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HILL'S EQUATIONS AND LIOUVILLE'S FORMULA

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Periodicity (and its consequence, i.e. oscillatority in case of combined effect of more forces) is surely the basic and the most important phenomenon of nature. The differential equation has periodic solutions only if coefficients are periodic with the same period (or specially, constants). This is the necessary condition for the periodicity of the differential equation. However, the sufficient conditions are related to the properties of classes of equations and can be rather different. In this work we have offered four theorems on periodic solutions of Hill's equation, which we have not found in familiar monographic works [1] and [2]. We have shown that theorems 1^o – 4^o can serve for establishing different classes of Hill's equations, but periodic solutions can be obtained only for very narrow sub-classes of equations and only for certain values of given coefficients in supposed particular integrals.

Keywords: Differential equations, periodicity, oscillatority, analiticality, Liouville's formula.

1. Introduction

The equation of the following form:

$$y'' + b(x) \cdot y = 0 \qquad (1.1.)$$

where *b(x)* is a periodic function, is called Hill's equation. This equation is a well-known differential equation which frequently occurs in physical, technical and astronomic problems. A great number of important equations, frequently or directly or after performing adequate transformations, belongs to the Hill's equation type. For example, these are some Legendre's equations, some hypergeometric equations and Bessel and Matthew's equations.

One of the basic questions related to the equation (1.1) is if it has periodic solutions, either one class of periodic solutions or all periodic solutions, when we say that the solutions are in co-existence.

The necessary condition for periodicity of solutions of the differential equations requires all coefficients to be periodic with the same or commensurable period. Since the coefficient *b(x)* is periodic, it means that the necessary condition is fulfilled. However, the literature does not emphasize enough the problem of the second condition, i.e. if the integral $\int b(x) \cdot dx$ is periodic or non-periodic and when this is condition for one or both solutions to be periodic.

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If y_1 is one particular integral of the differential equation:

$$y'' + a(x)\cdot y' + b(x)\cdot y = 0 \qquad (1.2)$$

then a second particular solution is found according to Liouville's formula:

$$y_2 = y_1 \cdot \int \frac{e^{-\int a(x)\cdot dx}}{y_1^2} \cdot dx \qquad (1.3)$$

However, since $a(x) = 0$ for Hill's equation, then the first Liouville's formula for the equation (1.1) has the following form:

$$y_2 = y_1 \cdot \int \frac{dx}{y_1^2} \qquad (1.4)$$

Nevertheless, the influence of the coefficient *b(x)* on the solution y_2 cannot be explicitly seen from the (1.4). That is why we use Liouville's connection between particular integrals and the coefficient *b(x)* of the equation (1.1) based on which we obtain:

$$b(x) = \frac{1}{W(x)} \cdot \begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix} = \frac{1}{W(x)} \cdot (y_1'' \cdot y_2' - y_1' \cdot y_2'') \qquad (1.5)$$

Since $W(x) = W(y_1 y_2) = W(x_0) \cdot e^{-\int_{x_0}^{x} a(x)dx} = W(x_0) = C = const. \neq 0$

because the Wronskian determinant is either identically equal to zero or different from zero for each value of the variable *x* from the observed interval, then, without reducing the general nature, we can take that $W(x) = 1$ and obtain the following:

$$b(x) = y_1'' \cdot y_2' - y_1' \cdot y'' \qquad (1.6)$$

After being divided with $y_2'^2$, it follows that: $\frac{b(x)}{y_2'^2} = (\frac{y_1'}{y_2'})'$, which leads to

$$\frac{y_1'}{y_2'} = \int \frac{b(x)\cdot dx}{y_2'^2} \qquad (1.7)$$

The influence of the coefficient *b(x)* and function $y_2'^2$ on periodicity of the solution can be seen from the relation (1.7). Therefore, if (1.7) is integrated, the following is obtained:

$$y_1 = \int (y_2' \cdot \int \frac{b(x)}{y_2'^2} dx) \cdot dx \qquad (1.8)$$

The relation (1.8) represents the second Liouville's formula. Such approach makes possible for us to make discussion when the integral $\int \frac{b(x)}{y_2'^2} dx$ is periodic, if the solution y_2 is periodic. This is the basic question related to the periodicity of the solution (1.8).

2. Main results

Since the particular integrals y_1 and y_2 of the Hill's equation fulfill the second Liouville's formula, there is the need to separate two kinds of coefficients. These are:

a) *b(x)* is a complete continuous analytic function, which can be developed into Fourier series, convergent along the whole period, where *b(x)* either has permanent sign or zeros and changes sign.

b) *b(x)* is periodic function which is not analytic, because it has interruptible or algebraic critical points and it cannot be developed into Fourier's series. Now the question of the periodicity of the solution (1.7) is being conditioned because

c) if particular integral y_2 does not have zeros and it is not periodic, based on some previous theorems, the derivative y_2' must have zeros.

It follows that $\frac{1}{y_2'^2}$ has poles of the second order, because $y_2'^2$ has zeros of the second order. This leads to the question if the integral

$$\int \frac{b(x)}{y_2'^2} \cdot dx \qquad (2.1)$$

is going to remain periodic and under what conditions.

d) also the question of the relation of zeroes and poles of coefficient *b(x)* with zeros and poles of the function $y_2'^2$ must be considered

(A) Let *b(x)* be periodic, complete continuous analytic function. Then, by substitution

$$\frac{y'}{y} = Z \text{ or } y = e^{\int Z \cdot dx} \qquad (2.2)$$

the equation (1.1) is reduced to canonic Riccati's equation of the first order

$$-b(x) = Z' + Z^2 \qquad (2.3)$$

which is more suitable for solving because it is of the first order and the coefficient *b(x)* is separated from the unknown function $Z = Z(x)$

If $y = y(x)$ is periodic solution, then from the first formula (2.2), it follows that the function Z is also periodic. However, from the second formula (2.2) it does not follow that *y(x)* must be periodic solution if *Z(x)* is periodic function.

Therefore, (2.3) cannot solve the issue of co-existence of periodicity. That is why we must act rather systematically and consequently.

Therefore, first we look for periodic solution Z of the equation (2.3). If (2.3) is integrated, we obtain:

$$-\int b(x)\cdot dx = Z + C + \int Z^2 dx \tag{2.4}$$

Since *b(x)* is a complete continuous analytic periodic function, then, based on Cauchy's theorem, solution *Z(x)* of the equation (2.3) is also analytic function. This means that if *b(x)* can be developed into convergent Fourier series

$$b(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx, \tag{2.5}$$

then also Z has its development into Fourier series

$$Z(x) = A_0 + \sum_{k=1}^{\infty} A_k \cos kx + \sum_{k=1}^{\infty} B_k \sin kx, \tag{2.6}$$

where the coefficients (A_k, B_k) depend on (a_k, b_k).

However, from the following relation

$$\begin{aligned}\int_0^x Z^2 dx &= \int_0^x (A_0 + \sum_{k=1}^{\infty} A_k \cos kx + \sum_{k=1}^{\infty} B_k \sin kx)^2 dx = \\ &= \int_0^x \Big[A_0{}^2 + (\sum_{k=1}^{\infty} A_k \cos kx)^2 + (\sum_{k=1}^{\infty} B_k \sin kx)^2 + 2A_0 \sum_{k=1}^{\infty} A_k \cos kx + 2A_0 \sum_{k=1}^{\infty} B_k \sin kx + \\ &+ 2(\sum_{k=1}^{\infty} A_k \cos kx)\cdot(\sum_{k=1}^{\infty} B_k \sin kx)\Big]dx\end{aligned} \tag{2.7}$$

it follows that each integral which figurate in the relation (2.7) is non periodic, because

$$\int_0^x \cos^2 kx \cdot dx = \int_0^x \frac{1+\cos 2kx}{2} dx = \frac{x}{2} + \frac{1}{4k}\sin kx \text{ and}$$

$$\int_0^x \sin^2 kx \cdot dx = \int_0^x \frac{1-\cos 2kx}{2} dx = \frac{x}{2} - \frac{1}{4k}\sin kx$$

Therefore, the integral $\int_0^x Z^2 dx$ remains periodic if $A_0 = A_1 = A_2 = ... = A_k = ... = 0$ and $B_0 = B_1 = B_2 = ... = B_k = ... = 0$, so that all other products in the observed series also fall. Consequently, if Z is periodic analytic solution of the equation (2.3), the integral $\int_0^x Z^2 dx$ cannot be periodic. From (2.4) it follows that:

$$-\int b(x)\cdot dx - Z - C = \int_0^x Z^2 dx \tag{2.8}$$

Let *b(x)* be such analytic periodic function, so that the interval $\int b(x)\cdot dx$ is also periodic. In that case, the left side of the equation (2.4) is periodic, while the right side is non-periodic. This leads to contradiction created by the premise that *Z* is periodic solution. Therefore, if *Z* is analytic function, it cannot be periodic solution of the equation (2.3). From (2.2) it follows that solution $y = y(x)$ cannot be periodic, either. Thus, it follows:

Theorem 2.1. Hill's equation (1.1) in which *b(x)* is a complete, continuous, analytic function and where $\int b(x)\cdot dx$ is also periodic, cannot have a completely analytic and periodic solution.

Hence, Hill's equation can have some periodic solution only if the integral $\int b(x)\cdot dx$ is non-periodic, or if the coefficient *b(x)* is not an analytic function.

Example 2.2. Hill's equation $y'' - (\cos^2 x - \sin x)y = 0$ has analytic periodic coefficient *b(x)*, whose integral

$$\int b(x)\cdot dx = \int (\cos^2 x - \sin x)dx = \frac{x}{2} + \frac{1}{4}\sin 2x + \cos x$$

is non-periodic. The equation obviously has one periodic particular solution $y_1 = e^{\sin x}$, while the other particular solution is not periodic.

We have been familiar with the theorem that the integral $\int b(x)\cdot dx$ is periodic if *b(x)* is periodic function which has at least one zero of the odd order within one period, i.e. if *b(x)* changes the sign. Therefore, the change of the sign of the coefficient *b(x)* within one period, can be signal that the integral $\int b(x)\cdot dx$ is periodic and that Hill's equation does not have complete periodic solutions in the form of Fourier series. However, these can be possible under certain conditions.

The situation is clearer if the coefficient *b(x)* has permanent sign. In this case the theorem on monotony and oscillatority of the solutions is valid. It means that if *b(x)* is analytic function which has permanent sign in the interval of one period length, then Hill's equation cannot have periodic solutions. If $b(x)<0$, the solutions of the equation (1.1) are monotonous and can possibly have definite number of zeros. If $b(x)>0$, solutions are oscillating.

Example 2.3. Matthew's equation $y'' + (a + b\cos 2x)\cdot y = 0$, where $a>b>0$ is special case of Hill's equation and it does not have periodic solutions. However Matthew's equation, as canonical equation of the second order meets the requirements of the classic theorem on oscillations, since:

$b(x) = a + b\cdot\cos 2x > 0$ for $x \in (0,+\infty)$, because $a>b>0$ and

$$\int_0^{+\infty} b(x)\cdot dx = \int_0^{+\infty}(a + b\cos x)\cdot dx = +\infty .$$

That is why it has continuous oscillating solutions y_1 and y_2 whose zeros are located in cross sections of the curve $F(x) = x\cdot\sqrt{a + b\cos 2x}$ with horizontal lines $y = n\pi$ and $y = (n - \frac{1}{2})\pi$, i.e. in the solutions of the following equations

$x\cdot\sqrt{a + b\cos 2x} = n\pi$, $n=0,1,2,\ldots$ and $x\cdot\sqrt{a + b\cos 2x} = (2n-1)\cdot\frac{\pi}{2}$, $n=1,2,3\ldots$

(B) Let now the integral $\int b(x)\cdot dx$ be non-periodic, while the coefficient $b(x)$ is a complete periodic function. We are going to consider the possibility if $b(x)$ is also analytic function, whether it is possible for Hill's equation (1.1) to have analytic solution.

Let $-b(x)$ be analytic function presented by Fourier series

$$-b(x) = a_0 + \sum_{k=1}^{\infty} a_k\cdot\cos kx + \sum_{k=1}^{\infty} b_k\cdot\sin kx,$$

but in such a way so that the integral $-\int b(x)\cdot dx$ should not be periodic function. However it should be:

$$-\int b(x)\cdot dx = a_0 x + \sum_{k=1}^{\infty}\frac{a_k}{k}\sin kx - \sum_{k=1}^{\infty}\frac{b_k}{k}\cos kx + \sum_{k=1}^{\infty}\frac{b_k}{k},$$

where $a_0 \neq 0$ and the series $\sum_{k=1}^{\infty}\frac{b_k}{k}$ converges.

Then, based on the relations (2.4.) and (2.6.), we obtain:

$$(a_0 x + \sum_{k=1}^{\infty}\frac{b_k}{k}) + \sum_{k=1}^{\infty}\frac{a_k}{k}\sin kx - \sum_{k=1}^{\infty}\frac{b_k}{k}\cos kx =$$

$$= (A_0 + \sum_{k=1}^{\infty} A_k\cos kx + \sum_{k=1}^{\infty} B_k\sin kx) + \int_0^x Z^2 dx \tag{2.10}$$

Based on the relation (2.6), we calculate the integral $\int_0^x Z^2 dx$, so it follows:

$$\int_0^x Z^2 dx = \int_0^x (A_0 + \sum_{k=1}^{\infty} A_k\cos kx + \sum_{k=1}^{\infty} B_k\sin kx)^2 dx =$$

$$= \int_0^x \left\{ {A_0}^2 + (\sum_{k=1}^{\infty} A_k\cos kx)^2 + (\sum_{k=1}^{\infty} B_k\sin kx)^2 + 2A_0\sum_{k=1}^{\infty} A_k\cos kx + 2A_0\sum_{k=1}^{\infty} B_k\sin kx + 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} A_k B_j\cdot\cos kx\cdot\sin jx \right\} dx =$$

$$= \int_0^x \left\{ A_0^{\,2} + \sum_{k=1}^{\infty} A_k^{\,2} \cos^2 kx + 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} A_k A_j \cdot \cos kx \cdot \cos jx + \sum_{k=1}^{\infty} B_k^{\,2} \sin^2 kx + 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} B_k B_j \cdot \sin kx \cdot \sin jx + \right.$$

$$\left. + 2A_0 \sum_{k=1}^{\infty} A_k \cos kx + 2A_0 \sum_{k=1}^{\infty} B_k \sin kx + 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} A_k B_j \cdot \cos kx \cdot \sin jx \right\} dx =$$

$$= \int_0^x \left\{ A_0^{\,2} + \sum_{k=1}^{\infty} A_k^{\,2} \cdot \frac{1+\cos 2kx}{2} + \sum_{k=1}^{\infty} B_k^{\,2} \frac{1-\cos 2kx}{2} + 2A_0 \sum_{k=1}^{\infty} A_k \cos kx + 2A_0 \sum_{k=1}^{\infty} B_k \sin kx + \right.$$

$$\left. + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} A_k A_j \cdot \cos kx \cdot \cos jx + 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} B_k B_j \cdot \sin kx \cdot \sin jx + 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} A_k B_j \cdot \cos kx \cdot \sin jx \right\} dx =$$

$$= A_0^{\,2} \cdot x + \frac{x}{2} \cdot \sum_{k=1}^{\infty} (A_k^{\,2} + B_k^{\,2}) + 2A_0 \sum_{k=1}^{\infty} \frac{B_k}{k} + \sum_{k=1}^{\infty} \frac{A_k^{\,2}}{4k} \sin 2kx - \sum_{k=1}^{\infty} \frac{B_k^{\,2}}{4k} \sin 2kx +$$

$$+ 2A_0 \cdot \sum_{k=1}^{\infty} \frac{A_k}{k} \sin kx - 2A_0 \cdot \sum_{k=1}^{\infty} \frac{B_k}{k} \cos kx + 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} A_k A_j \cdot \int_0^x \cos kx \cdot \cos jx \cdot dx +$$

$$+ 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} B_k B_j \cdot \int_0^x \sin kx \cdot \sin jx \cdot dx + 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} A_k B_j \cdot \int_0^x \cos kx \cdot \sin jx \cdot dx =$$

$$= x\left[A_0^{\,2} + \frac{1}{2} \sum_{k=1}^{\infty} (A_k^{\,2} + B_k^{\,2}) \right] + 2A_0 \sum_{k=1}^{\infty} \frac{B_k}{k} + Q(x)$$

where *Q(x)* is trigonometric series which must be convergent.

If we substitute the obtained result in (2.4), we are going to obtain

$$\left(a_0 \cdot x + \sum_{k=1}^{\infty} \frac{b_k}{k} \right) + P(x) = A_0 + R(x) +$$
$$+ x \cdot \left[A_0^{\,2} + \frac{1}{2} \sum_{k=1}^{\infty} \left(A_k^{\,2} + B_k^{\,2} \right) \right] + 2A_0 \sum_{k=1}^{\infty} \frac{B_k}{k} + Q(x) \qquad (2.11)$$

where *P, Q* and *R* are trigonometric series which have free members.

Now it is easy to separate linear part of the non-periodic integral from the periodic one of the trigonometric series: $L_1(x) + P(x) = L_2(x) + R(x) + Q(x)$, where there are:

$$\begin{cases} L_1(x) = a_0 \cdot x + \sum_{k=1}^{\infty} \frac{b_k}{k} \; and \\ L_2(x) = x \cdot \left[A_0^{\,2} + \frac{1}{2} \sum_{k=1}^{\infty} \left(A_k^{\,2} + B_k^{\,2} \right) \right] + A_0 + 2A_0 \cdot \sum_{k=1}^{\infty} \frac{b_k}{k} \end{cases} \qquad (2.12)$$

If the linear parts are identical to the equation, then the following equality of the series is valid

$$P(x) = R(x) + Q(x) \qquad (2.13)$$

which provides general connections between known coefficients $(a_k\,,\ b_k)$ and the unknown and wanted $(A_k\,,\ B_k)$.

The equality $L_1(x) \equiv L_2(x)$ annuls the deviation from the periodicity when talking about the integral $-\int b(x)\cdot dx$ and $\int Z^2(x)\cdot dx$

Hence, it follows:

$$\begin{cases} a_0 = A_0 + \dfrac{1}{2}\sum_{k=1}^{\infty}\left(A_k{}^2 + B_k{}^2\right) \\ \sum_{k=1}^{\infty}\dfrac{b_k}{k} = A_0\cdot\left(1+2\sum_{k=1}^{\infty}\dfrac{B_k}{k}\right) \end{cases} \text{and} \tag{2.14}$$

It is so, if the integrals of the products $\sin kx\cdot\sin jx$, $\cos kx\cdot\cos jx$, $\sin kx\cdot\cos jx$, on substitution of the lower limit, do not produce one more constant which would supplement the second of the last two formulae (2.14).

In this case, the convergence of the series is necessary above all:

$$\sum_{k=1}^{\infty}\frac{b_k}{k}\,,\ \sum_{k=1}^{\infty}\frac{B_k}{k}\,,\ \sum_{k=1}^{\infty}A_k{}^2 \text{ and } \sum_{k=1}^{\infty}B_k{}^2\,, \tag{2.15}$$

although the first formula from (2.14) is the most important for the procedure. Namely, it clearly shows that $a_0 = 0$ is not possible, since the integral $\int b(x)\cdot dx$ is non-periodic and it does not have to contain a free member a_0, which makes for the same reasons $A_0 \neq 0$. Therefore from (2.13) and (2.14) it is possible to establish the system of recurrent connections $(a_k, b_k) \leftrightarrow (A_k, B_k)$, which, in general, at least lead to determination of the periodic solution of Riccati's equation (2.3) with possible periodicity of the integral $\int Z^2(x)\cdot dx$ through (2.2) and periodicity of the solution $y=y(x)$ of Hill's equation.

It is obvious that, due to complicated sums and lack of symmetry , it is not possible to determine explicitly the coefficients A_k and B_k in the function of the known a_k and b_k. That is why it is not possible to set the general theorem and we must refer to the monograph [1], which achieved the most, but only some special sufficient conditions.

Example 2.4 The simplest and the most convenient example of Hill's equation is the equation of harmonious oscillations.

$$y'' + n^2\cdot y = 0$$

The coefficient $b(x) = n^2 = const.$ is periodic function whose period is any real number. However the integral $\int b(x)\cdot dx = \int n^2\cdot dx = n^2\cdot x$ is non-periodic, so periodic solutions of Riccati's equation can be expected.

$$-n^2 = Z' + Z^2$$

If we separate the variables, we obtain:

$Z = n \cdot tg(C - nx)$ and $y_1 = \cos(C - nx),$ because $Z = \frac{y'}{y}$

By introducing substitution $y = y_1 \cdot W$, we obtain the second particular integral $y_2 = \sin(C - nx)$. By linear combination we obtain general solution:

$$y = C_1 \cos nx + C_2 \sin nx$$

However, such quadratures are possible only in small number of cases.

C) Let us consider the case when the coefficient *b(x)* is a non-analytic function. In this case the coefficient *b(x)* has interruptions, poles, algebraic and logarithmic critical points and cannot be developed into Fourier's series. These are quite often cases when the periodicity of the integral of coefficient *b(x)* is maintained, no matter if the undefined integral exists or not within the interval of the one period's length, as in the following example:

$$b(x) = \frac{1}{\cos^2 x},\ \frac{1}{\sin^2 x},\ \frac{1}{\sin x},\ \dots$$

Then, based on Liouville's formula , the functions $\frac{b(x)}{y_1'^2}$ and $\int \frac{b(x)dx}{y_1'^2}$ should be investigated when they are periodic. This makes us look for the connection between coefficient *b(x)* and properties of the solution and properties of the derivative of the solution of Hill's equation. In that sense the following theorems should be proved:

Theorem 2.5. Zeroes of the first order of the solution y_1 of Hill's equation are simultaneously the poles of the first order of the coefficient *b(x)*.

Theorem 2.6. Zero of the complete order *n* of the solution y_1 of Hill's equation is the pole of the second order of the coefficient *b(x)*

Theorem 2.7. Every zero of any complete order, any algebraic critical point of the order *a* or *b* $(a \neq 1, b \neq 1)$ or pole of the order *a* or *b*, represent only pole of the second order for the coefficient *b(x)*.

The theorems are easily proven if Hill's equation is written in the form $b(x) = -\frac{y''(x)}{y(x)}$ and then

$$y_1 = (\sin x) \cdot P(x)$$

$$y_2 = (\sin x)^n \cdot P(x)$$

$$y_1 = (\sin x)^a \cdot P(x)$$

$$y_1 = (\sin x)^a \cdot (\cos x)^b \cdot Q(x)$$

where *P(x)* and *Q(x)* are only polynomial whose zeros are not zeros of the functions $\sin x$ i.e. $\cos x$, and where $a<0$ and $b<0$ can be possible.

However, the opposite case can also be valid, i.e. the poles of the first order of the coefficient *b(x)* are zeros of the first order of solution y_1 of Hill's equation. Equally, if the coefficient *b(x)* has poles of the second order, these can be either poles of the higher order or zeros of the higher order or algebraic critical points of any order *a* (except the first one), of the Hill's equation.

This method serves for solving probably the greatest number of Hill's equations.

Example 2.8. The equation $y'' - \frac{3}{\sin^2 x \cdot \cos^2 x} \cdot \left(\frac{1}{4} + \sin^2 x \right) \cdot y = 0$ has negative coefficient *b(x)* unlike the previous equation. Since the coefficient *b(x)* has the poles of the second order, the theorems on monotony and oscillatority are not valid here. According to our theorems, the given equation can have particular solution of the following form:

$$y_1 = (\sin x)^\alpha \cdot (\cos x)^\beta \cdot P(x)$$

However, owing to the symmetry, it is the best way to take that $P(x) \equiv 1$ and $\beta = -\alpha$, so the particular integral $y_1 = (tgx)^\alpha$. If we substitute y_1 and y_1'', we obtain the identity:

$$\left[\alpha(\alpha - 1) - \frac{3}{4} \right] \cdot (tgx)^{\alpha-2} + (2\alpha^2 - \frac{9}{2}) \cdot (tgx)^\alpha + (\alpha^2 + \alpha - \frac{15}{4}) \cdot (tgx)^{\alpha+2} \equiv 0$$

which will be fulfilled for every real *x*, if parameter α is the solution of the following system:

$$\alpha^2 - \alpha - \frac{3}{4} = 0, \quad 2\alpha^2 - \frac{9}{2} = 0, \quad \alpha^2 + \alpha - \frac{15}{4} = 0$$

The system is fulfilled for $\alpha = \frac{3}{2}$, so one particular integral is $y_1 = (tgx)^{3/2}$.

According to Liouville's formula the general integral:

$$y = C_1 y_1 + C_2 y_1 \int \frac{dx}{y_1^2} = C_1 \sqrt{(tgx)^3} + C_2 \sqrt{(tgx)^3} \cdot \int \frac{dx}{tg^3 x} =$$

$$= C_1 \sqrt{(tgx)^3} + C_2 \sqrt{(tgx)^3} \cdot (\ln \sin x + \frac{1}{2\sin^2 x}) =$$

$$= \sqrt{(tgx)^3} \left[C_1 + C_2 (\ln \sin x + \frac{1}{2\sin^2 x} \right]$$

is periodic function, interruptible in the following points: $x_1 = 0$, $x_2 = \frac{\pi}{2}$ and $x_3 = \pi$.

Example 2.9. Based on known theorems, for Hill's equation $y'' - \frac{3-2\sin x\cdot\cos x}{1+2\sin x\cdot\cos x}\cdot y = 0$, the solution of the following type $y_1 = (\sin x + \cos x)^{\alpha}\cdot P(x)$ has been suggested. $P(x)\equiv 1$ has the following solution:

$$y = (\sin x + \cos x)^{\alpha}$$

where α is a parameter which should be determined. If we substitute y and y'' in the given equation, we obtain the identity which is fulfilled for every real *x* if $\alpha = -1$. Therefore, one particular solution is $y_1 = \frac{1}{\sin x + \cos x}$. The general solution is easily found according to Liouville's formula.

$$y = C_1 y_1 + C_2 y_1 \int \frac{dx}{y_1^2} = \frac{1}{\sin x + \cos x} + \left[C_1 + C_2(x - \cos^2 x)\right]$$

However, it is not periodic, because it does not have two linearly independent periodic particular integrals, i.e. two integrals which are in coexistence. The given equation has only one periodic sub-class of particular integrals.

3. Conclusion

In this work we have shown that Hill's equation (1.1) can have periodic solutions if the integral $\int b(x)\cdot dx$ is non-periodic function or if the periodic coefficient *b(x)* is not analytic function (the function is analytic if it has derivative of any series, i.e. if it can derive into Taylor's series). That is why the Hill's equation is very often written in the following form:

$$y'' + (\lambda + Q(x)\cdot y = 0 \qquad (3.1)$$

where

$$b(x) = \lambda + Q(x) \qquad (3.2)$$

is periodic function as a sum of constant and periodic function, while the integral

$$\int b(x)dx = \int(\lambda + Q(x))dx = \lambda\cdot x + \int Q(x)dx \qquad (3.3)$$

is non-periodic, because it contains the linear part $\lambda\cdot x$

We have also shown, by using Liouville's formula, that the poles of the first order of the coefficient *b(x)* of the equation (1.1) are simultaneously the zeros of the first order of particular integral y_1. This is used for solving many Hill's equations. Namely, by substitution

$$y(x) = e^{\int Z(x)dx} \qquad (3.4)$$

where $Z = Z(x)$ is a new unknown function, the Hill's equation is most often reduced to Riccati's equation:

$$Z'' + a(x)Z^2 + b(x)Z + c(x) = 0 \qquad (3.5)$$

So it is very important to determine the form of its particular integral, because it is known that in general case, the equation (3.5) cannot be solved by applying quadratures.

We have shown that the poles of second order of the coefficient of the equation (1.1) can be either poles of higher order or zeros of higher order or algebraic critical points of any order, except the first one, solutions of Hill's equations.

By using this facts, periodic solutions for some narrow subclasses of Hill's equation can be determined, as well as for some values of the given coefficients in the supposed particular integrals.

Therefore, periodic coefficient *b(x)* in the Hill's equation (1.1) is a necessary but not sufficient condition for the equation to have one periodic particular solution or both periodic solutions, when we say that solutions are co-existing.

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