BEST PROXIMITY POINTS OF F-PROXIMAL CONTRACTIONS UNDER THE INFLUENCE OF AN α-FUNCTION

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In this paper, we introduce the notions of F-α-proximal contractions for Hardy-Rogers type mappings as well as for Ciric-type mappings. Then we discuss the existence of best proximity for nonself multivalued mappings satisfying at least one of these notions along with few other conditions. An example is also constructed to support the result.

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1. Introduction

With the introduction of Banach Contraction Principle to metric fixed poind theory, pure and applied mathematical research has taken new dimensions. Researchers around the world have done major developments in the field by generalizing this contraction principle.

Samet *et. al.* [1] introduced the notion of the α-ψ-contraction principle which generalized the Banach contraction in a different and unique way. With this break through the research, metric fixed point theory has widened to many different directions. The α-admissibility condition used by Samet, has been used quite frequently to discuss existence and uniqueness of fixed points by different researchers in different ways

Recently, Wardowski [2] introduced a new family of mappings so called F or $\mathfrak{F}$ family. Using the mappings from this family, he introduced a new contraction condition namely the F-contractions. Many researchers have generalized the concept of Wardowski [2], see for example: Ali *et al.* [3], Cosentino and Vetro [4], Kamran *et al.* [6], Minak *et al.* [7], Sgroi

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and Vetro [8], Paesano and Vetro [9], Piri and Kumam [10], Akar *et al.* [11], Batra and Vashistha [12].

Jleli *et al.* [13] introduced the notion of α-ψ-proximal contractive type mappings and proved some best proximity point theorems. Later on, Ali *et al.* [14] extended these notions to multivalued mappings. Many authors obtained best proximity point theorems in different settings, see for example: Abkar and Gabeleh [15, 16, 17], Choudhury *et al.* [5], Kamran *et al.* [18], Alghamdi [19], Al-Thagafi and Shahzad [20, 21], Derafshpour *et al.* [22], Di Bari *et al.* [23], Eldred and Veeramani [24], Jacob *et al.* [25], Markin and Shahzad [26], Rezapour *et al.* [27], Sadiq Basha [28], Shatanawi and Pitea [29], Vetro [30], Zhang [31]. Note that Abkar and Gabeleh [17] and Al-Thagafi and Shahzad [20, 21] investigated best proximity points for multi-valued mappings.

In this paper, we introduce the notions of F-α-proximal contractions for Hardy-Rogers type mappings as well as for Ciric-type mappings. Taking advantage of this framework, we discuss the existence of best proximity for nonself multivalued mappings satisfying at least one of these notions along with few other conditions. An example supports the result.

2. **Preliminaries**

First we recall the concept of F-contraction, see Wardowski [2]. In this respect, denote by $\mathfrak{F}$ the class of all functions $F\colon (0,\infty) \to \mathbb{R}$ satisfying:

(F_1) Function F is strictly increasing, that is, for each $a_1, a_2 \in (0,\infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$.

(F_2) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$.

(F_3) There exists $k \in (0,1)$ such that $\lim_{n\to\infty} \mathfrak{d}_\mathfrak{n}{}^k F(\mathfrak{d}_\mathfrak{n}) = 0$.

Following are some examples of such functions.

(i) $F(a) = \ln a$ for each $a \in (0,\infty)$.

(ii) $F(b) = b + \ln b$ for each $b \in (0,\infty)$.

(iii) $F(c) = -\frac{1}{\sqrt{c}}$ for each $c \in (0,\infty)$.

Wardowski [2] introduced F-contraction and proved corresponding fixed point theorem in the following way:

Definition 2.1 ([2]). *Let (X,d) be a metric space. A mapping $T\colon X \to X$ is F-contraction if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $x, y \in X$ with $d(Tx,Ty) > 0$, we have*

$$\tau + F(d(Tx,Ty)) \le F(d(x,y)).$$

Theorem 2.1 ([2]). *Let (X,d) be a complete metric space and let $T\colon X \to X$ is F-contraction. Then T has a unique fixed point.*

Minak *et al.* [7] generalized the above result in the following way:

Theorem 2.2 ([7]). *Let (X, d) be a complete metric space and let $T\colon X \to X$. Assume that there exists $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(d(Tx, Ty)) \leq F\Big(\max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\Big\}\Big),$$

for each $x, y \in X$ with $d(Tx, Ty) > 0$. If T or F is continuous, then T has a unique fixed point.

Sgroi and Vetro [8] gave the following generalization of [2].

Theorem 2.3 ([8]). *Let (X, d) be a complete metric space and let $T\colon X \to CB(X)$. Assume that there exists $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$2\tau + F(H(Tx, Ty)) \leq F(a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + L d(y, Tx)),$$

for each $x, y \in X$ with $Tx \neq Ty$, where $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$. Then T has a fixed point.

We make complete these preliminaries with other basic notions, definitions and results, which are necessary to state our results.

Let (X, d) be a metric space. For $A, B \subseteq X$, we use the notions:
$\operatorname{dist}(A, B) = \inf\{d(a, b) : \ a \in A, \ b \in B\}$, $\qquad d(x, B) = \inf\{d(x, b) : \ b \in B\}$,
$A_0 = \{a \in A : \ d(a, b) = \operatorname{dist}(A, B) \text{ for some } b \in B\}$,
$B_0 = \{b \in B : \ d(a, b) = \operatorname{dist}(A, B) \text{ for some } a \in A\}$.

If $CL(X)$ is the set of all nonempty closed subsets of X, then for every $A, B \in CL(X)$, let

$$H(A, B) = \begin{cases} \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases}$$

Such a map H is called generalized Hausdorff metric induced by d.

A point $x^* \in X$ is said to be a best proximity point of mapping $T\colon A \to CL(B)$ if $d(x^*, Tx^*) = \operatorname{dist}(A, B)$. When $A = B$, the best proximity point reduces to fixed point of the mapping T.

Definition 2.2 ([31]). *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,*

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{cases} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

Lemma 2.1. *Let (X, d) is a metric space and $B \in CL(X)$. Then for each $x \in X$ and $q > 1$, there exists $b \in B$ such that $d(x, b) \leq q d(x, B)$.*

3. Main results

We begin this section with the following definitions.

Definition 3.1. *Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T\colon A \to CL(X)$ is called strictly α-proximal admissible if there exists a mapping $\alpha\colon A \times A \to [0, \infty)$ such that*

$$\begin{cases} \alpha(x_1, x_2) > 1 \\ d(u_1, y_1) = \text{dist}(A, B) \\ d(u_2, y_2) = \text{dist}(A, B) \end{cases} \Rightarrow \alpha(u_1, u_2) > 1,$$

where $x_1, x_2, u_1, u_2 \in A$ and $y_1 \in Tx_1$, $y_2 \in Tx_2$.

Definition 3.2. *Let (X, d) be a metric space, $A, B \subseteq X$ and $\alpha\colon A \times A \to [0, \infty)$ be a function. A mapping $T\colon A \to CL(B)$ is F-α-proximal contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(N(x, y)), \tag{1}$$

for each $x, y \in A$, whenever $\min\{\alpha(x, y)H(Tx, Ty), N(x, y)\} > 0$, where

$$\begin{aligned} N(x, y) \quad = \quad & a_1 d(x, y) + a_2[d(x, Tx) - \text{dist}(A, B)] + a_3[d(y, Ty) - \text{dist}(A, B)] \\ & + a_4[d(x, Ty) - \text{dist}(A, B)] + L[d(y, Tx) - \text{dist}(A, B)], \end{aligned}$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 3.1. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T\colon A \to CL(B)$ is an F-α-proximal contraction of Hardy-Rogers-type satisfying the following conditions:*

- *(i) $Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P-property;*
- *(ii) T is strictly α-proximal admissible;*
- *(iii) there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$\alpha(x_0, x_1) > 1 \quad \textit{and} \quad d(x_1, y_1) = dist(A, B).$$

- *(iv) T is continuous, or, for any sequence $\{x_n\} \subseteq A$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.*

Then T has a best proximity point.

Proof. By hypothesis (iii), there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = \text{dist}(A, B) \text{ and } \alpha(x_0, x_1) > 1. \tag{2}$$

If $y_1 \in Tx_1$, then x_1 is a best proximity point of T.

Let $y_1 \notin Tx_1$. As $\alpha(x_0, x_1) > 1$, by Lemma 2.1 there exists $y_2 \in Tx_1$ such that

$$d(y_1, y_2) \leq \alpha(x_0, x_1)H(Tx_0, Tx_1).$$

Since F is strictly increasing, we have

$$F(d(y_1, y_2)) \leq F(\alpha(x_0, x_1)H(Tx_0, Tx_1)).$$

From (1), we have

$$\begin{aligned}
\tau + F(d(y_1, y_2)) &\leq \tau + F(\alpha(x_0, x_1)H(Tx_0, Tx_1)) \\
&\leq F\Big(a_1 d(x_0, x_1) + a_2[d(x_0, Tx_0) - \text{dist}(A, B)] + a_3[d(x_1, Tx_1) - \text{dist}(A, B)], \\
&\quad a_4[d(x_0, Tx_1) - \text{dist}(A, B)] + L[d(x_1, Tx_0) - \text{dist}(A, B)]\Big) \\
&\leq F\Big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(y_1, y_2), \\
&\quad a_4[d(x_0, x_1) + d(y_1, y_2)] + L \cdot 0\Big) \\
&= F\Big((a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(y_1, y_2)\Big). \qquad (3)
\end{aligned}$$

Since

$$d(x_0, Tx_0) \leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, Tx_0) = d(x_0, x_1) + \text{dist}(A, B) + 0,$$

$$d(x_1, Tx_1) \leq d(x_1, y_1) + d(y_1, y_2) + d(y_2, Tx_1) = \text{dist}(A, B) + d(y_1, y_2) + 0,$$

$$d(x_0, Tx_1) \leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, y_2) + d(y_2, Tx_1) = d(x_0, x_1) + \text{dist}(A, B) + d(y_1, y_2) + 0$$

$$d(x_1, Tx_0) \leq d(x_1, y_1) + d(y_1, Tx_0) = \text{dist}(A, B) + 0.$$

Since F is strictly increasing, we get from (3) that

$$d(y_1, y_2) < (a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(y_1, y_2).$$

That is,

$$(1 - a_3 - a_4)d(y_1, y_2) < (a_1 + a_2 + a_4)d(x_0, x_1).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(y_1, y_2) < d(x_0, x_1).$$

Now, from (3), we have

$$\tau + F(d(y_1, y_2)) \leq F(d(x_0, x_1)). \qquad (4)$$

As $y_2 \in Tx_1 \subseteq B_0$, there exists $x_2 \neq x_1 \in A_0$ such that

$$d(x_2, y_2) = \text{dist}(A, B), \qquad (5)$$

for otherwise x_1 is a best proximity point. As (A, B) satisfies the weak P-property, from (2) and (5), we have

$$0 < d(x_1, x_2) \leq d(y_1, y_2).$$

By applying F, we get

$$F(d(x_1, x_2)) \leq F(d(y_1, y_2)). \qquad (6)$$

Thus from (4) and (6), we have

$$\tau + F(d(x_1, x_2)) \leq \tau + F(d(y_1, y_2)) \leq F(d(x_0, x_1)).$$

As T is strictly α-proximal admissible, since $\alpha(x_0, x_1) > 1$, $d(x_1, y_1) = \text{dist}(A, B)$ and $d(x_2, y_2) = \text{dist}(A, B)$, then $\alpha(x_1, x_2) > 1$. Thus we have

$$d(x_2, y_2) = \text{dist}(A, B) \text{ and } \alpha(x_1, x_2) > 1.$$

If $y_2 \in Tx_2$, then x_2 is a best proximal point of T. Let $y_2 \notin Tx_2$. As $\alpha(x_1, x_2) > 1$. There exists $y_3 \in Tx_2$ such that

$$d(y_2, y_3) \leq \alpha(x_1, x_2)H(Tx_1, Tx_2).$$

Since, F is strictly increasing, we have

$$F(d(y_2, y_3)) \leq F(\alpha(x_1, x_2)H(Tx_1, Tx_2)).$$

From (1), we have

$$\begin{aligned} \tau + F(d(y_2, y_3)) &\leq \tau + F(\alpha(x_1, x_2)H(Tx_1, Tx_2)) \\ &\leq F\Big(a_1 d(x_1, x_2) + a_2[d(x_1, Tx_1) - \text{dist}(A, B)] + a_3[d(x_2, Tx_2) - \text{dist}(A, B)], \\ &\quad a_4[d(x_1, Tx_2) - \text{dist}(A, B)] + L[d(x_2, Tx_1) - \text{dist}(A, B)]\Big) \\ &\leq F\Big(a_1 d(x_1, x_2) + a_2 d(x_1, x_2) + a_3 d(y_2, y_3), \\ &\quad a_4[d(x_1, x_2) + d(y_2, y_3)] + L \cdot 0\Big) \\ &= F\Big((a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(y_2, y_3)\Big). \end{aligned} \tag{7}$$

Since F is strictly increasing, we get from above that

$$d(y_2, y_3) < (a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(y_2, y_3).$$

That is,

$$(1 - a_3 - a_4)d(y_2, y_3) < (a_1 + a_2 + a_4)d(x_1, x_2).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(y_2, y_3) < d(x_1, x_2).$$

Now from (7), we have

$$\tau + F(d(y_2, y_3)) \leq F(d(x_1, x_2)).$$

As $y_3 \in Tx_2 \subseteq B_0$, there exists $x_3 \neq x_2 \in A_0$ such that

$$d(x_3, y_3) = \text{dist}(A, B), \tag{8}$$

for otherwise x_2 is a best proximity point. As (A, B) satisfies the weak P-property. From (5) and (8), we have

$$0 < d(x_2, x_3) \leq d(y_2, y_3).$$

By applying F, we obtain

$$F(d(x_2, x_3)) \leq F(d(y_2, y_3)).$$

Thus, we have

$$\tau + F(d(x_2, x_3)) \leq \tau + F(d(y_2, y_3)) \leq F(d(x_1, x_2)).$$

So we get

$$F(d(x_2, x_3)) \leq F(d(y_2, y_3)) \leq F(d(x_1, x_2)) - \tau \leq F(d(x_0, x_1)) - 2\tau.$$

As T is strictly α-proximal admissible, since $\alpha(x_1, x_2) > 1$, $d(x_2, y_2) = \text{dist}(A, B)$ and $d(x_3, y_3) = \text{dist}(A, B)$, then $\alpha(x_2, x_3) > 1$.

Continuing in the same way, we get sequences $\{x_n\}$ in A_0 and $\{y_n\}$ in B_0, where $y_n \in Tx_{n-1}$ for each $n \in \mathbb{N}$ such that

$$d(x_n, y_n) = \text{dist}(A, B) \text{ and } \alpha(x_{n-1}, x_n) > 1. \tag{9}$$

Furthermore,

$$F(d(x_n, x_{n+1})) \leq F(d(y_n, y_{n+1})) \leq F(d(x_0, x_1)) - n\tau \text{ for each } n \in \mathbb{N}. \tag{10}$$

Letting $n \to \infty$ in (10), we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = \lim_{n\to\infty} F(d(y_n, y_{n+1})) - \infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (10) we have

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \tag{11}$$

Letting $n \to \infty$ in (11), we get

$$\lim_{n\to\infty} n d_n^k = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n d_n^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1. \tag{12}$$

To prove that $\{x_n\}$ is a Cauchy sequence in A, consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangle inequality and (12), we obtain

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series, then $\lim_{n\to\infty} d(x_n, x_m) = 0$, which implies that $\{x_n\}$ is a Cauchy sequence in A. Similarly, we see that $\{y_n\}$ is a Cauchy sequence in B. Since A and B are closed subsets of a complete metric space, there exist x^* in A and y^* in B such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. By the (9), we conclude that $d(x^*, y^*) = \text{dist}(A, B)$ as $n \to \infty$. By hypothesis (iv), when T is continuous, we have $y^* \in Tx^*$, since $y_n \in Tx_{n-1}$. Hence $\text{dist}(A, B) \leq d(x^*, Tx^*) \leq d(x^*, y^*) = \text{dist}(A, B)$. Therefore x^* is a best proximity point of the mapping T. By hypothesis (iv), when $\alpha(x_n, x^*) > 1$ for each $n \in \mathbb{N}$, then by using triangular property, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, y_{n+1}) + d(y_{n+1}, Tx^*) \\ &< d(x^*, y_{n+1}) + \alpha(x_n, x^*)H(Tx_n, Tx^*) \\ &< d(x^*, y_{n+1}) + a_1 d(x_n, x^*) + a_2[d(x_n, Tx_n) - \text{dist}(A, B)] \\ &\quad + a_3[d(x^*, Tx^*) - \text{dist}(A, B)] + a_4[d(x_n, Tx^*) - \text{dist}(A, B)] \\ &\quad + L[d(x^*, Tx_n) - \text{dist}(A, B)] \\ &\leq d(x^*, y_{n+1}) + a_1 d(x_n, x^*) + a_2[d(x_n, y_{n+1}) - \text{dist}(A, B)] \\ &\quad + a_3[d(x^*, Tx^*) - \text{dist}(A, B)] + a_4[d(x_n, Tx^*) - \text{dist}(A, B)] \\ &\quad + L[d(x^*, y_{n+1}) - \text{dist}(A, B)]. \end{aligned} \tag{13}$$

Letting $n \to \infty$ in (13), we have

$$d(x^*, Tx^*) \leq \text{dist}(A, B) + (a_3 + a_4)[d(x^*, Tx^*) - \text{dist}(A, B)].$$

This implies that

$$d(x^*, Tx^*) \leq \text{dist}(A, B).$$

Thus, we conclude that $d(x^*, Tx^*) = \text{dist}(A, B)$. □

Definition 3.3. *Let (X, d) be a metric space, $A, B \subseteq X$ and $\alpha\colon A \times A \to [0, \infty)$ be a function. A mapping $T\colon A \to CL(B)$ is F-α-proximal contraction of Ciric-type, if there exist continuous F in $\mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(M(x, y)), \tag{14}$$

for each $x, y \in A$, whenever $\min\{\alpha(x, y)H(Tx, Ty), M(x, y)\} > 0$, where

$$\begin{aligned} M(x, y) &= \max\Big\{d(x, y), d(x, Tx) - \text{dist}(A, B), d(y, Ty) - \text{dist}(A, B), \\ &\qquad \frac{d(x, Ty) + d(y, Tx) - 2dist(A, B)}{2}\Big\} + L[d(y, Tx) - \text{dist}(A, B)] \end{aligned}$$

and $L \geq 0$.

Theorem 3.2. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T\colon A \to CL(B)$ is an F-α-proximal contraction of Ciric-type satisfying the following conditions:*

- *(i) $Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P-property;*
- *(ii) T is strictly α-proximal admissible;*
- *(iii) there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$\alpha(x_0, x_1) > 1 \quad \textit{and} \quad d(x_1, y_1) = \text{dist}(A, B).$$

- *(iv) T is continuous, or, for any sequence $\{x_n\} \subseteq A$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.*

Then T has a best proximity point.

Proof. By hypothesis (iii), there exist $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = \text{dist}(A, B) \text{ and } \alpha(x_0, x_1) > 1, \tag{15}$$

If $y_1 \in Tx_1$, then x_1 is a best proximity point of T. Let $y_1 \notin Tx_1$. As $\alpha(x_0, x_1) > 1$, by Lemma 2.1 there exists $y_2 \in Tx_1$ such that

$$d(y_1, y_2) \leq \alpha(x_0, x_1)H(Tx_0, Tx_1).$$

Since F is strictly increasing, we have

$$F(d(y_1, y_2)) \leq F(\alpha(x_0, x_1)H(Tx_0, Tx_1)).$$

From (14), we have

$$\begin{aligned}
\tau + F(d(y_1, y_2)) &\leq \tau + F(\alpha(x_0, x_1)H(Tx_0, Tx_1)) \\
&\leq F\Big(\max\Big\{d(x_0, x_1), d(x_0, Tx_0) - \text{dist}(A, B), d(x_1, Tx_1) - \text{dist}(A, B), \\
&\qquad \frac{d(x_0, Tx_1) + d(x_1, Tx_0) - 2\text{dist}(A, B)}{2}\Big\} + L[d(x_1, Tx_0) - \text{dist}(A, B)]\Big) \\
&\leq F\Big(\max\Big\{d(x_0, x_1), d(x_0, x_1), d(y_1, y_2), \frac{d(x_0, x_1) + d(y_1, y_2)}{2}\Big\} + L \cdot 0\Big) \\
&= F\Big(\max\{d(x_0, x_1), d(y_1, y_2)\}\Big) \\
&= F(d(x_0, x_1)),
\end{aligned} \tag{16}$$

for other choose of max, we have a contraction. Note that, we use the following facts in above inequalities:

$$d(x_0, Tx_0) \leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, Tx_0) = d(x_0, x_1) + \text{dist}(A, B) + 0,$$

$$d(x_1, Tx_1) \leq d(x_1, y_1) + d(y_1, y_2) + d(y_2, Tx_1) = \text{dist}(A, B) + d(y_1, y_2) + 0,$$

$$d(x_0, Tx_1) \leq d(x_0, x_1) + d(x_1, y_1) + d(y_1, y_2) + d(y_2, Tx_1) = d(x_0, x_1) + \text{dist}(A, B) + d(y_1, y_2) + 0$$

$$d(x_1, Tx_0) \leq d(x_1, y_1) + d(y_1, Tx_0) = \text{dist}(A, B) + 0.$$

As $y_2 \in Tx_1 \subseteq B_0$, there exists $x_2 \neq x_1 \in A_0$ such that

$$d(x_2, y_2) = \text{dist}(A, B), \tag{17}$$

for otherwise x_1 is a best proximity point. As (A, B) satisfies the weak P-property. From (15) and (17), we have

$$0 < d(x_1, x_2) \leq d(y_1, y_2).$$

By applying F, we get

$$F(d(x_1, x_2)) \leq F(d(y_1, y_2)). \tag{18}$$

Thus from (16) and (18), we have

$$\tau + F(d(x_1, x_2)) \leq \tau + F(d(y_1, y_2)) \leq F(d(x_0, x_1)).$$

As mapping T is strictly α-proximal admissible, since $\alpha(x_0,x_1)>1$, $d(x_1,y_1)=\text{dist}(A,B)$ and $d(x_2,y_2)=\text{dist}(A,B)$, then $\alpha(x_1,x_2)>1$. Thus we have

$$d(x_2,y_2)=\text{dist}(A,B) \textit{ and } \alpha(x_1,x_2)>1.$$

If $y_2\in Tx_2$, then x_2 is a best proximity point of T. Let $y_2\notin Tx_2$. As $\alpha(x_1,x_2)>1$. There exists $y_3\in Tx_2$ such that

$$d(y_2,y_3)\le \alpha(x_1,x_2)H(Tx_1,Tx_2).$$

Since, F is strictly increasing, we have

$$F(d(y_2,y_3))\le F(\alpha(x_1,x_2)H(Tx_1,Tx_2)).$$

From (14), we have

$$\begin{aligned}
\tau+F(d(y_2,y_3)) &\le \tau+F(\alpha(x_1,x_2)H(Tx_1,Tx_2))\\
&\le F\Big(\max\Big\{d(x_1,x_2),d(x_1,Tx_1)-\text{dist}(A,B),d(x_2,Tx_2)-\text{dist}(A,B),\\
&\quad \frac{d(x_1,Tx_2)+d(x_2,Tx_1)-2\text{dist}(A,B)}{2}\Big\}+L[d(x_2,Tx_1)-\text{dist}(A,B)]\Big)\\
&\le F\Big(\max\Big\{d(x_1,x_2),d(x_1,x_2),d(y_2,y_3),\frac{d(x_1,x_2)+d(y_2,y_3)}{2}\Big\}+L\cdot 0\Big)\\
&= F\Big(\max\{d(x_1,x_2),d(y_2,y_3)\}\Big)\\
&= F(d(x_1,x_2)),
\end{aligned}$$

otherwise we have a contradiction. As $y_3\in Tx_2\subseteq B_0$, there exists $x_3\neq x_2\in A_0$ such that

$$d(x_3,y_3)=\text{dist}(A,B), \tag{19}$$

for otherwise x_2 is a best proximity point. As (A,B) satisfies the weak P-property. From (17) and (19), we have

$$0<d(x_2,x_3)\le d(y_2,y_3).$$

By applying F, we get

$$F(d(x_2,x_3))\le F(d(y_2,y_3)).$$

Thus, we have

$$\tau+F(d(x_2,x_3))\le\tau+F(d(y_2,y_3))\le F(d(x_1,x_2)).$$

So we get

$$F(d(x_2,x_3))\le F(d(y_2,y_3))\le F(d(x_1,x_2))-\tau\le F(d(x_0,x_1))-2\tau.$$

As T is strictly α-proximal admissible, since $\alpha(x_1,x_2)>1$, $d(x_2,y_2)=\text{dist}(A,B)$ and $d(x_3,y_3)=\text{dist}(A,B)$, then $\alpha(x_2,x_3)>1$. Continuing in the same way, we get sequences $\{x_n\}$ in A_0 and $\{y_n\}$ in B_0, where $y_n\in Tx_{n-1}$ for each $n\in\mathbb{N}$ such that

$$d(x_n,y_n)=\text{dist}(A,B) \textit{ and } \alpha(x_{n-1},x_n)>1. \tag{20}$$

Furthermore,

$$F(d(x_n, x_{n+1})) \leq F(d(y_n, y_{n+1})) \leq F(d(x_0, x_1)) - n\tau \text{ for each } n \in \mathbb{N}. \quad (21)$$

Letting $n \to \infty$ in (21), we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = \lim_{n\to\infty} F(d(y_n, y_{n+1})) - \infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (21) we have

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \quad (22)$$

Letting $n \to \infty$ in (22), we get

$$\lim_{n\to\infty} n d_n^k = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n d_n^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1. \quad (23)$$

To prove that $\{x_n\}$ is a Cauchy sequence in A. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (23), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n\to\infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is a Cauchy sequence in A. Similarly, we see that $\{y_n\}$ is a Cauchy sequence in B. Since A and B are closed subsets of a complete metric space, there exist x^* in A and y^* in B such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. By the (20), we conclude that $d(x^*, y^*) = \text{dist}(A, B)$ as $n \to \infty$. By hypothesis (iv), when T is continuous, we have $y^* \in Tx^*$, since $y_n \in Tx_{n-1}$. Hence $\text{dist}(A, B) \leq d(x^*, Tx^*) \leq d(x^*, y^*) = \text{dist}(A, B)$. Therefore x^* is a best proximity point of the mapping T. By hypothesis (iv), when $\alpha(x_n, x^*) > 1$ for each $n \in \mathbb{N}$. We claim that $\text{dist}(A, B) = d(x^*, Tx^*)$. On contrary assume that $\text{dist}(A, B) \neq d(x^*, Tx^*)$. By using the triangular inequality, we have

$$\begin{aligned}
\tau + F(d(y_{n+1}, Tx^*)) &\leq \tau + F(\alpha(x_n, x^*)H(Tx_n, Tx^*)) \\
&\leq F\Big(\max\Big\{ d(x_n, x^*), d(x_n, Tx_n) - \operatorname{dist}(A,B), d(x^*, Tx^*) - \operatorname{dist}(A,B), \\
&\quad \frac{d(x^*, Tx_n) + d(x_n, Tx^*) - 2\operatorname{dist}(A,B)}{2} \Big\} \\
&\quad + L[d(x^*, Tx_n) - \operatorname{dist}(A,B)] \Big) \\
&\leq F\Big(\max\Big\{ d(x_n, x^*), d(x_n, y_{n+1}) - \operatorname{dist}(A,B), d(x^*, Tx^*) - \operatorname{dist}(A,B), \\
&\quad \frac{d(x^*, y_{n+1}) + d(x_n, Tx^*) - 2\operatorname{dist}(A,B)}{2} \Big\} \\
&\quad + L[d(x^*, y_{n+1}) - \operatorname{dist}(A,B)] \Big).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we have

$$\tau + F(d(y^*, Tx^*)) \leq F\Big(\max\Big\{ d(x^*, Tx^*) - \operatorname{dist}(A,B), \frac{d(x^*, Tx^*) - \operatorname{dist}(A,B)}{2} \Big\} \Big). \tag{24}$$

As

$$d(x^*, Tx^*) \leq d(x^*, y^*) + d(y^*, Tx^*).$$

Thus, we have

$$d(x^*, Tx^*) - \operatorname{dist}(A,B) \leq d(y^*, Tx^*).$$

By using the above inequality, (F_1) and (24), we get

$$\begin{aligned}
\tau + F(d(x^*, Tx^*) - \operatorname{dist}(A,B)) &\leq \tau + F(d(y^*, Tx^*)) \\
&\leq F\Big(\max\Big\{ d(x^*, Tx^*) - \operatorname{dist}(A,B), \\
&\quad \frac{d(x^*, Tx^*) - \operatorname{dist}(A,B)}{2} \Big\} \Big).
\end{aligned}$$

This implies that

$$d(x^*, Tx^*) - \operatorname{dist}(A,B) < d(x^*, Tx^*) - \operatorname{dist}(A,B).$$

This is a contradiction to our assumption. Thus, we conclude that $d(x^*, Tx^*) = \operatorname{dist}(A,B)$.

□

Example 3.1. *Let $X = \mathbb{R} \times \mathbb{R}$ be endowed with a metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for each $x, y \in X$. Take $A = \{(0, x) : x \in \mathbb{R}\}$ and $B = \{(1, x) : x \in \mathbb{R}\}$. Define*

$$T \colon A \to CL(B), \quad T(0,x) = \begin{cases} \{(1, x^2)\} \text{ if } x < 0 \\ \{(1,0), (1,1)\} \text{ if } 0 \leq x \leq 1 \\ \{(1, x-1), (1, x)\} \text{ if } x > 1 \end{cases}$$

and

$$\alpha\colon A\times A\to[0,\infty),\quad \alpha((0,x),(0,y))=\begin{cases}2 \ \textit{if } x,y\in[0,1]\\ \frac{1}{2} \ \textit{if } x,y\in\mathbb{N}-\{1\}\\ 0 \ \textit{otherwise.}\end{cases}$$

Take $F(x)=x+\ln x$ *for each* $x\in(0,\infty)$. *Under this* F, *condition* (1) *reduces to*

$$\frac{\alpha(x,y)H(Tx,Ty)}{N(x,y)}e^{\alpha(x,y)H(Tx,Ty)-N(x,y)}\le e^{-\tau} \tag{25}$$

for each $x,y\in A$ *with* $\min\{\alpha(x,y)H(Tx,Ty),N(x,y)\}>0$. *Assume that* $a_1=1$, $a_2=a_3=a_4=L=0$ *and* $\tau=\frac{1}{2}$. *Clearly,* $\min\{\alpha(x,y)H(Tx,Ty),d(x,y)\}>0$ *for each* $x,y\in\mathbb{N}\setminus\{1\}$ *with* $x\neq y$. *From* (25) *for each* $x,y\in\mathbb{N}\setminus\{1\}$ *with* $x\neq y$, *we have*

$$\frac{1}{2}e^{-\frac{1}{2}|x-y|}<e^{-\frac{1}{2}}.$$

Thus, T *is* F-α*-proximal contraction of Hardy-Rogers-type with* $F(x)=x+\ln x$. *Note that* $A_0=A$, $B_0=B$ *and* $Tx\subseteq B_0$ *for each* $x\in A_0$. *Also, the pair* (A,B) *satisfies the weak* P*-property. If* $x_0,x_1\in\{(0,x):0\le x\le 1\}$, *then* $Tx_0,Tx_1\in\{(1,0),(1,1)\}$. *Take* $y_1\in Tx_0$, $y_2\in Tx_1$ *and* $u_1,u_2\in A$ *such that* $d(u_1,y_1)=\operatorname{dist}(A,B)$ *and* $d(u_2,y_2)=\operatorname{dist}(A,B)$. *Then we have* $u_1,u_2\in\{(0,0),(0,1)\}$. *Hence* T *is strictly* α*-proximal admissible mapping. For* $x_0=(0,1)\in A_0$ *and* $y_1=(1,0)\in Tx_0$ *in* B_0, *we have* $x_1=(0,0)\in A_0$ *such that* $d(x_1,y_1)=\operatorname{dist}(A,B)$ *and* $\alpha(x_0,x_1)=2>1$. *Moreover, for any sequence* $\{x_n\}\subseteq A$ *such that* $x_n\to x$ *as* $n\to\infty$ *and* $\alpha(x_n,x_{n+1})>1$ *for each* $n\in\mathbb{N}$, *we have* $\alpha(x_n,x)>1$ *for each* $n\in\mathbb{N}$. *Therefore, by Theorem 3.1,* T *has a best proximity point.*

When we take $X=A=B$, we get the following fixed point theorems from our results:

Theorem 3.3. *Let* (X,d) *be a complete metric space. Assume* $T\colon X\to CL(X)$ *is a mapping for which there exist* $F\in\mathfrak{F}$ *and* $\tau>0$ *such that*

$$\tau+F(\alpha(x,y)H(Tx,Ty))\le F(N(x,y)),$$

for each $x,y\in X$, *whenever* $\min\{\alpha(x,y)H(Tx,Ty),N(x,y)\}>0$, *where*

$$N(x,y) = a_1d(x,y)+a_2d(x,Tx)+a_3d(y,Ty)+a_4d(x,Ty)+Ld(y,Tx),$$

with $a_1,a_2,a_3,a_4,L\ge 0$ *satisfying* $a_1+a_2+a_3+2a_4=1$ *and* $a_3\neq 1$. *Further assume that the following conditions hold:*

- **:** *(i)* T *is strictly* α*-admissible, that is, if* $\alpha(x,y)>1$, *then* $\alpha(a,b)>1$ *for each* $a\in Tx$ *and* $b\in Ty$;
- **:** *(ii) there exist* $x_0\in X$ *and* $x_1\in Tx_0$ *such that* $\alpha(x_0,x_1)>1$;
- **:** *(iii)* T *is continuous, or, for any sequence* $\{x_n\}\subseteq X$ *such that* $x_n\to x$ *as* $n\to\infty$ *and* $\alpha(x_n,x_{n+1})>1$ *for each* $n\in\mathbb{N}$, *we have* $\alpha(x_n,x)>1$ *for each* $n\in\mathbb{N}$.

Then T *has a fixed point.*

Theorem 3.4. *Let (X, d) be a complete metric space. Assume $T\colon X \to CL(X)$ is a mapping for which there exist continuous F in $\mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(M(x, y)),$$

for each $x, y \in X$, whenever $\min\{\alpha(x, y)H(Tx, Ty), M(x, y)\} > 0$, where

$$M(x, y) \quad = \quad \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} + Ld(y, Tx)$$

and $L \geq 0$. Further assume that the following conditions hold:

: *(i) T is strictly α-admissible, that is, if $\alpha(x, y) > 1$, then $\alpha(a, b) > 1$ for each $a \in Tx$ and $b \in Ty$;*

: *(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) > 1$;*

: *(iii) T is continuous, or, for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

4. Conclusions

In this paper, we introduced the notions of F-α-proximal contractions for Hardy-Rogers type mappings as well as for Ciric-type mappings. Within this framework, we studied the existence of best proximity for nonself multivalued mappings satisfying at least one of these notions along with few other conditions. Nontrivial example supports the results herein.

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