FIXED POINT THEOREMS FOR MULTIVALUED γ-FG-CONTRACTIONS WITH (α_*, β_*)-ADMISSIBLE MAPPINGS IN PARTIAL b-METRIC SPACES AND APPLICATION

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In this paper, we introduce and study the notion of cyclic (α_*, β_*)-type-γ-FG-contractive mapping and establish some fixed point theorems for such mappings of rational type defined on a partial b-metric space. Our work generalizes several recent results existing in the literature. We set up an example to elucidate our main result. As application of our findings, we demonstrate the existence of the solution of class of nonlinear integral equations.

Keywords: fixed point, partial b-metric, multivalued cyclic (α_*, β_*)-type-γ-FG-contractions.
MSC2010: 47H10; 54H25.

1. Introduction

The well known Banach's Contraction Principle, has many fruitful generalizations in various directions. One of these generalizations is for F-contraction presented by Wardowski [6]: every F-contraction defined on a complete metric space has a unique fixed point. So the concept of an F-contraction proved to be a milestone in fixed point theory. Numerous research papers on F-contractions have been published (see for instant, ([9, 10, 14, 8, 16]). Recently, Cosentino *et al.* [9] established a fixed point result for Hardy-Rogers type F-contraction and Piri and Kumam [13] generalized the concept of F-contraction and proved certain fixed and common fixed point results. Minak *et al.* [10] presented a fixed point result for Ćirić type generalized F-contraction. Parvaneh *et al.* [12] used slightly modified the family of functions, denoted by $\Delta_{G,\beta}$ and generalized the Wardowski fixed point results in b-metric and ordered b-metric spaces. Very recently, Padhan *et al.* [25] introduced a new concept of cyclic (α, β)-type-γ-FG-contractive mapping and proved some fixed point theorems for such mappings in b-metric spaces. Following this line of work, Alizadeh *et al.* [5] introduced the notion of cyclic (α, β)-admissible mapping and proved several fixed point results. On the other hand, Bakhtin [3] investigated the concept of b-metric spaces. Subsequently, Czerwik [2] initiated the study of fixed point results in b-metric spaces and proved an analogue of Banach's fixed point theorem. Afterwards, numerous research articles have been published on fixed point theorems for various classes of single-valued and multivalued operators in b-metric spaces (see for example, ([17, 18, 19, 20, 21, 22, 23]). In this article, we shall investigate fixed points of cyclic (α_*, β_*)-type-γ-FG-contractive mappings

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defined on a partial b-metric space.
Czerwik generalized the notion of metric as follows:

Definition 1.1. [2] *Let X be a nonempty set and $s \geq 1$ be a real number. A mapping $d : X \times X \to [0, \infty)$ is said to be a b-metric if for all $x, y, z \in X$, d satisfies following axioms;*

(b_1) $d(x, y) = 0$ *if and only if* $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space (with coefficient s).

Matthews generalized the notion of metric as follows:

Definition 1.2. [1] *Let X be a nonempty set. A mapping $P : X \times X \to [0, \infty)$ is said to be a partial metric if for all $x, y, z \in X$, P satisfies following axioms;*

(P_1) $P(x, x) = P(x, y) = P(y, y)$ *if and only if* $x = y$,
(P_2) $P(x, x) \leq P(x, y)$,
(P_3) $P(x, y) = P(y, x)$,
(P_4) $P(x, y) \leq P(x, z) + P(z, y) - P(z, z)$.

The pair (X, P) is called a partial metric space.

Shukla generalized the notion of partial metric as follows:

Definition 1.3. [15] *Let X be a nonempty set and $s \geq 1$ a real number. A mapping $P_b : X \times X \to [0, \infty)$ is said to be a partial b-metric if for all $x, y, z \in X$, P_b satisfies following axioms;*

(P_1) $P_b(x, x) = P_b(x, y) = P_b(y, y)$ *if and only if* $x = y$,
(P_2) $P_b(x, x) \leq P_b(x, y)$,
(P_3) $P_b(x, y) = P_b(y, x)$,
(P_4) $P_b(x, y) \leq s[P_b(x, z) + P_b(z, y)] - P_b(z, z)$.

The pair (X, P_b) is called a partial b-metric space (with coefficient s).

Remark 1.1. *The self distance $P_b(x, x)$, referred to the size or weight of x, is a feature used to describe the amount of information contained in x.*

Remark 1.2. *Obviously, every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with zero self-distance. However, the converse of this fact need not to hold.*

Example 1.1. Let $X = \mathbb{R}^+$ and $k > 1$, the mapping $P_b : X \times X \to \mathbb{R}^+$ defined by

$$P_b(x, y) = \left\{ (x \vee y)^k + |x - y|^k \right\} \text{ for all } x, y \in X$$

is a partial b-metric on X with $s = 2^k$. For $x = y$, $P_b(x, x) = x^k \neq 0$, so, P_b is not a b-metric on X.
Let $x, y, z \in X$ such that $x > z > y$. Then following inequality always holds

$$(x - y)^k > (x - z)^k + (z - y)^k.$$

Since, $P_b(x, y) = x^k + (x - y)^k$ and $P_b(x, z) + P_b(z, y) - P_b(z, z) = x^k + (x - z)^k + (z - y)^k$, therefore,

$$P_b(x, y) > P_b(x, z) + P_b(z, y) - P_b(z, z).$$

This shows that P_b is not a partial metric on X.

Definition 1.4. [15] *Let (X, P_b) be a partial b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then*

(i) $\{x_n\}$ *is said to be convergent to* x *if* $\lim_{n\to\infty} P_b(x_n, x) = P_b(x, x)$.
(ii) $\{x_n\}$ *is said to be Cauchy sequence if* $\lim_{n,m\to\infty} P_b(x_n, x_m)$ *exists and is finite.*
(iii) (X, P_b) *is said to be complete if every Cauchy sequence is convergent in* X.

Alizadeh *et al.* [5] introduced the concept of cyclic (α, β)-admissible mapping as follows:

Definition 1.5. [5] *Let* X *be a nonempty set and* $\alpha, \beta : X \to [0, \infty)$ *be mappings. A self-mapping* T *on* X *is called a cyclic* (α, β)*-admissible mapping if,*

$$\alpha(x) \geq 1 \ (x \in X) \Rightarrow \beta(T(x)) \geq 1,$$

and

$$\beta(x) \geq 1 \ (x \in X) \Rightarrow \alpha(T(x)) \geq 1.$$

Let (X, P_b) be a partial b-metric space and $CB_{P_b}(X)$ denote the family of all bounded and closed subsets of X. For $x \in X$ and $A, B \in CB_{P_b}(X)$, we define

$$P_b(x, A) = \inf_{a\in A} P_b(x, a), \ \ \delta(A, B) = \sup_{a\in A} P_b(a, B).$$

Define a mapping $H : CB_{P_b}(X) \times CB_{P_b}(X) \to [0, \infty)$ by

$$H_{P_b}(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

for every $A, B \in CB_{P_b}(X)$. It clear that for $A, B \in CB_{P_b}(X)$ and $a \in A$, one has

$$P_b(a, B) = \inf_{b\in B} P_b(a, b) \leq \delta(A, B) \leq H_{P_b}(A, B).$$

Lemma 1.1. [4] *Let* A *and* B *be nonempty closed, bounded subsets of a partial* b*-metric space* (X, P_b) *and* $q > 1$*. Then, for all* $a \in A$*, there exists* $b \in B$ *such that* $P_b(a, b) \leq qH_{P_b}(A, B)$.

Lemma 1.2. [4] *Let* (X, P_b) *be a partial* b*-metric space with coefficient* $s \geq 1$*. For* $A \in CB_{P_b}(X)$ *and* $x \in X$*, then* $P_b(x, A) = P_b(x, x)$ *if and only if* $x \in \overline{A}$*, where* $\overline{A}$ *is the closure of* A.

Lemma 1.3. [4] *Let* (X, P_b) *be a partial* b*-metric space. For any* $A, B, C \in CB_{P_b}(X)$*, ones have*
(H_1) $H_{P_b}(A, A) \leq H_{P_b}(A, B)$,
(H_2) $H_{P_b}(A, B) = H_{P_b}(B, A)$,
(H_3) $H_{P_b}(A, B) \leq s[H_{P_b}(A, C) + H_{P_b}(C, B)] - \inf_{c\in C} P_b(c, c)$.

Lemma 1.4. [4] *Let* (X, P_b) *be a partial* b*-metric space with coefficient* s *and* $B \in CB_{P_b}(X)$*. If* $x \in X$ *and* $P_b(x, B) < c$ *where* $c > 0$*, then there exists* $y \in B$ *such that* $P_b(x, y) < c$.

In light with ([12],[24],[11]), we denote the set of all mappings $F : \mathbb{R}_+ \to \mathbb{R}$ by Δ_F which satisfy following axioms;
(Δ_1) F is strictly increasing,
(Δ_2) for each sequence $\{t_n\} \subseteq \mathbb{R}_+$, $\lim_{n\to\infty} t_n = 0 \Leftrightarrow \lim_{n\to\infty} F(t_n) = -\infty$,
(Δ_3) there exists $k \in (0, 1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$,
(Δ_4) $F(st_n) \leq F(t_{n-1}) + G(\gamma(t_{n-1})) \Rightarrow F(s^n t_n) \leq F(s^{n-1}t_{n-1}) + G(\gamma(t_{n-1}))$, $(G, \gamma) \in \Delta_{G,\gamma}$,
(Δ_5) $F(\inf A) = \inf F(A)$ for all $A \subseteq \mathbb{R}_+$ with $\inf A > 0$.

Let $\Delta_{G,\gamma}$ represents the set of pairs (G, γ), where $G : \mathbb{R} \to \mathbb{R}$ and $\gamma : [0, \infty) \to [0, 1)$ are mappings such that
(Δ_6) for each sequence $\{t_n\} \subseteq \mathbb{R}_+$, $\lim_{n\to\infty} \sup G(t_n) \geq 0 \Leftrightarrow \lim_{n\to\infty} \sup t_n \geq 1$,
(Δ_7) for each sequence $\{t_n\} \subseteq [0, \infty)$, $\lim_{n\to\infty} \sup \gamma(t_n) = 1$ implies that

$$\lim_{n\to\infty} t_n = 0,$$

(Δ_8) for each sequence $\{t_n\} \subseteq \mathbb{R}_+$, $\sum_{n=1}^{\infty} G(\gamma(t_n)) = -\infty$.

Example 1.2. *Let* $F : \mathbb{R}_+ \to \mathbb{R}$ *be defined by*

(1) $F_1(r) = r + \ln r$;
(2) $F_2(r) = \ln r$.

It is easy to check that $F_1, F_2 \in \Delta_{Fs}$.

2. Main results

In this section, we introduce the concept of cyclic (α_*, β_*)-type-γ-FG-contraction. We also set up a fixed point theorem for such contraction. Moreover, we explain this theorem by an supportive example.

Definition 2.1. *Let* (X, p_b) *be a partial b-metric space. The mapping* $T : X \to CB_{P_b}(X)$ *is called multivalued* P_b*-continuous at point* $x \in X$ *if* $\lim_{n\to\infty} P_b(x_n, x) = P_b(x, x)$ *implies that* $\lim_{n\to\infty} H_{P_b}T(x_n, T(x)) = H_{P_b}(Tx, Tx)$.

Definition 2.2. *Let* X *be a nonempty set,* $\alpha, \beta : X \to [0, \infty)$ *be mappings and* A, B *be subsets of* X. *A mapping* $T : X \to CB_{P_b}(X)$ *is called a cyclic* (α_*, β_*)*-admissible mapping if,*

$$\alpha(x) \geq 1 \ (x \in X) \Rightarrow \beta_*(T(x)) \geq 1, \ \textit{where } \beta_*(A) = \inf_{a\in A} \beta(a),$$

and

$$\beta(x) \geq 1 \ (x \in X) \Rightarrow \alpha_*(T(x)) \geq 1, \ \textit{where } \alpha_*(B) = \inf_{b\in B} \alpha(b).$$

Definition 2.3. *Let* (X, P_b) *be a partial b-metric space with coefficient* $s \geq 1$, $T : X \to CB_{P_b}(X)$ *and* $\alpha, \beta : X \to [0, \infty)$ *be mappings. Then* T *is called cyclic* (α_*, β_*)*-type-*γ*-*FG*-contraction, if there exist* $F \in \Delta_F$, $(G, \gamma) \in \Delta_{G,\gamma}$ *such that for all* $x, y \in X$, $\alpha(x)\beta(y) \geq 1$ *and* $H_{P_b}(T(x), T(y)) > 0$ *imply*

$$F(\alpha(x)\beta(y)sH_{P_b}(T(x), T(y))) \leq F(\mathcal{M}_s(x, y)) + G(\gamma(\mathcal{M}_s(x, y))), \tag{1}$$

where

$$\mathcal{M}_s(x, y) = \max\left\{P_b(x, y), P_b(y, T(y)), P_b(x, T(x)), \frac{P_b(x, T(y)) + P_b(y, T(x))}{2s}\right\}. \tag{2}$$

Theorem 2.1. *Let* (X, P_b) *be a complete partial b-metric space with coefficient* $s \geq 1$, $\alpha, \beta : X \to [0, \infty)$ *be mappings and* $T : X \to CBP_{P_b}(X)$ *be a cyclic* (α_*, β_*)*-type-*γ*-*FG*-contractive mapping satisfying the following conditions:*

(1) *either there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geq 1$ *or there exists* $y_0 \in X$ *such that* $\beta(y_0) \geq 1$,
(2) T *is multivalued* P_b*-continuous,*
(3) T *is cyclic* (α_*, β_*)*-admissible.*

Then T *has a fixed point.*

Proof. Let $x_0 \in X$ be such that $\alpha(x_0) \geq 1$. Since T is cyclic (α_*, β_*)-admissible mapping, there exist $x_1 \in T(x_0)$, $x_2 \in T(x_1)$ such that

$$\alpha(x_0) \geq 1 \Rightarrow \beta(x_1) \geq \beta_*(T(x_0)) \geq 1 \Rightarrow \alpha(x_2) \geq \alpha_*(T(x_1)) \geq 1. \tag{3}$$

Because of $\alpha(x_0)\beta(x_1) \geq 1$, it is easy to see that

$$sP_b(x_1, T(x_1)) \leq sH_{P_b}(T(x_0), T(x_1)) \leq \alpha(x_0)\beta(x_1)sH_{P_b}(T(x_0), T(x_1))$$

by F_1), we have

$$\begin{aligned} F(sP_b(x_1, T(x_1))) &\leq F(\alpha(x_0)\beta(x_1)sH_{P_b}(T(x_0), T(x_1))) \\ &\leq F(\mathcal{M}_s(x_0, x_1)) + G(\gamma(\mathcal{M}_s(x_0, x_1))). \end{aligned} \tag{4}$$

The axiom (Δ_5) implies that $F(sP_b(x_1, T(x_1))) = \inf_{x\in T(x_1)} F(sP_b(x_1, x))$. Thus, there exists $x = x_2 \in T(x_1)$ such that $F(sP_b(x_1, T(x_1))) = F(sP_b(x_1, x_2))$ and the inequality (4) implies

$$F(sP_b(x_1, x_2)) \leq F(\mathcal{M}_s(x_0, x_1)) + G(\gamma(\mathcal{M}_s(x_0, x_1))), \tag{5}$$

where

$$\begin{aligned}
&\mathcal{M}_s(x_0, x_1)\\
&= \max\left\{P_b(x_0, x_1), P_b(x_1, T(x_1)), P_b(x_0, T(x_0)), \frac{P_b(x_0, T(x_1)) + P_b(x_1, T(x_0))}{2s}\right\}\\
&\leq \max\left\{P_b(x_0, x_1), P_b(x_1, x_2), \frac{P_b(x_0, x_2) + P_b(x_1, x_1)}{2s}\right\}\\
&\leq \max\left\{P_b(x_0, x_1), P_b(x_1, x_2), \frac{s[P_b(x_0, x_1)) + P_b(x_1, x_2)]}{2s}\right\}\\
&\leq \max\left\{P_b(x_0, x_1), P_b(x_1, x_2)\right\}.
\end{aligned}$$

If $\mathcal{M}(x_0, x_1) \leq P_b(x_1, x_2)$, then (5) yields that

$$F(sP_b(x_1, x_2)) \leq F(P_b(x_1, x_2)) + G(\gamma(\mathcal{M}_s(x_0, x_1))),$$

which implies $G(\gamma(\mathcal{M}_s(x_0, x_1))) \geq 0$ and by (Δ_6) we get $\gamma(\mathcal{M}_s(x_0, x_1)) \geq 1$. This is a contradiction to definition of γ. Thus, $\mathcal{M}(x_0, x_1) \leq P_b(x_0, x_1)$. By (5), we get

$$F(sP_b(x_1, x_2)) \leq F(P_b(x_0, x_1)) + G(\gamma(\mathcal{M}_s(x_0, x_1))).$$

Similarly, there exists $x_3 \in T(x_2)$ such that

$$F(sP_b(x_2, x_3)) \leq F(P_b(x_1, x_2)) + G(\gamma(\mathcal{M}_s(x_1, x_2))).$$

Continuing this process, we construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$, $\alpha(x_n)\beta(x_{n+1}) \geq 1$, and

$$F(sP_b(x_n, x_{n+1})) \leq F(P_b(x_{n-1}, x_n)) + G(\gamma(\mathcal{M}_s(x_{n-1}, x_n))). \tag{6}$$

By (6) and axiom (Δ_4), we have

$$F(s^n P_b(x_n, x_{n+1})) \leq F(s^{n-1} P_b(x_{n-1}, x_n)) + G(\gamma(\mathcal{M}_s(x_{n-1}, x_n)))),$$

for all $n \in \mathbb{N}$. which further implies,

$$\begin{aligned}
F(s^n P_b(x_n, x_{n+1})) \leq{}& F(s^{n-2} P_b(x_{n-2}, x_{n-1})) + G(\gamma(\mathcal{M}_s(x_{n-2}, x_{n-1})))\\
&+ G(\gamma(\mathcal{M}_s(x_{n-1}, x_n))).
\end{aligned}$$

Thus,

$$F(s^n P_b(x_n, x_{n+1})) \leq F(P_b(x_0, x_1)) + \sum_{i=1}^{n} G(\gamma(\mathcal{M}_s(x_{i-1}, x_i))). \tag{7}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} s^n F(P_b(x_n, x_{n+1})) = -\infty.$$

By (Δ_2), we get

$$\lim_{n\to\infty} s^n P_b(x_n, x_{n+1}) = 0.$$

By (Δ_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (s^n P_b(x_n, x_{n+1}))^k F(s^n P_b(x_n, x_{n+1})) = 0.$$

By (7), for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} & (s^n P_b(x_n, x_{n+1}))^k F(s^n P_b(x_n, x_{n+1})) - (s^n P_b(x_n, x_{n+1}))^k F(P_b(x_0, x_1)) \\ \leq \quad & (s^n P_b(x_n, x_{n+1}))^k \sum_{i=1}^{n} G(\gamma(\mathcal{M}_s(x_{i-1}, x_i))) \leq 0. \end{aligned}$$

On taking Limit $n \to \infty$ in above inequality, we have

$$\lim_{n\to\infty} (\sum_{i=1}^{n} G(\gamma(\mathcal{M}_s(x_{i-1}, x_i)))(s^n P_b(x_n, x_{n+1}))^k) = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{i=1}^{n} G(\gamma(\mathcal{M}_s(x_{i-1}, x_i)))(s^n P_b(x_n, x_{n+1}))^k \leq 1 \text{ for all } n \geq n_1, \text{ or}$$

$$s^n P_b(x_n, x_{n+1}) \leq \frac{1}{A_n^{1/k}}, \text{ for all } n \geq n_1, \text{ where, } A_n = \sum_{i=1}^{n} G(\gamma(\mathcal{M}_s(x_{i-1}, x_i))). \tag{8}$$

To prove $\{x_n\}$ is a Cauchy sequence, we use (8) and for $m \geq n \geq n_1$, we consider,

$$\begin{aligned} P_b(x_n, x_m) &\leq \sum_{i=n}^{m-1} s^i P_b(x_i, x_{i+1}) - \sum_{i=n+1}^{m-1} s^{i-(n+1)} P_b(x_i, x_i) \\ &\leq \sum_{i=n}^{m-1} s^i pb(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} s^i P_b(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{A_i{}^{1/k}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{A_i{}^{1/k}}$ entails $\lim_{n,m\to\infty} P_b(x_n, x_m) = 0$. Therefore $\{x_n\}$ is a Cauchy sequence in (X, P_b), so, there exists $x \in X$ such that

$$P_b(x, x) = \lim_{n\to\infty} P_b(x_n, x) = \lim_{n,m\to\infty} P_b(x_n, x_m) = 0.$$

By multivlued P_b-continuity of T we get,

$$\lim_{n\to\infty} pb(x_{n+1}, T(x)) \leq \lim_{n\to\infty} H_{P_b}(T(x_n), T(x)) = H_{P_b}(T(x), T(x)). \tag{9}$$

Using the triangular inequality, we have

$$\begin{aligned} P_b(x, T(x)) \quad &\leq \quad s[P_b(x, x_{n+1}) + P_b(x_{n+1}, T(x))] - P_b(x_{n+1}, x_{n+1}) \\ &\leq \quad s[P_b(x, x_{n+1}) + P_b(x_{n+1}, T(x))] \end{aligned}$$

Letting $n \to \infty$ and using (9),

$$\begin{aligned} P_b(x, T(x)) \quad &\leq \quad \lim_{n\to\infty} sP_b(x, x_{n+1}) + \lim_{n\to\infty} sP_b(x_{n+1}, T(x))] \\ &\leq \quad sH_{P_b}(T(x), T(x)). \end{aligned}$$

So we have $P_b(x, T(x)) \leq sH_{P_b}(T(x), T(x))$. We will show that $x \in Tx$. Suppose that $x \notin Tx$. By Lemma 1.2, we obtain that $P_b(x, Tx) \neq 0$, which implies that

$$\begin{aligned} F(sH_{P_b}(T(x), T(x))) \quad &\leq \quad F(\alpha(x)\beta(x)sH_{P_b}(T(x), T(x))) \\ &\leq \quad F(\mathcal{M}_s(x, x)) + G(\gamma(\mathcal{M}_s(x, x))), \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}_s(x, x) \quad &= \quad \max\left\{P_b(x, x), P_b(x, T(x)), P_b(x, T(x)), \frac{P_b(x, T(x)) + P_b(x, T(x))}{2s}\right\} \\ &= \quad P_b(x, T(x)). \end{aligned}$$

We get

$$\begin{aligned} F(H_{P_b}(T(x),T(x))) &\le F(P_b(x,T(x))) + G(\gamma(P_b(x,T(x)))) \\ &\le F(sH_{P_b}(T(x),T(x))) + G(\gamma(sH_{P_b}(T(x),T(x)))). \end{aligned}$$

Since $G(\gamma(sH_{P_b}(T(x),T(x)))) \ge 0$, which yields that $\gamma(sH_{P_b}(T(x),T(x))) \ge 1$, a contradiction. Therefore, $x \in Tx$ and hence T has a fixed point in X. □

Example 2.1. *Let $X = \{0,1,2,3\}$ endowed with the partial b-metric $P_b : X \times X \to [0,\infty)$ defined by*

$$P_b(x,y) = |x-y|^2 + \max\{x,y\}^2, \text{ for all } x,y \in X.$$

Clearly, (X,P_b) is a complete partial b-metric space with coefficient $s = 4$, but it is not a partial metric space since $P_b(0,3) = 18 > 14 = P_b(0,1) + P_b(1,3) - P_b(1,1)$. Define mappings $T : X \to CB_{P_b}(X)$ and $\gamma : [0,\infty) \to [0,1)$, by

$$T(0) = T(1) = \{1,2\}, T(2) = T(3) = \{1\} \text{ and } \gamma(t) = \frac{9}{10}.$$

Define mappings $\alpha,\beta : X \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} \frac{x+4}{4} & \text{if } x \in \{2,3\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\beta(x) = \begin{cases} \frac{x+5}{5} & \text{if } x \in \{2,3\}, \\ 0 & \text{otherwise.} \end{cases}$$

For, $x \in \{2,3\}$, we have

$$\alpha(x) \ge 1 \Rightarrow \beta_*(Tx) = \beta_*(\{1\}) = \frac{1+5}{5} \ge 1,$$

and

$$\beta(x) \ge 1 \Rightarrow \alpha_*(Tx) = \alpha_*(\{1\}) = \frac{1+4}{4} \ge 1.$$

Hence, T is cyclic (α_*,β_*)-admissible mapping. Now, assume that $x,y \in X$ are such that $\alpha(x)\beta(y) \ge 1$, then we have $x,y \in \{2,3\}$ and $H_{P_b}(Tx,Ty) > 0$ imply

$$\begin{aligned} F(\alpha(x)\beta(y)sH_{P_b}(Tx,Ty)) &= F(\alpha(x)\beta(y)sH_{P_b}(T(1),T(1))) \\ &= F(\alpha(x)\beta(y)sP_b(1,1)) \\ &\le F(P_b(x,y)) + G(\gamma(P_b(x,y))) \\ &\le F(\mathcal{M}_s(x,y)) + G(\gamma(\mathcal{M}_s(x,y))). \end{aligned}$$

Thus, all conditions of Theorem 2.1 are satisfied with $F(t) = G(t) = \ln(t), t > 0$, and T has a fixed point.

Following Corollary provides a generalization of the results in [6], [25] in the set up of a partial b-metric space.

Corollary 2.1. *Let (X,P_b) be a complete partial b-metric space with coefficient $s \ge 1$, $\alpha,\beta : X \to [0,\infty)$ and $T : X \to CB_{P_b}(X)$ be a multivalued mapping satisfying following conditions:*

(1) *there exists $x_0 \in X$ such that $\alpha(x_0) \ge 1$ or there exists $y_0 \in X$ such that $\beta(y_0) \ge 1$;*
(2) *T is multivalued P_b-continuous;*
(3) *there exist $\tau > 0$, for all $x,y \in X$; $\alpha(x)\beta(y) \ge 1$ and $H_{P_b}(T(x),T(y)) > 0$ such that*

$$\Rightarrow \tau + F(\alpha(x)\beta(y)sH_{P_b}(T(x),T(y))) \le F(\mathcal{M}_s(x,y)),$$

(4) *T is cyclic (α_*,β_*)-admissible.*

Then T has a fixed point.

Proof. Set $\gamma(t) = k, G(t) = \ln(t)$, where $k \in (0,1)$ and $\tau = -\ln(k)$ in the Theorem 2.1. □

Corollary 2.2. *Let (X, P_b) be a complete partial b-metric space with coefficient $s \geq 1$, $\alpha, \beta : X \to [0, \infty)$ and $T : X \to CB_{P_b}(X)$ be a multivalued mapping satisfying following conditions:*

(1) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ or there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$;*
(2) *T is multivalued P_b-continuous;*
(3) *for all $x, y \in X$ with $\alpha(x)\beta(y) = 1$, we have*

$$sH_{P_b}(T(x), T(y)) \leq \gamma(\mathcal{M}_s(x, y)))\mathcal{M}_s(x, y)),$$

(4) *T is cyclic (α_*, β_*)-admissible.*

Then T has a fixed point.

Proof. Set $F(t) = G(t) = \ln(t)$ in Theorem 2.1. □

Definition 2.4. *Let (X, P_b) be a partial b-metric space with coefficient $s \geq 1$, $T : X \to X$ be a self-mapping. Then T is called γ-FG-contraction, if there exist $F \in \Delta_F$, $(G, \gamma) \in \Delta_{G,\gamma}$ such that*

$$F(sP_b(T(x), T(y))) \leq F(\mathcal{M}_s(x, y)) + G(\gamma(\mathcal{M}_s(x, y))),$$

for all $x, y \in X$, $P_b(T(x), T(y)) > 0$, where

$$\mathcal{M}_s(x, y) = \max\left\{P_b(x, y), P_b(y, T(y)), P_b(x, T(x)), \frac{P_b(x, T(y)) + P_b(y, T(x))}{2s}\right\}.$$

Corollary 2.3. *Let (X, P_b) be a complete partial b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a γ-FG-contraction. If T is multivalued P_b-continuous, then T has a fixed point.*

Proof. Consider Picard iterative sequence $\{x_n : \ x_n = T(x_{n-1})\}_{n \in \mathbb{N}}$ in the proof of Theorem 2.1. This proof contains similar steps as in the proof of Theorem 2.1, so, we omit details. □

Corollary 2.4. *Let (X, P_b) be a complete partial b-metric space with coefficient $s \geq 1$ and $T : X \to CB_{P_b}(X)$ be a multivalued mapping, such that*

$$sH_{P_b}(T(x), T(y)) \leq \ r\mathcal{M}_s(x, y), \tag{10}$$

for some $r \in [0, 1)$ and for all $x, y \in X$. Then T has a fixed point.

Proof. Set $F(t) = t, G(t) = (1 - k)t, \gamma(t) = k$, $k \in (0, 1)$ and $\alpha(x) = \beta(y) = 1$ in Theorem 2.1. □

Corollary 2.5. *Let (X, p_b) be a complete partial b-metric space with coefficient $s \geq 1$ and $T : X \to CB_{P_b}(X)$ be a multivalued mapping, such that*

$$H_{P_b}(T(x), T(y)) \leq \ r\mathcal{M}_s(x, y), \tag{11}$$

for some $r \in [0, 1)$ and for all $x, y \in X$. Then T has a fixed point.

Proof. Set $F(t) = t$, $G(t) = (r - 1)t$, $\gamma(t) = r$, $r \in [0, \infty)$, $k = 1$ aand $\alpha(x) = \beta(y) = 1$ in Theorem 2.1. □

3. **Some fixed point results of multivalued cyclic mappings**

Definition 3.1. *Let A and B be a nonempty subsets of a set X. $T : A \cup B \to CB_{P_b}(A) \cup CB_{P_b}(B)$ is called a multivalued cyclic if $T(A) \subseteq CB_{P_b}(B)$ and $T(B) \subseteq CB_{P_b}(A)$.*

Definition 3.2. *Let (X, P_b) be a partial b-metric space with coefficient $s \geq 1$. We sat that $T : A \cup B \to CB_{P_b}(A) \cup CB_{P_b}(B)$ is a multivalued $(A, B) - \gamma - FG$- contraction, if there exist $F \in \Delta_F$, $(G, \gamma) \in \Delta_{G,\gamma}$ such that, for all $x \in A$ and $y \in B$,*

$$\begin{aligned} &A(x)B(y) \geq 1, H_{P_b}(T(x), T(y)) > 0 \\ &\Rightarrow F(A(x)B(y)sH_{P_b}(T(x), T(y))) \leq F(\mathcal{M}_s(x, y)) + G(\gamma(\mathcal{M}_s(x, y))), \end{aligned} \tag{12}$$

where

$$\mathcal{M}_s(x,y) = \max\left\{P_b(x,y), P_b(y,T(y)), P_b(x,T(x)), \frac{P_b(x,T(y)) + P_b(y,T(x))}{2s}\right\}.$$

Theorem 3.1. *Let A and B be two nonempty subsets of the partial b-metric space (X, P_b) with coefficient $s \geq 1$ and $T : A \cup B \to CB_{P_b}(A) \cup CB_{P_b}(B)$ is a multivalued $(A,B) - \gamma - FG$-contraction, Then T has a fixed point in $CB_{P_b}(A) \cap CB_{P_b}(B)$.*

Proof. Define mappings $\alpha, \beta :\to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

$$\beta(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

For $x, y \in A \cup B$ such that $\alpha(x)\beta(y) \geq 1$, we get $x \in A$ and $y \in B$. Then
$A(x)B(y) \geq 1, H_{P_b}(T(x), T(y)) > 0$

$$\Rightarrow F(A(x)B(y)sH_{P_b}(T(x),T(y))) \leq F(\mathcal{M}_s(x,y)) + G(\gamma(\mathcal{M}_s(x,y))),$$

and thus contractive condition (12) holds. Therefore, T is a multivalued $(A,B) - \gamma - FG$-contractive mapping. It is easy to see that T is a cyclic (α_*, β_*)-admissible mapping. Since A and B are nonempty closed bounded subsets, there exists $x_0 \in A$ such that $\alpha(x_0) \geq 1$ and there exists $y_0 \in B$ such that $\beta(y_0) \geq 1$. Now, all conditions of Theorem 3.1 hold, so T has a fixed point in $A \cup B$, say z. If $z \in A$, then $z = T(z) \in CB_{P_b}(B)$. Similarly, if $z \in B$ then $z \in CB_{P_b}(A)$. Hence $z \in CB_{P_b}(A) \cap CB_{P_b}(B)$. □

We can obtain the following corollaries.

Corollary 3.1. *Let A and B be two nonempty subsets of the partial b-metric space (X, P_b) with coefficient $s \geq 1$ and $T : A \cup B \to CB_{P_b}(A) \cup CB_{P_b}(B)$ be a multivalued mapping such that,*
$A(x)B(y) \geq 1, H_{P_b}(T(x), T(y)) > 0$

$$\Rightarrow F(A(x)B(y)sH_{P_b}(T(x),T(y))) \leq F(P_b(x,y)) + G(\gamma(P_b(x,y))).$$

Then T has a fixed point in $CB_{P_b}(A) \cap CB_{P_b}(B)$.

Corollary 3.2. *Let A and B be two nonempty subsets of the partial b-metric space (X, P_b) with coefficient $s \geq 1$ and $T : A \cup B \to CB_{P_b}(A) \cup CB_{P_b}(B)$ be a multivalued mapping such that,*

$$sH_{P_b}(T(x),T(y)) \leq \mathcal{M}_s(x,y)\gamma(\mathcal{M}_s(x,y)), \tag{13}$$

for all $x \in A$ and $y \in B$. Then T has a fixed point in $CB_{P_b}(A) \cap CB_{P_b}(B)$.

Proof. Taking $F(t) = G(t) = \ln(t)$ and $A(x)B(y) = 1$ in Theorem 3.1, we obtain this proof. □

4. **Application**

We consider the following nonlinear integral equation:

$$x(t) = g(t) + \lambda \int_0^1 \kappa(t,\theta) f(\theta, x(\theta)) d\theta, t, \theta \in I = [0,1], \lambda \geq 0. \tag{14}$$

Let $X = C(I)$ represents the space of all real valued mappings defined on I. Assume the following conditions:

(a) $g : I \to \mathbb{R}$ is a continuous mapping;

(b) $f : I \times X \to \mathbb{R}$ is a continuous mapping and there exists a constant $\delta \in [0,1)$ such that for all $x, y \in X$;

$$|f(t,x(t)) - f(t,y(t))| \leq \delta \sqrt{\ln \left(\mathcal{M}_s(x(t),y(t)) e^{G(\gamma(\mathcal{M}_s(x(t),y(t))))}\right)};$$

(c) $\kappa : I \times I \to \mathbb{R}$ is continuous at $t \in I$ for every $\theta \in I$ and measurable at $\theta \in I$ for all $t \in I$ such that

$$\sup_{t \in I} \int_0^1 \kappa(t,\theta) d\theta \leq \mathbb{K};$$

(d) $\lambda \mathbb{K} \delta \leq 1$;
(e) $\ln(sP_b(T(x),T(y))) \leq \sup_{t\in I} |T(x(t)) - T(y(t))|^2$.

Let $X = C(I)$ be the space of all continuous real valued mappings defined on I. Define mapping $P_b : X \times X \to [0,\infty)$ by

$$P_b(x,y) = \left(\sup_{t \in I} |x(t) - y(t)| + \eta\right)^2 \text{ for all } x, y \in X \text{ and } \eta > 0. \tag{15}$$

Then (X, P_b) is a complete partial b-metric space with $s = 2 > 1$. Also, define

$$\mathcal{M}_s(x,y) = \max\left\{P_b(x,y), P_b(y,T(y)), P_b(x,T(x)), \frac{P_b(x,T(y)) + P_b(y,T(x))}{2s}\right\}.$$

Lemma 4.1. *Let $X = C(I)$. Define mapping $P_b : X \times X \to [0,\infty)$ as in (15). Then*

$$\mathcal{M}_s(x,y) = \sup_{t \in I} \mathcal{M}_s(x(t),y(t)).$$

$$\mathcal{M}_s(x(t),y(t)) = \left\{ \begin{array}{l} \left(\sup_{t\in I} |x(t) - y(t)| + \eta\right)^2, \left(\sup_{t\in I} |y(t) - T(y(t))| + \eta\right)^2, \\ \left(\sup_{t\in I} |x(t) - T(x(t))| + \eta\right)^2, \\ \dfrac{\left(\sup_{t\in I} |x(t) - T(y(t))| + \eta\right)^2 + \left(\sup_{t\in I} |y(t) - T(x(t))| + \eta\right)^2}{2s} \end{array} \right\}.$$

Proof. Since,

$$P_b(x,y) = \left(\sup_{t \in I} |x(t) - y(t)| + \eta\right)^2 \text{ for all } x, y \in X \text{ and } \eta > 0.$$

Result follows. □

Theorem 4.1. *Let $X = C(I)$. Define the mapping $T : X \to X$ by*

$$Tx(t) = g(t) + \lambda \int_0^1 \kappa(t,\theta) f(\theta, x(\theta)) d\theta, t \in I = [0,1], \lambda \geq 0. \tag{16}$$

If the assumptions (a)-(e) hold, then the nonlinear integral equation (14) has a unique solution in X.

Proof. We note that $x^*(\cdot) \in X$ is a solution of (14) if and only if $x^*(\cdot) \in X$ is a fixed point of the mapping T defined in (16).

By assumption (b), we have

$$|Tx(t)-Ty(t)|^2 = \left|\lambda\int_0^1 \kappa(t,\theta)f(\theta,x(\theta))d\theta - \lambda\int_0^1 \kappa(t,\theta)f(\theta,y(\theta))d\theta\right|^2$$
$$\leq \lambda^2\left(\int_0^1 \kappa(t,\theta)\left|f(\theta,x(\theta))-f(\theta,y(\theta))\right|d\theta\right)^2$$
$$\leq \lambda^2\left(\int_0^1 \kappa(t,\theta)\delta\sqrt{\ln\left(\mathcal{M}_s(x(\theta),y(\theta))e^{G(\gamma(\mathcal{M}_s(x(\theta),y(\theta))))}\right)}d\theta\right)^2$$
$$\leq \lambda^2\delta^2\ln\left(\sup_{\theta\in I}\mathcal{M}_s(x(\theta),y(\theta))e^{G(\gamma(\sup_{\theta\in I}\mathcal{M}_s(x(\theta),y(\theta))))}\right)$$
$$\left(\int_0^1 \kappa(t,\theta)d\theta\right)^2.$$

By assumption (c) and (d), we have

$$\sup_{t\in I}|Tx(t)-Ty(t)|^2 \leq \lambda^2\delta^2\mathbb{K}^2\ln\left(\sup_{\theta\in I}\mathcal{M}_s(x(\theta),y(\theta))e^{G(\gamma(\sup_{\theta\in I}\mathcal{M}_s(x(\theta),y(\theta))))}\right)$$
$$\leq \left(\sup_{\theta\in I}\mathcal{M}_s(x(\theta),y(\theta))e^{G(\gamma(\sup_{\theta\in I}\mathcal{M}_s(x(\theta),y(\theta))))}\right).$$

By assumption (e) and Lemma 4.1, we have

$$\ln(sP_b(T(x),T(y))) \leq \ln\left(\mathcal{M}_s(x,y)e^{G(\gamma(\mathcal{M}_s(x,y)))}\right)$$
$$\leq \ln(\mathcal{M}_s(x,y)) + G(\gamma(\mathcal{M}_s(x,y))).$$

Now $F(r)=\ln(r)$ satisfies all the hypotheses of Corollary 2.3 and so the integral equations given by (14) has a solution. □

5. **Conclusions**

In this paper, we obtained fixed point theorem for cyclic (α_*,β_*)-type-γ-FG-contraction type for multivalued mappings in partial b-metric spaces. Our results are extensions of recent fixed point theorems of Wardowski [6], Padhan *et al.* [25] and some other results in the literature. Moreover, We also applied our main results to study existence of a solution for a nonlinear integral equation. The new concepts lead to further investigations and applications.

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