involving higher order derivatives. Using the notion of (ρ, b)-quasiinvexity, we formulate sufficient efficiency conditions for a feasible solution in (MFP).

Keywords: (ρ, b)-quasiinvexity; multiobjective fractional variational problem; (normal) efficient solution.
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1. Introduction and problem description

In a previous work (Treanţă and Udrişte, [12]), strongly motivated by its applications in natural phenomena, mechanics and engineering problems (which imply derivatives of order higher than one or two), we extended and further developed some optimization results connected to the efficiency of a feasible solution for a class of multiobjective nonfractional variational problems. As natural continuation of these results, the present research paper introduces a study of efficiency conditions for a feasible solution in a multiobjective fractional optimization problem of minimizing a vector of simple integral functionals subject to certain higher order differential equations and/or inequations. Our study is encouraged by many practical optimization problems with simple integral functional quotients as objective vectors.

There is a long story of multiobjective (fractional) programming problems (see [2], [11], [5], [10], [6], [1], [7], etc.) which involve a generalized convexity. We make an abuse mentioning only a little part: [11], [4], [10], [6], [7], [9], [8]. In [11], Singh and Hanson derive duality results using invex functions in vector ratio problems. Jeyakumar and Mond (see [4]) generalize these results for V-invex functions. Later, a unified formulation of the generalized convexity, in order to derive duality results and optimality conditions, was provided by Z. A. Liang, H. X. Huang and P. M. Pardalos (see [6]).

Our vector minimization problem, required by practical reasons, despite of the previous mentioned advances in optimization, has not been studied so far. The present paper is organized as follows: Section 1 motivates the study and describes the vector ratio problem (MFP), while Section 2 provides, for a better coherence of this paper, the main ingredients derived in a previous work (see [12]); Section 3 includes the original results and the final section contains the conclusions of this work.

[1]Teaching Assistant, Department of Mathematics-Informatics, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania, E-mail: savin_treanta@yahoo.com

In our subsequent theory, we shall use the following notations:

$$u = v \Leftrightarrow u_i = v_i, \quad u \leq v \Leftrightarrow u_i \leq v_i$$

$$u < v \Leftrightarrow u_i < v_i, \quad u \preceq v \Leftrightarrow u \leq v, \ u \neq v, \quad i = \overline{1, s}$$

for any two vectors $u = (u_1, ..., u_s), v = (v_1, ..., v_s)$ in R^s. Let us consider $\chi_x(t) := (t, x(t), x^{(1)}(t), ..., x^{(k)}(t))$, where $x^{(k)}(t) := \frac{d^k}{dt^k}x(t)$, with $k \geq 1$ a fixed natural number. Let be given

$$f = (f_\alpha) : I \times R^{n(k+1)} \to R^p, \quad e = (e_\alpha) : I \times R^{n(k+1)} \to R^p, \quad \alpha = \overline{1, p}$$

$$g = (g_1, ..., g_m) : I \times R^{n(k+1)} \to R^m, \quad h = (h_1, ..., h_r) : I \times R^{n(k+1)} \to R^r$$

(see $m < n, \ r < n$) four C^{k+1}-class functions, where $I := [t_0, t_1] \subseteq R$ is a real interval. The previous C^{k+1}-class Lagrangians, $f_\alpha(\chi_x(t)), \ e_\alpha(\chi_x(t)), \ \alpha = \overline{1, p}$, generate the following simple integral functionals

$$F_\alpha(x(t)) := \int_{t_0}^{t_1} f_\alpha(\chi_x(t))dt, \quad E_\alpha(x(t)) := \int_{t_0}^{t_1} e_\alpha(\chi_x(t))dt > 0, \quad \alpha = \overline{1, p}.$$

Consider the space $C^\infty([t_0, t_1], R^n)$ of all functions $x \colon [t_0, t_1] \to R^n$ of C^∞-class, having the norm $\|x\| := \|x\|_\infty + \sum_{\beta=1}^{k} \|x^{(\beta)}\|_\infty$. Define $\mathtt{F}(I)$ as being the set

$$x \in C^\infty(I, R^n), \quad g(\chi_x(t)) \leq 0, \quad h(\chi_x(t)) = 0, \quad t \in I$$

$$x(t_\varepsilon) = x_\varepsilon, \quad x^{(\beta)}(t_\varepsilon) = x_{\beta\varepsilon}, \quad \varepsilon = 0, 1, \quad \beta = \overline{1, k-1}$$

of all *feasible solutions (domain)* of the following multiobjective fractional variational problem (MFP),

$$\min_{x(\cdot)} \left(\frac{\int_{t_0}^{t_1} f_1(\chi_x(t))dt}{\int_{t_0}^{t_1} e_1(\chi_x(t))dt}, \frac{\int_{t_0}^{t_1} f_2(\chi_x(t))dt}{\int_{t_0}^{t_1} e_2(\chi_x(t))dt}, ..., \frac{\int_{t_0}^{t_1} f_p(\chi_x(t))dt}{\int_{t_0}^{t_1} e_p(\chi_x(t))dt} \right)$$

$$\text{subject to } x(\cdot) \in \mathtt{F}(I),$$

an ODEs & ODIs constrained vector optimization problem.

Next section introduces auxiliary tools which will be further used for proving the main results of this work, that is: necessary and sufficient efficiency conditions for (MFP).

2. **Preliminaries**

Using the same mathematical data as in the previous section, let consider the following multiobjective variational problem (MVP),

$$\min_{x(\cdot)} \left(\int_{t_0}^{t_1} f_1(\chi_x(t))dt, \int_{t_0}^{t_1} f_2(\chi_x(t))dt, ..., \int_{t_0}^{t_1} f_p(\chi_x(t))dt \right)$$

$$\text{subject to } x(\cdot) \in \mathtt{F}(I).$$

This problem was formulated and studied in [12] and, for a better understanding and coherence of all that will follow in this paper, we shall recall the basic results.

Let ρ be a real number and $b\colon \left[C^\infty\left([t_0,t_1],R^n\right)\right]^{k+1} \to [0,\infty)$ a functional. Denote

$$b\left(x,x^0,x^{0(1)},...,x^{0(k-1)}\right) := b_{xx^0},\quad \eta\left(t,x,x^{(1)},...,x^{(k-1)},x^{0(k)}\right) := \eta_{txx^0}.$$

Also, let $a\colon I\times R^{n(k+1)} \to R$ be a real function that determines the following simple integral functional $A(x(t)) = \int_{t_0}^{t_1} a\left(\chi_x(t)\right)dt$.

Definition 2.1. ([12]) The functional $A(x)$ is *[strictly] (ρ,b)-quasiinvex at x^0* if there exist the vector functions $\eta = (\eta_1,...,\eta_n)$, with the property $\frac{d^\zeta \eta_{tx^0x^0}}{dt^\zeta} = 0$, $\zeta \in \{0,1,...,k-1\}$, $(\forall)\ t\in I$, and $\theta\colon \left[C^\infty\left([t_0,t_1],R^n\right)\right]^{k+1} \to R^n$ such that, for any x $[x\neq x^0]$, we have

$$\left(A(x)\le A(x^0)\right) \Longrightarrow (b_{xx^0}\int_{t_0}^{t_1}\Big\{\eta_{txx^0}\frac{\partial a}{\partial x}(\chi_{x^0}(t)) + \frac{d\eta_{txx^0}}{dt}\frac{\partial a}{\partial x^{(1)}}(\chi_{x^0}(t))$$

$$+...+\frac{d^k\eta_{txx^0}}{dt^k}\frac{\partial a}{\partial x^{(k)}}(\chi_{x^0}(t))\Big\}dt\ [<]\le -\rho b_{xx^0}\parallel \theta_{xx^0}\parallel^2).$$

Theorem 2.1. ([12]) ([Normal] necessary efficiency conditions for (MVP)) *If $x^0(\cdot)\in \mathtt{F}(I)$ is a [normal] efficient solution of the problem (MVP) then there are $\lambda\in R^p$, $p\colon I\to R^m$ and $q\colon I\to R^r$ satisfying the following conditions:*

$$\sum_{j=1}^p \lambda_j\frac{\partial f_j}{\partial x}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x}(\chi_{x^0}(t)) + q(t)\frac{\partial h}{\partial x}(\chi_{x^0}(t))$$

$$-\frac{d}{dt}\left\{\sum_{j=1}^p \lambda_j\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t)) + q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t))\right\}$$

$$+...+(-1)^k\frac{d^k}{dt^k}\left\{\sum_{j=1}^p \lambda_j\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}(t)) + q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}(t))\right\}$$

$$= 0 \text{ (higher order Euler-Lagrange ODEs)}$$

$$p(t)g(\chi_{x^0}(t)) = 0,\quad p(t)\ge 0,\quad (\forall)t\in I$$

$$\lambda\ge 0,\quad e^t\lambda = 1,\quad e^t = (1,1,...,1)\in R^p.$$

Theorem 2.2. ([12]) (Sufficient efficiency conditions for (MVP)) *Assume that Theorem* 2.1 *is fulfilled and there exist the vector functions η and θ satisfying Definition* 2.1. *Also, consider that the following statements are true:*

a) the functionals $\int_{t_0}^{t_1} f_l(\chi_x(t))dt$, $l\in\{1,...,p\}$, are (ρ_l^1,b)-quasiinvex at $x^0(\cdot)$ with respect to η and θ;

b) $\int_{t_0}^{t_1} p(t)g(\chi_x(t))dt$ is (ρ^2,b)-quasiinvex at $x^0(\cdot)$ with respect to η and θ;

c) $\int_{t_0}^{t_1} q(t)h(\chi_x(t))dt$ is (ρ^3,b)-quasiinvex at $x^0(\cdot)$ with respect to η and θ;

d) one of the integrals $\int_{t_0}^{t_1} f_l(\chi_x(t))dt,\ l \in \{1,...,p\},\ \int_{t_0}^{t_1} p(t)g(\chi_x(t))dt,$ $\int_{t_0}^{t_1} q(t)h(\chi_x(t))dt$ *is strictly* (ρ, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

e) $\sum_{l=1}^{p} \lambda_l \rho_l^1 + \rho^2 + \rho^3 \geq 0 \quad (\rho_l^1, \rho^2, \rho^3 \in R).$

Then $x^0(\cdot)$ *is an efficient solution for (MVP).*

3. Main results

This section aims to formulate and prove necessary and sufficient efficiency conditions for (MFP). In this direction, we establish the following auxiliary results and definitions.

Definition 3.1. A feasible solution $x^0(\cdot) \in \mathtt{F}(I)$ is called *efficient solution* (or *Pareto minimum*) in (MFP) if there exists no other feasible solution $x(\cdot) \in \mathtt{F}(I)$ such that $K(x(t)) \preceq K\left(x^0(t)\right)$, where

$$K(x(t)) := \left(\frac{\int_{t_0}^{t_1} f_1(\chi_x(t))dt}{\int_{t_0}^{t_1} e_1(\chi_x(t))dt}, \frac{\int_{t_0}^{t_1} f_2(\chi_x(t))dt}{\int_{t_0}^{t_1} e_2(\chi_x(t))dt}, ..., \frac{\int_{t_0}^{t_1} f_p(\chi_x(t))dt}{\int_{t_0}^{t_1} e_p(\chi_x(t))dt} \right).$$

Lemma 3.1. *The feasible solution* $x^0(\cdot) \in \mathtt{F}(I)$ *is an efficient solution in (MFP) if and only if* $x^0(\cdot) \in \mathtt{F}(I)$ *is an optimal solution to problems* $P_l(x^0),\ l = \overline{1,p}$*,*

$$\min_{x(\cdot)} \frac{\int_{t_0}^{t_1} f_l(\chi_x(t))dt}{\int_{t_0}^{t_1} e_l(\chi_x(t))dt}$$

subject to

$$x(t_\varepsilon) = x_\varepsilon, \quad x^{(\beta)}(t_\varepsilon) = x_{\beta\varepsilon}, \quad \varepsilon = 0,1, \quad \beta = \overline{1,k-1}$$

$$g(\chi_x(t)) \leq 0, \quad h(\chi_x(t)) = 0, \quad (\forall) t \in I$$

$$\frac{\int_{t_0}^{t_1} f_j(\chi_x(t))dt}{\int_{t_0}^{t_1} e_j(\chi_x(t))dt} \leq \frac{\int_{t_0}^{t_1} f_j(\chi_{x^0}(t))dt}{\int_{t_0}^{t_1} e_j(\chi_{x^0}(t))dt}, \quad j = \overline{1,p},\ j \neq l.$$

Proof. " $\Longrightarrow$ " Let $x^0(\cdot) \in \mathtt{F}(I)$ be an efficient solution of (MFP) and let suppose there exists $k \in \{1,...,p\}$ such that $x^0(\cdot) \in \mathtt{F}(I)$ is not an optimal solution of the scalar problem $P_k(x^0)$. Therefore, there exists a function $y(\cdot) \in \mathtt{F}(I)$ such that

$$\frac{\int_{t_0}^{t_1} f_j(\chi_y(t))\,dt}{\int_{t_0}^{t_1} e_j(\chi_y(t))\,dt} \leq \frac{\int_{t_0}^{t_1} f_j(\chi_{x^0}(t))\,dt}{\int_{t_0}^{t_1} e_j(\chi_{x^0}(t))\,dt}, \quad j = \overline{1,p},\ j \neq k;$$

and

$$\frac{\int_{t_0}^{t_1} f_k(\chi_y(t))\,dt}{\int_{t_0}^{t_1} e_k(\chi_y(t))\,dt} < \frac{\int_{t_0}^{t_1} f_k(\chi_{x^0}(t))\,dt}{\int_{t_0}^{t_1} e_k(\chi_{x^0}(t))\,dt}.$$

This fact contradicts the efficiency of the function $x^0(\cdot) \in F(I)$ in (MFP). Therefore, the direct implication is proved.

" $\Longleftarrow$ " Consider $x^0(\cdot) \in F(I)$ an optimal solution of each scalar problem $P_l(x^0)$, $l = \overline{1,p}$. Assume that $x^0(\cdot) \in F(I)$ is not an efficient solution in (MFP). Consequently, there exists a function $y(\cdot) \in F(I)$ such that

$$\frac{\displaystyle\int_{t_0}^{t_1} f_j(\chi_y(t))\,dt}{\displaystyle\int_{t_0}^{t_1} e_j(\chi_y(t))\,dt} \leq \frac{\displaystyle\int_{t_0}^{t_1} f_j(\chi_{x^0}(t))\,dt}{\displaystyle\int_{t_0}^{t_1} e_j(\chi_{x^0}(t))\,dt}, \quad j = \overline{1,p}$$

and there exists $k \in \{1, ..., p\}$ such that

$$\frac{\displaystyle\int_{t_0}^{t_1} f_k(\chi_y(t))\,dt}{\displaystyle\int_{t_0}^{t_1} e_k(\chi_y(t))\,dt} < \frac{\displaystyle\int_{t_0}^{t_1} f_k(\chi_{x^0}(t))\,dt}{\displaystyle\int_{t_0}^{t_1} e_k(\chi_{x^0}(t))\,dt}.$$

But, the function $x^0(\cdot) \in F(I)$ minimizes the functional $\dfrac{\displaystyle\int_{t_0}^{t_1} f_k(\chi_x(t))\,dt}{\displaystyle\int_{t_0}^{t_1} e_k(\chi_x(t))\,dt}$ on the set of all feasible solutions of problem $P_k(x^0)$. The proof is complete.

Remark 3.1. Denoting $R_l^0 := \dfrac{\displaystyle\int_{t_0}^{t_1} f_l(\chi_{x^0}(t))dt}{\displaystyle\int_{t_0}^{t_1} e_l(\chi_{x^0}(t))dt} = \min_{x(\cdot)} \dfrac{\displaystyle\int_{t_0}^{t_1} f_l(\chi_x(t))dt}{\displaystyle\int_{t_0}^{t_1} e_l(\chi_x(t))dt}$, $l = \overline{1,p}$, the previous lemma can be rewritten under the next equivalent form:

The feasible solution $x^0(\cdot) \in F(I)$ is an efficient solution in (MFP) if and only if $x^0(\cdot) \in F(I)$ is an optimal solution to problems $P_l(x^0)$, $l = \overline{1,p}$,

$$\min_{x(\cdot)} \frac{\displaystyle\int_{t_0}^{t_1} f_l(\chi_x(t))dt}{\displaystyle\int_{t_0}^{t_1} e_l(\chi_x(t))dt} \quad [= R_l^0]$$

subject to

$$x(t_\varepsilon) = x_\varepsilon, \quad x^{(\beta)}(t_\varepsilon) = x_{\beta\varepsilon}, \quad \varepsilon = 0,1, \quad \beta = \overline{1,k-1}$$

$$g(\chi_x(t)) \leq 0, \quad h(\chi_x(t)) = 0, \quad (\forall) t \in I$$

$$\int_{t_0}^{t_1} \left[f_j(\chi_x(t)) - R_j^0 e_j(\chi_x(t)) \right] dt \leq 0, \quad j = \overline{1,p},\ j \neq l.$$

Next, we shall enunciate a Jagannathan-type lemma (see [3]).

Lemma 3.2. *The feasible solution $x^0(\cdot) \in F(I)$ is an optimal solution of $P_l(x^0)$, $l = \overline{1,p}$, if and only if $x^0(\cdot) \in F(I)$ is an optimal solution of $\boldsymbol{P}_l(x^0)$, $l = \overline{1,p}$,*

$$\min_{x(\cdot)} \int_{t_0}^{t_1} \left[f_l(\chi_x(t)) - R_l^0 e_l(\chi_x(t)) \right] dt$$

subject to

$$x(t_\varepsilon) = x_\varepsilon, \quad x^{(\beta)}(t_\varepsilon) = x_{\beta\varepsilon}, \quad \varepsilon = 0,1, \quad \beta = \overline{1,k-1}$$

$$g(\chi_x(t)) \leq 0, \quad h(\chi_x(t)) = 0, \quad (\forall) t \in I$$

$$\int_{t_0}^{t_1} \left[f_j(\chi_x(t)) - R_j^0 e_j(\chi_x(t)) \right] dt \leq 0, \quad j = \overline{1,p}, \ j \neq l.$$

Now, we have all the necessary ingredients to provide the following

Lemma 3.3. *Consider $l \in \{1,...,p\}$ fixed and $x^0(\cdot) \in F(I)$ an optimal solution of the scalar problem $\boldsymbol{P}_l(x^0)$. Then there exist the real scalars $\lambda_{jl} \geq 0$ and the piecewise smooth functions $p_l(t)$ and $q_l(t)$ such that*

$$\sum_{j=1}^{p} \lambda_{jl} \left[\frac{\partial f_j}{\partial x}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x}(\chi_{x^0}) \right] + p_l(t) \frac{\partial g}{\partial x}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x}(\chi_{x^0})$$

$$- \frac{d}{dt} \left\{ \sum_{j=1}^{p} \lambda_{jl} \left[\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(1)}}(\chi_{x^0}) \right] + p_l(t) \frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}) \right\}$$

$$+ ... + \ (-1)^k \frac{d^k}{dt^k} \left\{ \sum_{j=1}^{p} \lambda_{jl} \left[\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(k)}}(\chi_{x^0}) \right] \right\}$$

$$+ (-1)^k \frac{d^k}{dt^k} \left\{ p_l(t) \frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}) \right\} = 0$$

(higher order Euler-Lagrange ODEs)

$$p_l(t) g(\chi_{x^0}(t)) = 0, \quad p_l(t) \geq 0, \ (\forall) t \in I.$$

Proof. Define the real C^{k+1}-class functions, $\phi_j : I \times R^{n(k+1)} \to R$, $\phi_j(\chi_x(t)) \geq 0$, $j = \overline{1,p}$, $j \neq l$, as follows

$$G_j(x(t)) := \int_{t_0}^{t_1} \left[f_j\left(\chi_x(t)\right) - R_j^0 e_j\left(\chi_x(t)\right) + \phi_j(\chi_x(t)) \right] dt = 0.$$

Therefore, the scalar problem $\mathbf{P}_l(x^0)$, $l \in \{1,...,p\}$ fixed, is changed into

$$\max_{x(\cdot)} \int_{t_0}^{t_1} \left[f_l(\chi_x) - R_l^0 e_l(\chi_x) \right] dt$$

subject to

$$x \in F(I), \quad G_j(x) = 0$$

$$\phi_j(\chi_x) \geq 0, \quad j = \overline{1,p}, \quad j \neq l$$

or, equivalently,

$$\max_{x(\cdot)} \int_{t_0}^{t_1} \left\{ f_l\left(\chi_x\right) - R_l^0 e_l(\chi_x) + \sum_{j=1;\, j \neq l}^{p} \lambda_{jl} \left[f_j\left(\chi_x\right) - R_j^0 e_j(\chi_x) + \phi_j(\chi_x) \right] \right\} dt \qquad (1)$$

subject to

$$x(t_\varepsilon) = x_\varepsilon, \quad x^{(\beta)}(t_\varepsilon) = x_{\beta\varepsilon}, \quad \varepsilon \in \{0,1\}, \quad \beta \in \{1, ..., k-1\}$$
$$g(\chi_x) \leq 0, \quad h(\chi_x) = 0, \quad (\forall) t \in I$$
$$-\phi_j(\chi_x) \leq 0, \quad j = \overline{1,p}, \quad j \neq l.$$

Consider

$$V_l(\chi_x, p_l, q_l, \gamma_l, a_j)$$
$$= \gamma_l \left\{ f_l(\chi_x) - R_l^0 e_l(\chi_x) + \sum_{j=1;\, j\neq l}^{p} \lambda_{jl} \left[f_j(\chi_x) - R_j^0 e_j(\chi_x) + \phi_j(\chi_x) \right] \right\}$$
$$+ p_l(t) g(\chi_x) + q_l(t) h(\chi_x) - \sum_{j=1;\, j\neq l}^{p} a_j(t) \phi_j(\chi_x),$$

where $\gamma_l \in R_+$, and $p_l \colon I \to R_+^m$, $q_l \colon I \to R_+^r$, $a_j \colon I \to R_+$, $j = \overline{1,p}$, $j \neq l$, are piecewise smooth functions. The function x^0 being an optimal solution for (1), the following Valentine's necessary conditions (see [13]) are fulfilled

$$\frac{\partial V_l}{\partial x}(\chi_{x^0}, p_l, q_l, \gamma_l, a_j) - \frac{d}{dt}\frac{\partial V_l}{\partial x^{(1)}}(\chi_{x^0}, p_l, q_l, \gamma_l, a_j)$$
$$+ ... + (-1)^k \frac{d^k}{dt^k}\frac{\partial V_l}{\partial x^{(k)}}(\chi_{x^0}, p_l, q_l, \gamma_l, a_j) = 0$$
$$p_l(t) g(\chi_{x^0}(t)) = 0, \quad p_l(t) \geq 0, \quad t \in I$$
$$a_j(t) \phi_j(\chi_{x^0}(t)) = 0, \quad a_j(t) \geq 0, \quad j = \overline{1,p}, \quad j \neq l$$
$$\gamma_l \geq 0, \quad \lambda_{jl} \geq 0, \quad j = \overline{1,p}, \quad j \neq l.$$

Concretely, we have

$$\gamma_l \frac{\partial f_l}{\partial x}(\chi_{x^0}) - \gamma_l R_l^0 \frac{\partial e_l}{\partial x}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \left[\frac{\partial f_j}{\partial x}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x}(\chi_{x^0}) + \frac{\partial \phi_j}{\partial x}(\chi_{x^0}) \right]$$
$$+ p_l(t) \frac{\partial g}{\partial x}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x}(\chi_{x^0}) - \sum_{j=1;\, j\neq l}^{p} a_j(t) \frac{\partial \phi_j}{\partial x}(\chi_{x^0})$$
$$- \frac{d}{dt}\left\{ \gamma_l \frac{\partial f_l}{\partial x^{(1)}}(\chi_{x^0}) - \gamma_l R_l^0 \frac{\partial e_l}{\partial x^{(1)}}(\chi_{x^0}) \right\}$$
$$- \frac{d}{dt}\left\{ \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \left[\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(1)}}(\chi_{x^0}) + \frac{\partial \phi_j}{\partial x^{(1)}}(\chi_{x^0}) \right] \right\}$$
$$- \frac{d}{dt}\left\{ p_l(t) \frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}) - \sum_{j=1;\, j\neq l}^{p} a_j(t) \frac{\partial \phi_j}{\partial x^{(1)}}(\chi_{x^0}) \right\}$$
$$+ ... + (-1)^k \frac{d^k}{dt^k}\left\{ \gamma_l \frac{\partial f_l}{\partial x^{(k)}}(\chi_{x^0}) - \gamma_l R_l^0 \frac{\partial e_l}{\partial x^{(k)}}(\chi_{x^0}) \right\}$$
$$+ (-1)^k \frac{d^k}{dt^k}\left\{ \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \left[\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(k)}}(\chi_{x^0}) + \frac{\partial \phi_j}{\partial x^{(k)}}(\chi_{x^0}) \right] \right\}$$
$$+ (-1)^k \frac{d^k}{dt^k}\left\{ p_l(t) \frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}) - \sum_{j=1;\, j\neq l}^{p} a_j(t) \frac{\partial \phi_j}{\partial x^{(k)}}(\chi_{x^0}) \right\} = 0$$

or, equivalently,

$$\gamma_l \frac{\partial f_l}{\partial x}(\chi_{x^0}) - \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} R_j^0 \frac{\partial e_j}{\partial x}(\chi_{x^0}) - \gamma_l R_l^0 \frac{\partial e_l}{\partial x}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \frac{\partial f_j}{\partial x}(\chi_{x^0}) \tag{2}$$

$$+ \sum_{j=1;\, j\neq l}^{p} [\gamma_l \lambda_{jl} - a_j(t)] \frac{\partial \phi_j}{\partial x}(\chi_{x^0}) + p_l(t) \frac{\partial g}{\partial x}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x}(\chi_{x^0})$$

$$- \frac{d}{dt}\left\{ \gamma_l \frac{\partial f_l}{\partial x^{(1)}}(\chi_{x^0}) - \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} R_j^0 \frac{\partial e_j}{\partial x^{(1)}}(\chi_{x^0}) - \gamma_l R_l^0 \frac{\partial e_l}{\partial x^{(1)}}(\chi_{x^0}) \right\}$$

$$- \frac{d}{dt}\left\{ \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} [\gamma_l \lambda_{jl} - a_j(t)] \frac{\partial \phi_j}{\partial x^{(1)}}(\chi_{x^0}) \right\}$$

$$- \frac{d}{dt}\left\{ p_l(t) \frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}) \right\}$$

$$+ ... + (-1)^k \frac{d^k}{dt^k}\left\{ \gamma_l \frac{\partial f_l}{\partial x^{(k)}}(\chi_{x^0}) - \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} R_j^0 \frac{\partial e_j}{\partial x^{(k)}}(\chi_{x^0}) - \gamma_l R_l^0 \frac{\partial e_l}{\partial x^{(k)}}(\chi_{x^0}) \right\}$$

$$+ (-1)^k \frac{d^k}{dt^k}\left\{ \sum_{j=1;\, j\neq l}^{p} \gamma_l \lambda_{jl} \frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}) + \sum_{j=1;\, j\neq l}^{p} [\gamma_l \lambda_{jl} - a_j(t)] \frac{\partial \phi_j}{\partial x^{(k)}}(\chi_{x^0}) \right\}$$

$$+ (-1)^k \frac{d^k}{dt^k} \frac{d}{dt}\left\{ p_l(t) \frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}) \right\} = 0.$$

We impose the following conditions: $\gamma_l \lambda_{jl} - a_j(t) = 0,\ j = \overline{1,p},\ j \neq l$, for any $t \in I$, $\gamma_l = \lambda_{ll} \geq 0,\ \lambda_{jl} = \gamma_l \lambda_{jl} \geq 0,\ j = \overline{1,p},\ j \neq l$. Rewriting (2), we obtain

$$\sum_{j=1}^{p} \lambda_{jl} \left[\frac{\partial f_j}{\partial x}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x}(\chi_{x^0}) \right] + p_l(t) \frac{\partial g}{\partial x}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x}(\chi_{x^0})$$

$$- \frac{d}{dt}\left\{ \sum_{j=1}^{p} \lambda_{jl} \left[\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(1)}}(\chi_{x^0}) \right] + p_l(t) \frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) \right\}$$

$$- \frac{d}{dt}\left\{ q_l(t) \frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}) \right\} + ... + (-1)^k \frac{d^k}{dt^k}\left\{ \sum_{j=1}^{p} \lambda_{jl} \left[\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(k)}}(\chi_{x^0}) \right] \right\}$$

$$+ (-1)^k \frac{d^k}{dt^k}\left\{ p_l(t) \frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + q_l(t) \frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}) \right\} = 0$$

and the proof is complete.

Theorem 3.1. *The feasible solution $x^0(\cdot) \in F(I)$ is an efficient solution to (MFP) if and only if it is an optimal solution of each scalar problem $\boldsymbol{P}_l(x^0),\ l = \overline{1,p}$.*

Proof. Using Lemmas 3.1 and 3.2, the proof is immediately.

Definition 3.2. The feasible solution $x^0(\cdot) \in F(I)$ is a *normal efficient solution* of the problem (MFP) if it is a normal optimal solution for at least one of the scalar problems $\mathbf{P}_l(x^0),\ l = \overline{1,p}$.

Let establish one of the main results of this section, that is, the normal necessary efficiency conditions of the multiobjetive fractional variational program (MFP).

Theorem 3.2. ([Normal] necessary efficiency conditions for (MFP)) *Let* $x^0(\cdot) \in \mathcal{F}(I)$ *be a [normal] efficient solution of the fractional program (MFP). Then there are* $\lambda \in R^p$, $p\colon I \to R^m$ *and* $q\colon I \to R^r$ *satisfying the following conditions:*

$$\sum_{j=1}^{p} \lambda_j \left[\frac{\partial f_j}{\partial x}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x}(\chi_{x^0}) \right] + p(t)\frac{\partial g}{\partial x}(\chi_{x^0}) + q(t)\frac{\partial h}{\partial x}(\chi_{x^0})$$

$$-\frac{d}{dt}\left\{ \sum_{j=1}^{p} \lambda_j \left[\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(1)}}(\chi_{x^0}) \right] + p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}) \right\}$$

$$+ ... + \ (-1)^k \frac{d^k}{dt^k}\left\{ \sum_{j=1}^{p} \lambda_j \left[\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(k)}}(\chi_{x^0}) \right] \right\}$$

$$+ (-1)^k \frac{d^k}{dt^k}\left\{ p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) + q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}) \right\} = 0$$

(higher order Euler-Lagrange ODEs)

$$p(t)g(\chi_{x^0}(t)) = 0, \quad p(t) \geq 0, \quad (\forall)t \in I$$

$$\lambda \geq 0, \quad e^t\lambda = 1, \quad e^t = (1, 1, ..., 1) \in R^p.$$

Proof. Taking into account Theorem 3.1, we get that $x^0(\cdot) \in \mathcal{F}(I)$ is an optimal solution of each problem $\mathbf{P}_l(x^0)$, $l = \overline{1,p}$. Therefore, if $x^0(\cdot) \in \mathcal{F}(I)$ is [normal] optimal solution in $\mathbf{P}_l(x^0)$, $l \in \{1, ..., p\}$ fixed, then the relations which appear in Lemma 3.3 are true $[\lambda_{ll} = 1]$. Making summation over $l = \overline{1,p}$ of all relations in Lemma 3.3 and setting

$$\sum_{l=1}^{p} \lambda_{jl} = \tilde{\lambda}_j, \quad \sum_{l=1}^{p} p_l(t) = \tilde{p}(t), \quad \sum_{l=1}^{p} q_l(t) = \tilde{q}(t),$$

we get the following relations

$$\sum_{j=1}^{p} \tilde{\lambda}_j \left[\frac{\partial f_j}{\partial x}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x}(\chi_{x^0}) \right] + \tilde{p}(t)\frac{\partial g}{\partial x}(\chi_{x^0}) + \tilde{q}(t)\frac{\partial h}{\partial x}(\chi_{x^0})$$

$$-\frac{d}{dt}\left\{ \sum_{j=1}^{p} \tilde{\lambda}_j \left[\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(1)}}(\chi_{x^0}) \right] + \tilde{p}(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}) + \tilde{q}(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}) \right\}$$

$$+ ... + \ (-1)^k \frac{d^k}{dt^k}\left\{ \sum_{j=1}^{p} \tilde{\lambda}_j \left[\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x^{(k)}}(\chi_{x^0}) \right] + \tilde{p}(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}) \right\}$$

$$+ (-1)^k \frac{d^k}{dt^k}\left\{ \tilde{q}(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}) \right\} = 0$$

$$\tilde{p}(t)g(\chi_{x^0}(t)) = 0, \quad \tilde{p}(t) \geq 0, \quad \tilde{\lambda}_j \geq 0, \quad (\forall)t \in I, \quad [\tilde{\lambda}_j \geq 1].$$

By dividing with $S = \sum_{j=1}^{p} \tilde{\lambda}_j \geq 1$ and denoting $\lambda_j = \tilde{\lambda}_j/S$, $p(t) = \tilde{p}(t)/S$, $q(t) = \tilde{q}(t)/S$, we obtain

$$\sum_{j=1}^{p} \lambda_j \left[\frac{\partial f_j}{\partial x}(\chi_{x^0}) - R_j^0 \frac{\partial e_j}{\partial x}(\chi_{x^0}) \right] + p(t)\frac{\partial g}{\partial x}(\chi_{x^0}) + q(t)\frac{\partial h}{\partial x}(\chi_{x^0})$$

$$-\frac{d}{dt}\left\{\sum_{j=1}^{p}\lambda_j\left[\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0})-R_j^0\frac{\partial e_j}{\partial x^{(1)}}(\chi_{x^0})\right]+p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0})+q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0})\right\}$$

$$+...+\ (-1)^k\frac{d^k}{dt^k}\left\{\sum_{j=1}^{p}\lambda_j\left[\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0})-R_j^0\frac{\partial e_j}{\partial x^{(k)}}(\chi_{x^0})\right]\right\}$$

$$+(-1)^k\frac{d^k}{dt^k}\left\{p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0})+q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0})\right\}=0$$

(higher order Euler-Lagrange ODEs)

$$p(t)g(\chi_{x^0}(t))=0,\quad p(t)\geq 0,\quad (\forall)t\in I$$

$$\lambda\geq 0,\quad e^t\lambda=1,\quad e^t=(1,1,...,1)\in R^p$$

and the proof is complete.

The previous theorem represents the corresponding result to Theorem 2.1 in Section 2. We have (see Section 1)

$$F_l\left(x^0(t)\right):=\int_{t_0}^{t_1}f_l(\chi_{x^0}(t))dt,\quad E_l\left(x^0(t)\right):=\int_{t_0}^{t_1}e_l(\chi_{x^0}(t))dt>0$$

and, using Remark 3.1, we get $R_l^0=\dfrac{\int_{t_0}^{t_1}f_l(\chi_{x^0}(t))dt}{\int_{t_0}^{t_1}e_l(\chi_{x^0}(t))dt}=\dfrac{F_l\left(x^0(t)\right)}{E_l\left(x^0(t)\right)},\ l=\overline{1,p}$. By replacing the above given numbers R_l^0, $l=\overline{1,p}$, and redefining the functions p and q in Theorem 3.2, we obtain the following result:

Theorem 3.3. ([Normal] necessary efficiency conditions for (MFP)) *Consider* $x^0(\cdot)\in \mathtt{F}(I)$ *a [normal] efficient solution in (MFP). Then there exist* $\lambda\in R^p$, $p\colon I\to R^m$ *and* $q\colon I\to R^r$ *satisfying the following conditions:*

$$\sum_{j=1}^{p}\lambda_j\left[E_j\left(x^0\right)\frac{\partial f_j}{\partial x}(\chi_{x^0})-F_j\left(x^0\right)\frac{\partial e_j}{\partial x}(\chi_{x^0})\right]+p(t)\frac{\partial g}{\partial x}(\chi_{x^0})+q(t)\frac{\partial h}{\partial x}(\chi_{x^0})$$

$$-\frac{d}{dt}\left\{\sum_{j=1}^{p}\lambda_j\left[E_j\left(x^0\right)\frac{\partial f_j}{\partial x^{(1)}}(\chi_{x^0})-F_j\left(x^0\right)\frac{\partial e_j}{\partial x^{(1)}}(\chi_{x^0})\right]+p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0})\right\}$$

$$-\frac{d}{dt}\left\{q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0})\right\}$$

$$+...+\ (-1)^k\frac{d^k}{dt^k}\left\{\sum_{j=1}^{p}\lambda_j\left[E_j\left(x^0\right)\frac{\partial f_j}{\partial x^{(k)}}(\chi_{x^0})-F_j\left(x^0\right)\frac{\partial e_j}{\partial x^{(k)}}(\chi_{x^0})\right]\right\}$$

$$+(-1)^k\frac{d^k}{dt^k}\left\{p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0})+q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0})\right\}=0$$

(higher order Euler-Lagrange ODEs)

$$p(t)g(\chi_{x^0}(t)) = 0, \quad p(t) \geq 0, \quad (\forall)t \in I$$
$$\lambda \geq 0, \quad e^t\lambda = 1, \quad e^t = (1, 1, ..., 1) \in R^p.$$

In order to obtain sufficient efficiency conditions for the multiobjective fractional variational problem (MFP) we shall use the (ρ, b)-quasiinvexity notion that we recalled in the previous section (see Definition 2.1). The corresponding theorem to Theorem 2.2 is the following

Theorem 3.4. (Sufficient efficiency conditions for (MFP)) *Assume that Theorem* 3.2 *is fulfilled and there exist the vector functions* η *and* θ *satisfying Definition* 2.1. *Also, consider that the following statements are true:*

a) the functionals $\int_{t_0}^{t_1} \left[f_l(\chi_x(t)) - R_l^0 e_l(\chi_x(t))\right] dt,\ l \in \{1, ..., p\}$, *are* (ρ_l^1, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

b) $\int_{t_0}^{t_1} p(t)g(\chi_x(t))dt$ *is* (ρ^2, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

c) $\int_{t_0}^{t_1} q(t)h(\chi_x(t))dt$ *is* (ρ^3, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

d) one of the integrals $\int_{t_0}^{t_1} \left[f_l(\chi_x(t)) - R_l^0 e_l(\chi_x(t))\right] dt,\ l \in \{1, ..., p\}$, $\int_{t_0}^{t_1} p(t)g(\chi_x(t))dt$, $\int_{t_0}^{t_1} q(t)h(\chi_x(t))dt$ *is strictly* (ρ, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ; $(\rho = \rho_l^1, \rho^2$ *or* ρ^3, *respectively)*

e) $\sum_{l=1}^{p} \lambda_l\rho_l^1 + \rho^2 + \rho^3 \geq 0 \quad (\rho_l^1, \rho^2, \rho^3 \in R)$.

Then $x^0(\cdot)$ *is an efficient solution for (MFP).*

Proof. The proof follows in the same manner as in Theorem 2.2. The functionals $\int_{t_0}^{t_1} f_l(\chi_x(t))dt$ are replaced by $\int_{t_0}^{t_1} \left[f_l(\chi_x(t)) - R_l^0 e_l(\chi_x(t))\right] dt,\ l \in \{1, ..., p\}$.

Let assume that $x^0(\cdot)$ is not an efficient solution in (MFP). Taking into account the hypotheses a) (multiplied by $\lambda_l \geq 0$ and making summation over $l = \overline{1,p}$), b) and c) we get three similar relations as in Definition 2.1. Making the sum, side by side, of the three relations previously obtained and applying d), e) and the formula of integration by parts, we get

$$\int_{t_0}^{t_1} \eta_{txx^0} \left[\lambda\frac{\partial\varphi}{\partial x}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x}(\chi_{x^0}(t)) + q(t)\frac{\partial h}{\partial x}(\chi_{x^0}(t))\right] dt$$
$$+\eta_{txx^0} \left[\lambda\frac{\partial\varphi}{\partial x^{(1)}}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t)) + q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t))\right] |_{t_0}^{t_1}$$
$$-\int_{t_0}^{t_1} \eta_{txx^0}\frac{d}{dt} \left[\lambda\frac{\partial\varphi}{\partial x^{(1)}}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x^{(1)}}(\chi_{x^0}(t)) + q(t)\frac{\partial h}{\partial x^{(1)}}(\chi_{x^0}(t))\right] dt$$
$$+ ... + (-1)^k \int_{t_0}^{t_1} \eta_{txx^0}\frac{d^k}{dt^k} \left[\lambda\frac{\partial\varphi}{\partial x^{(k)}}(\chi_{x^0}(t)) + p(t)\frac{\partial g}{\partial x^{(k)}}(\chi_{x^0}(t))\right] dt$$
$$+(-1)^k \int_{t_0}^{t_1} \eta_{txx^0}\frac{d^k}{dt^k} \left[q(t)\frac{\partial h}{\partial x^{(k)}}(\chi_{x^0}(t))\right] dt < -\left(\sum_{l=1}^{p} \lambda_l\rho_l^1 + \rho^2 + \rho^3\right) \parallel \theta_{xx^0} \parallel^2$$

where $\varphi_l(\chi_x(t)) := f_l(\chi_x(t)) - R_l^0 e_l(\chi_x(t))$.

Considering the boundary conditions $x(t_\varepsilon) = x_\varepsilon$, $x^{(\beta)}(t_\varepsilon) = x_{\beta\varepsilon}$, $\varepsilon = 0,1$, $\beta = \overline{1,k-1}$ (see $x(t_\varepsilon) = x_\varepsilon = x^0(t_\varepsilon)$, $x^{(\beta)}(t_\varepsilon) = x_{\beta\varepsilon} = x^{0(\beta)}(t_\varepsilon)$), and knowing that $\dfrac{d^\zeta \eta_{tx^0x^0}}{dt^\zeta} = 0$, $\zeta \in \{0,1,2,...,k-1\}$, $(\forall)\ t \in I$ (see Definition 2.1), and applying Theorem 3.2, we get

$$0 < -\left(\sum_{l=1}^{p} \lambda_l \rho_l^1 + \rho^2 + \rho^3\right) \parallel \theta_{xx^0} \parallel^2 .$$

According to hypothesis e) and $\parallel \theta_{xx^0} \parallel^2 \geq 0$, we find a contradiction. Therefore, the point x^0 is an efficient solution to (MFP). The proof is complete.

Theorem 3.5. (Sufficient efficiency conditions for (MFP)) *Assume that Theorem* 3.3 *is fulfilled and there exist the vector functions* η *and* θ *satisfying Definition* 2.1. *Also, consider that the following statements are true:*

a) the functionals $\displaystyle\int_{t_0}^{t_1} \left[E_l\left(x^0(t)\right) f_l(\chi_x(t)) - F_l\left(x^0(t)\right) e_l(\chi_x(t))\right] dt$, $l = \overline{1,p}$, *are* (ρ_l^1, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

b) $\displaystyle\int_{t_0}^{t_1} p(t)g(\chi_x(t))dt$ *is* (ρ^2, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

c) $\displaystyle\int_{t_0}^{t_1} q(t)h(\chi_x(t))dt$ *is* (ρ^3, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

d) one of the integrals $\displaystyle\int_{t_0}^{t_1} \left[E_l\left(x^0(t)\right) f_l(\chi_x(t)) - F_l\left(x^0(t)\right) e_l(\chi_x(t))\right] dt$, $l \in \{1,...,p\}$, $\displaystyle\int_{t_0}^{t_1} p(t)g(\chi_x(t))dt$, $\displaystyle\int_{t_0}^{t_1} q(t)h(\chi_x(t))dt$ *is strictly* (ρ, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;* $(\rho = \rho_l^1, \rho^2$ *or* ρ^3*, respectively)*

e) $\displaystyle\sum_{l=1}^{p} \lambda_l \rho_l^1 + \rho^2 + \rho^3 \geq 0 \quad (\rho_l^1, \rho^2, \rho^3 \in R)$.

Then $x^0(\cdot)$ *is an efficient solution for (MFP).*

Proof. The proof follows in the same manner as in Theorem 2.2. The functionals $\displaystyle\int_{t_0}^{t_1} f_l(\chi_x(t))dt$ are replaced by $\displaystyle\int_{t_0}^{t_1} \left[E_l\left(x^0(t)\right) f_l(\chi_x(t)) - F_l\left(x^0(t)\right) e_l(\chi_x(t))\right] dt$, $l = \overline{1,p}$.

Corollary 3.1. (Sufficient efficiency conditions for (MFP)) *Let suppose that Theorem* 3.2 *is fulfilled and there exist the vector functions* η *and* θ *satisfying Definition* 2.1. *Also, consider that the following statements are true:*

a) the functionals $\displaystyle\int_{t_0}^{t_1} \left[f_l(\chi_x(t)) - R_l^0 e_l(\chi_x(t))\right] dt$, $l = \overline{1,p}$, *are* (ρ_l^1, b)*-quasi-invex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

b) $\displaystyle\int_{t_0}^{t_1} \left[p(t)g(\chi_x(t)) + q(t)h(\chi_x(t))\right] dt$ *is* (ρ^2, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;*

c) one of the integrals $\displaystyle\int_{t_0}^{t_1} \left[f_l(\chi_x(t)) - R_l^0 e_l(\chi_x(t))\right] dt$, $l \in \{1,...,p\}$, $\displaystyle\int_{t_0}^{t_1} \left[p(t)g(\chi_x(t)) + q(t)h(\chi_x(t))\right] dt$ *is strictly* (ρ, b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ*;* $(\rho = \rho_l^1$ *or* ρ^2*, respectively)*

d) $\displaystyle\sum_{l=1}^{p} \lambda_l \rho_l^1 + \rho^2 \geq 0 \quad (\rho_l^1, \rho^2 \in R)$.

Then $x^0(\cdot)$ *is an efficient solution for (MFP).*

Corollary 3.2. (Sufficient efficiency conditions for (MFP)) *Assume that Theorem* 3.3 *is fulfilled and there exist the vector functions* η *and* θ *satisfying Definition* 2.1. *Also, we suppose that the following statements are true:*

a) the functionals $\int_{t_0}^{t_1}\left[E_l\left(x^0(t)\right)f_l(\chi_x(t))-F_l\left(x^0(t)\right)e_l(\chi_x(t))\right]dt$, $l=\overline{1,p}$, *are* (ρ_l^1,b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ;

b) $\int_{t_0}^{t_1}\left[p(t)g(\chi_x(t))+q(t)h(\chi_x(t))\right]dt$ *is* (ρ^2,b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ;

c) one of the integrals $\int_{t_0}^{t_1}\left[E_l\left(x^0(t)\right)f_l(\chi_x(t))-F_l\left(x^0(t)\right)e_l(\chi_x(t))\right]dt$, $l\in\{1,...,p\}$, $\int_{t_0}^{t_1}\left[p(t)g(\chi_x(t))+q(t)h(\chi_x(t))\right]dt$ *is strictly* (ρ,b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ; $(\rho=\rho_l^1$ *or* ρ^2, *respectively*$)$

d) $\sum_{l=1}^{p}\lambda_l\rho_l^1+\rho^2\geq 0 \quad (\rho_l^1,\rho^2\in R)$.

Then $x^0(\cdot)$ *is an efficient solution for (MFP).*

Remark 3.2. The hypotheses b) and c) in Theorems 3.4 and 3.5 are replaced by

$\int_{t_0}^{t_1}\left[p(t)g(\chi_x(t))+q(t)h(\chi_x(t))\right]dt$ *is* (ρ^2,b)*-quasiinvex at* $x^0(\cdot)$ *with respect to* η *and* θ

and, in this way, we have obtained the previous two corollaries.

4. **Conclusions**

We introduced and studied a class of single-time vector fractional variational problems involving higher order derivatives (see (MFP)). Within this framework, we formulated and proved necessary and sufficient efficiency conditions for a feasible solution in (MFP).

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