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ON THE WELL-POSEDNESS OF RELAY SYSTEMS WITH FRACTIONAL ORDER

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In aceasta lucrare introducem o noua clasă de sisteme discontinue fractionale şi stabilim condiţii suficiente pentru unicitatea soluţiilor. Utilizând transformata Laplace, arătăm că tehnica propusa în [1]*, pentru systeme de tip relay lineare, poate fi utilizată cu success în studiul unicităţii solutiilor analitice pe porţiuni. Stabilim de asemenea o condiţie suficientă pentru existenţa şi unicitatea soluţiilor in spaţiul funcţiilor continuu diferentiabile.Un exemplu este prezentat în acest sens.*

In this paper a new class of fractional-order discontinuous-time systems containing relays is introduced and sufficient conditions for the uniqueness of solutions are established. By using the Laplace transform, we show that the technique proposed in [1] *for linear relay systems may be suitably applied to this new class, in order to state the uniqueness of piecewise analytic solutions. Then we derive a sufficient condition for the existence and uniqueness of solutions in the space of continuously differentiable functions. An example is also presented.*

Keywords: Relay systems, fractional discontinuous systems, well-posedness.

1. Introduction

Many linear viscoelastic damping materials exhibit a macroscopic constitutive behavior involving fractional order derivatives. Such behavior has been the subject of many investigations. Also, some dynamical processes such as gas diffusion and heat conduction can be more precisely modeled using fractional-order models than using integer-order models.

Generally speaking, there are three mostly used definitions of the fractional derivative of a function: Grünwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative and Caputo's fractional derivative. While the pure mathematicians work with the first two definitions (which have certain disadvantages when trying to model real-world phenomena), the last one seems to be more convenient in engineering applications and thus, adopted by the applied scientists.

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The physical interpretation of fractional derivatives and the solutions of fractional differential equations have been discussed in [2]. In this paper we shall deal with the following Caputo definition.

Definition 1. Let $0 < \alpha \le 1$. The Riemann-Liouville integral of order α for a function $x(.) \in L^1((0;\infty);\mathbb{R}^n)$ is defined as

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(s)ds,\ t \ge 0,$$

where $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt,\ p > 0$ is the Gamma function.

Definition 2. The Caputo derivative of order α for a function $x(.)$ is given by

$$D^\alpha x(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{x'(\tau)}{(t-\tau)^\alpha} d\tau, & \alpha \in (0,1) \\ \dfrac{d}{dt} x(t), & \alpha = 1 \end{cases},$$

provided that the expressions on the right-hand side exist. For example, for $\alpha \in (0,1)$ and an absolutely continuous function $x(.)$, the fractional derivative exists.

An important function that finds widespread use in the world of fractional calculus is the *Mittag-Leffler function*. The standard definition of the Mittag-Leffler function with complex argument $z \in \mathbb{C}$ is the following

$$E_\alpha(z) = \sum_{k \ge 0} \frac{z^k}{\Gamma(\alpha k + 1)}$$

which is, in fact, a (one-parameter) generalization of the exponential function (for a more general definition of Mittag-Leffler function with two parameters and its applications in fractional evolution processes, see e.g. [3], [4]). As the exponential function plays an important role in the theory of integer-order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. Among other properties, we mention the following:

a) $E_\alpha(0) = 1$ and $E_1(z) = e^z$ (that is, the exponential function corresponds to $\alpha = 1$).

b) $\dfrac{E_\alpha(z) + E_\alpha(-z)}{2} = \sum_{k \ge 0} \dfrac{z^{2k}}{\Gamma(2k\alpha + 1)}$.

c) $\frac{E_\alpha(z)-E_\alpha(-z)}{2}=\sum_{k\geq 0}\frac{z^{2k+1}}{\Gamma[(2k+1)\alpha+1]}$.

d) If $z\in\mathbb{C}$ with $|\arg(z)|\in\left(\alpha\frac{\pi}{2},\pi\right]$ then $\lim_{|z|\to\infty}E_\alpha(z)=0$. If $z\in\mathbb{C}$ with $|\arg(z)|<\alpha\frac{\pi}{2}$, then $\lim_{|z|\to\infty}E_\alpha(z)=\infty$. In particular, if $z\in\mathbb{R}_+$, one has:

$$\lim_{z\to\infty}E_\alpha(z)=\infty \text{ and } \lim_{z\to\infty}E_\alpha(-z)=0.$$

Given a matrix $M\in\mathcal{M}_{n,n}(\mathbb{R})$, we say that M is a P-matrix if all its principal minors are strictly positive. M is said to be a P_0-matrix if all its principal minors are non-negative. In the sequel, I_n will stand for the unit matrix of size n. For a vector v we write $v\succeq 0$ if the inequality holds component wise. The script $\mathcal{L}$ will stand for the Laplace transform.

Definition 3. Let $h:\mathbb{R}^l\to\mathbb{R}^l$ be a continuous function such that there exists a finite family of affine functions $\{h^1,...,h^k\}$ that maps $\mathbb{R}^m$ into itself and for every $x\in\mathbb{R}^l$ there is an $i=\overline{1,k}$ such that $h(x)=h^i(x)$. Then, h is said to be piecewise affine (PWA). If, in addition, $\det(Jh^i)$ has the same nonzero sign for all $i=\overline{1,k}$, then the PWA function h is said to be coherently oriented.

A switched system is said to have an accumulation point $\tau\geq 0$ of switches at the right (left) of τ if for any switched point $T>\tau$ ($T<\tau$), there exists another one $T^{'}>\tau$ ($T^{'}<\tau$) such that $T^{'}<T$ ($T^{'}>T$) and the sequence of these switches tends to τ.

In the remainder of this note, we formulate the problem and then the approach proposed in [1] is adopted in order to state the global well-posedness of this new class of relay systems, relatively to the space of piecewise analytic functions. We also establish a condition under which the local uniqueness of solution holds in a larger class, that of continuously differentiable functions. The note is concluded in Section 3.

2. Linear relay fractional-order systems.

For $0<\alpha\leq 1$, we consider the following fractional-order, multi-input multi-output relay system (FO-MIMO relay system, on short):

$$\begin{cases} D^{\alpha}x(t) = Ax(t) + B\overline{u}(t) \\ \overline{y}(t) = Cx(t) + D\overline{u}(t) \\ \overline{u}_i(t) \in -Sgn(\overline{y}_i(t)), \quad i = \overline{1,m} \end{cases} \tag{1.1}$$

with the initial condition $x(0) = x_0 \in \mathbb{R}^n$, where $A \in \mathcal{M}_{n,n}(\mathbb{R})$, $B \in \mathcal{M}_{n,m}(\mathbb{R})$, $C \in \mathcal{M}_{m,n}(\mathbb{R})$, $D \in \mathcal{M}_{m,m}(\mathbb{R})$; $\overline{u}_i(t)$ and $\overline{y}_i(t)$ stand for the i-th component of the control input vector $\overline{u}(t)$ and of the output $\overline{y}(t)$, respectively. For $i = \overline{1,m}$, each pair $\left(\overline{u}_i, \overline{y}_i\right)$ satisfies an ideal relay characteristic, via the multi-valued $Sgn(.)$ function, where $Sgn(0) = [-1,1]$.

It is well known (see, for instance, [5]) that the above system is equivalent to the following Volterra system coupled to relays

$$\begin{cases} x(t) = x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \dfrac{Ax(\tau) + B\overline{u}(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ \overline{y}(t) = Cx(t) + D\overline{u}(t) \\ \overline{u}_i(t) \in -Sgn(\overline{y}_i(t)), \quad i = \overline{1,m}. \end{cases} \tag{1.2}$$

The equivalent form in (1.2) will be useful later in the study of C^1 solutions.

Previous works. For $m = n$ and $C = I_n$, $D = 0_n$, the system (1.1) becomes

$$D^{\alpha}x(t) \in Ax(t) - Bs(x(t)),$$

where

$$s(x) = \begin{pmatrix} Sgn(x_1) \\ \vdots \\ Sgn(x_n) \end{pmatrix}.$$

A class of such discontinuous systems was investigated in [6]. More precisely, using a selection theorem due to Cellina, the author approximates the above inclusion by a single-valued fractional order problem, which in turn, is solved by using a fractional numerical scheme proposed in [7]. This scheme is a generalization of the classical multistep method Adams-Bashforth-Moulton.

Another class of FO-MIMO systems but without relays, continuous in time, was studied in [8]. Due to the absence of relays, the realization of these systems as the so called ”cone fractional systems” is obtained.

Let us now introduce the following

Definition 4. A triple $(\bar{u},\bar{x},\bar{y}):[0,\infty)\to\mathbb{R}^l\times\mathbb{R}^l\times\mathbb{R}^n$ is called a forward solution to the relay system (1), if $t\mapsto x(t)$ is continuous on $[0,\infty)$ and there exists a countable number of switching times $0=t_0<t_1<...<t_j<...$ such that, for every interval $[t_j,t_{j+1})$, the triple $(\bar{u},\bar{x},\bar{y})$ satisfies the following conditions

i) For any $i=\overline{1,l}$ and $t\in[t_j,t_{j+1})$, $\bar{u}_i(t)$ and $\bar{y}_i(t)$ correspond to one and only one of the following three branches:

$$\left[\bar{y}_i(t)>0 \text{ and } \bar{u}_i(t)=-1\right]$$
$$\left[\bar{y}_i(t)<0 \text{ and } \bar{u}_i(t)=1\right]$$
$$\left[\bar{y}_i(t)=0 \text{ and } \bar{u}_i(t)\in\left[\alpha_i,\beta_i\right]\right].$$

i) $(\bar{u},\bar{x},\bar{y})$ is analytic on $[t_j,t_{j+1})$.

ii) $(\bar{u},\bar{x},\bar{y})$ verifies (1) with initial condition $x(t_j)=\lim\limits_{t\nearrow t_j}x(t)$. For $j=0$ the initial condition is as for (1): $x(0)=x_0$.

It is our purpose to give a short but self-contained approach in the study of the well-posedness of FO-MIMO relay system (1.1). In what follows, considering $\alpha\in(0,1)$, we investigate the well-posedness of system (1.1) in the sense of Definition 4. The first result of this section is contained in the following theorem.

Theorem 1. If there exists $s_0\geq 0$ such that $G(s)=C(s^\alpha I_n-A)^{-1}B+D$ is an invertible P_0-matrix for $s\geq s_0$ then, for any initial condition $x(0)=x_0$, the relay system (1.1) admits a unique forward solution $(\bar{u}(.),\bar{x}(.),\bar{y}(.)),\ \ t\geq 0$.

Proof. Following [1], we show that system (1.1) may be transformed into a rational complementarity system. First, let us introduce ([9]) the following vectors in $\mathbb{R}^{2m}$

$$u(t)=\begin{pmatrix}1_m+\bar{u}(t)\\ 1_m-\bar{u}(t)\end{pmatrix} \text{ and } y(t)=\begin{pmatrix}\left(\bar{y}(t)\right)^+\\ \left(\bar{y}(t)\right)^-\end{pmatrix},$$

where 1_m denotes the constant vector in $\mathbb{R}^m$ with all components being equal to one; $\left(\bar{y}(t)\right)^+$ is the non-negative part of $y(t)$ and $\left(\bar{y}(t)\right)^-$ is the non-positive part of this vector, that is $y(t)=\left(\bar{y}(t)\right)^+-\left(\bar{y}(t)\right)^-$. With the above notations, one has

$$
\begin{aligned}
u^T(t)\cdot y(t) &= \left[1_m+\overline{u}(t)\right]^T\left(\overline{y}(t)\right)^+ + \left[1_m-\overline{u}(t)\right]^T\left(\overline{y}(t)\right)^- \\
&= 1_m^T\left[\left(\overline{y}(t)\right)^+ + \left(\overline{y}(t)\right)^-\right] + \left(\overline{u}(t)\right)^T\cdot\overline{y}(t) \\
&\in \sum_{\overline{y}_i(t)>0}\overline{y}_i(t) - \sum_{\overline{y}_i(t)<0}\overline{y}_i(t) - \sum_{i=\overline{1,m}}\overline{y}_i(t)Sgn\left(\overline{y}_i(t)\right) \\
&= \{0\}.
\end{aligned}
$$

Then, the relay system can be described by the former two equations of (1.1) with the additional complementarity constraints

$$0 \preceq u(t) \perp y(t) \succeq 0, \tag{1.3}$$

where $u(t)\perp y(t)$ indicates the orthogonality of $u(t)$ and $y(t)$, i.e. $u(t)^T y(t)=0$. In order to apply the Laplace transform to (1.1)-(1.2), let us adopt the following notations:

$$X(s)=\mathcal{L}[x(t)](s),\quad \overline{Y}(s)=\mathcal{L}\left[\overline{y}(t)\right](s),\quad \overline{U}(s)=\mathcal{L}\left[\overline{u}(t)\right](s),$$

$$U(s)=\begin{pmatrix} s^{-1}1_m+\overline{U}(s) \\ s^{-1}1_m-\overline{U}(s)\end{pmatrix} \text{ and } Y(s)=\begin{pmatrix}\left(\overline{Y}(s)\right)^+ \\ \left(\overline{Y}(s)\right)^-\end{pmatrix}.$$

The relay system in (1) can be rewritten in the frequency domain as follows:

$$\begin{cases} X(s)\left[s^\alpha I_n - A\right] = s^{\alpha-1}x_0 + B\overline{U}(s) \\ \overline{Y}(s) = CX(s) + D\overline{U}(s)\end{cases}, \tag{1.4}$$

together with the constraints

$$0 \preceq U(s) \perp Y(s) \succeq 0, \tag{1.5}$$

for $s\in\mathbb{R}_+$ sufficiently large (see Remark 2.1. in [10]). It is easy to see that $s^\alpha I_n - A$ is invertible and $\left[s^\alpha I_n - A\right]^{-1} = \sum_{k\geq 0} A^k s^{-(k+1)\alpha}$. Define the following matrices:

$$G(s) = C(s^\alpha I_n - A)^{-1}B + D \in \mathcal{M}_{m,m}(\mathbb{R})$$

$$T(s) = C(s^\alpha I_n - A)^{-1}s^{\alpha-1} \in \mathcal{M}_{m,n}(\mathbb{R}).$$

Hence system (1.3) is equivalent to

$$\begin{cases} X(s) = \left[s^\alpha I_n - A\right]^{-1}s^{\alpha-1}x_0 + \left[s^\alpha I_n - A\right]^{-1}B\overline{U}(s) \\ \overline{Y}(s) = T(s)x_0 + G(s)\overline{U}(s)\end{cases},$$

Now, assuming $G(s)$ to be invertible, we notice that the study of the well-posedness of the initial system on a small time interval $[0,\varepsilon]$ is reduced ([10], [1]) to the study of the so-called rational complementarity problem $\left(RCP\left(q(s),M(s)\right)\right)$:

$$U(s)=q(s)+M(s)Y(s), \tag{1.6}$$

under the constraints given in (1.5), where

$$q(s)=\begin{bmatrix}-G^{-1}(s)T(s)x_0+s^{-1}1_m\\ G^{-1}(s)T(s)x_0+s^{-1}1_m\end{bmatrix} \text{ and } M(s)=\begin{bmatrix}G^{-1}(s) & -G^{-1}(s)\\ -G^{-1}(s) & G^{-1}(s)\end{bmatrix} \tag{1.7}$$

For $s\in\mathbb{R}$ fixed, the data of $\left(RCP\left(q(s),M(s)\right)\right)$ in (1.6) and (1.7) define a standard linear complementarity problem (LCP on short). Next, due to Theorem 4.1 and Theorem 4.9 in [11], the existence and uniqueness of solutions to $\left(RCP\left(q(s),M(s)\right)\right)$ is equivalent to the existence and uniqueness of solutions to $\left(LCP\left(q(s),M(s)\right)\right)$ for all s sufficiently large.

Now, supposing that there exists $s_0\geq 0$ such that $G(s)$ is an invertible P_0-matrix for all $s\geq s_0$, the existence of solutions to $\left(LCP\left(q(s),M(s)\right)\right)$ follows immediately from Theorem 3.1 in [10]. Based on the correspondence between strictly proper rational functions and real-analytic time functions, we may conclude that the system (1.1) admits a unique forward solution. □

Remark 1. In fact, once the solution $(U(.),Y(.))$ of the rational problem (1.6), (1.4) is identified for $s\geq s_0$, due to Theorem 1 in [12], the solution to the state equation in (1.1) is given by the formula

$$x(t)=E_\alpha(At^\alpha)x_0+\int_0^t\phi(t-\tau)B\overline{u}(\tau)d\tau \tag{1.8}$$

on some interval $[0,\varepsilon)$, where $\phi(t)=\sum_{k\geq 0}\frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}$, $\overline{u}(t)=\mathcal{L}^{-1}\left[\overline{U}(s)\right](t)$, $\overline{U}(s)=\frac{1}{2}U(s)$ and $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

The global solution starting from x_0 will be constructed by taking the maximal interval $[0,\varepsilon)$, where $\overline{u}$ and $\overline{y}$ satisfy (1.1) and then, solving the system with the new initial condition $x(\varepsilon)=\lim_{t\nearrow\varepsilon}x(t)$.

Remark 2. As already remarked in [1] (see also [13]), the existence of finite accumulation points of relay switching instants (a kind of Zeno behavior) is allowed, but only at the left. The accumulations at the right of an instant should never occur in order to prove the uniqueness of forward solutions. In the case of the accumulations at the left, $(\varepsilon_n)_{n\ge 1} \nearrow \varepsilon$, we define $x(\varepsilon) = \lim_{n\to\infty} x(\varepsilon_n)$.

Example 1. Consider the system (1.1) with initial state $x_0^T = (x_0^1, x_0^2)$ and the following coefficients

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ C = (1 \ \ -1) \text{ and } D = 1.$$

Notice first that $A^k = \begin{cases} A, & k = 2l+1 \\ I_2, & k = 2l \end{cases}$. In view of the properties b), c) of the Mittag-Leffler function mentioned in the first section, after some computations we obtain

$$E_\alpha(At^\alpha) = \begin{pmatrix} \dfrac{E_\alpha(t^\alpha) + E_\alpha(-t^\alpha)}{2} & \dfrac{E_\alpha(t^\alpha) - E_\alpha(-t^\alpha)}{2} \\ \dfrac{E_\alpha(t^\alpha) - E_\alpha(-t^\alpha)}{2} & \dfrac{E_\alpha(t^\alpha) + E_\alpha(-t^\alpha)}{2} \end{pmatrix} \text{ and } \int_0^t \phi(t-\tau)d\tau = E_\alpha(t^\alpha) - 1.$$

Next, depending on the initial condition x_0, three possibilities may occur on a small interval $(0, \varepsilon)$:

Mode 1. $\overline{y}(t) < 0$ and $\overline{u}(t) \equiv 1$. Then, according to (1.8), the state equation will be of the form

$$x(t) = \begin{pmatrix} -1 + E_\alpha(t^\alpha)\left(1 + \dfrac{x_0^1 + x_0^2}{2}\right) + E_\alpha(-t^\alpha)\dfrac{x_0^1 - x_0^2}{2} \\ -1 + E_\alpha(t^\alpha)\left(1 + \dfrac{x_0^1 + x_0^2}{2}\right) - E_\alpha(-t^\alpha)\dfrac{x_0^1 - x_0^2}{2} \end{pmatrix}$$

The inequality $\overline{y}(t) < 0$ implies $E_\alpha(-t^\alpha)\left(x_0^1 - x_0^2\right) < -1$.

Mode 2. $\overline{y}(t) > 0$ and $\overline{u}(t) \equiv 1$. Then the state equation will be of the form

$$x(t) = \begin{pmatrix} 1 + E_\alpha(t^\alpha)\left(1 + \dfrac{x_0^1 + x_0^2}{2}\right) + E_\alpha(-t^\alpha)\dfrac{x_0^1 - x_0^2}{2} \\ 1 + E_\alpha(t^\alpha)\left(1 + \dfrac{x_0^1 + x_0^2}{2}\right) - E_\alpha(-t^\alpha)\dfrac{x_0^1 - x_0^2}{2} \end{pmatrix}$$

The inequality $\bar{y}(t) > 0$ implies $E_\alpha(-t^\alpha)\left(x_0^1 - x_0^2\right) > 1$.

Mode 3. $\bar{y}(t) \equiv 0$ and $\bar{u}(t) \in [-1,1]$. Then the state equation will be of the form

$$x(t) = \begin{pmatrix} -\bar{u}(t) + E_\alpha(t^\alpha)\left(1 + \dfrac{x_0^1 + x_0^2}{2}\right) + E_\alpha(-t^\alpha)\dfrac{x_0^1 - x_0^2}{2} \\ -\bar{u}(t) + E_\alpha(t^\alpha)\left(1 + \dfrac{x_0^1 + x_0^2}{2}\right) - E_\alpha(-t^\alpha)\dfrac{x_0^1 - x_0^2}{2} \end{pmatrix}$$

The equality $\bar{y}(t) \equiv 0$ implies $E_\alpha(-t^\alpha)\left(x_0^1 - x_0^2\right) = -\bar{u}(t) \in [-1,1]$.

If the initial state satisfies $x_0^1 - x_0^2 \in [-1,1]$ then, taking into account the properties a) and d) of the Mittag-Leffler function (see also [3]), we have

$$E_\alpha(-t^\alpha)\left(x_0^1 - x_0^2\right) \in [-1,1] \quad \forall t > 0,$$

and the global solution corresponds to Mode 3, with $\bar{u}(t) = -E_\alpha(-t^\alpha)\left(x_0^1 - x_0^2\right)$. Further, for any initial condition $x_0^T = (x_0^1, x_0^2)$ satisfying $x_0^1 - x_0^2 < -1$ (or $x_0^1 - x_0^2 > 1$), let us consider the maximal interval $[0,\varepsilon)$ such that the solution remains in Mode 1 (or Mode 2, respectively). At $t = \varepsilon$ we put $x(\varepsilon) = \lim_{t \nearrow \varepsilon} x(t)$. Once arrived on the surface $\bar{y}(t) = x^1(t) - x^2(t) + \bar{u}(t) = 0$, the continuation on $[\varepsilon, \infty)$ will be possible only in Mode 3, since $\left|E_\alpha(-t^\alpha)\left(x_0^1 - x_0^2\right)\right| \le 1$. That is, the relay system has a unique global solution.

The uniqueness of the solutions can also be obtained by computing the matrix $G(s)$, $G(s) = C(s^\alpha I_2 - A)^{-1}B + D = 1$ and observing that $G(s)$ is an invertible P_0-matrix.

Remark 3. The uniqueness of the solutions for the system considered above also holds if one replaces $D = 1$ with $D = 0$. Indeed, for any initial condition x_0, there is a global solution corresponding to only one mode of the relay system (Note that in the case $\bar{y}(0) = x_0^1 - x_0^2 = 0$, the solution will correspond to Mode 3 with $\bar{u}(t) \equiv 0$).

Nevertheless, for $D = 0$ we cannot invoke Theorem 1 above in order to establish the well-posedness, since we obtain $G(s) = 0$. This shows that the condition of invertibility of $G(s)$ in Theorem 1 is not necessary in the study of the wellposedness of system (1.1).

Remark 4. We point out that the method presented above (and that proposed by [1]), reduces the study of system (1.1) to the class of piecewise continuous functions $x(.)$ of exponential order, that is the well-posedness of (1.1) is studied in the space of all $x(.)$ for which there exist M, $a>0$ such that $|x(t)| \le Me^{at}$ for all $t \ge 0$. In particular, as stated in Theorem 1, the matter of the uniqueness of solutions holds in the space of continuous piecewise analytic functions.

In what follows we establish a condition under which the local uniqueness of solutions to system (1.1) holds in the space of continuously differentiable functions. Contrarily to the above linear FO-MIMO relay system, the next result allows for some nonlinearity in both equations of the state variable and of the output. Consider the following nonlinear FO-MIMO relay system

$$\begin{cases} D^{\alpha}x(t) = f(x(t)) + B\bar{u}(t) \\ \bar{y}(t) = g(x(t)) + \bar{D}u(t) \\ \bar{u}_i(t) \in -Sgn\left(\bar{y}_i(t)\right), \quad i = \overline{1,m} \end{cases} \tag{1.9}$$

with the initial condition $x(0) = x_0 \in \mathbb{R}^n$, where $f(.), g(.) : \mathbb{R}^n \to \mathbb{R}^n$ are two given functions.

The main idea is to rewrite the input-output condition in (1.9) as an affine variational inequality and then to apply the constructive theory of Affine Variational Problems. Further we treat the new system in the framework of fractional differential equations, using a uniqueness result due to [14].

Theorem 2. Suppose that $f(.)$ and $g(.)$ are Lipschitz continuous and $D \in \mathcal{M}_{m,m}(\mathbb{R})$ is a P-matrix. Then, the relay system (1.9) has a unique C^1-solution on some interval $[0,\varepsilon), \varepsilon > 0$.

Proof. Let us denote $K = [-1,1]^m$. Solving inequality $\bar{u}_i(t) \in -Sgn\left(\bar{y}_i(t)\right)$, $i = \overline{1,m}$ is equivalent with the following affine variational problem: For $t \ge 0$ find $\bar{u}(t) \in K$ such that $\left(s - \bar{u}(t)\right)^T y(t) \ge 0, \ \forall s \in K$. Since D is a P-matrix, according to Example 4.2.9 in [15], the normal map associated to the pair (K, D), given by

$$M_K^{nor} : \mathbb{R}^l \to \mathbb{R}^l, \ M_K^{nor}(\lambda) = D \cdot \Pi_K(\lambda) + \lambda - \Pi_K(\lambda)$$

where $\prod_K(\lambda) = proj(K;\lambda)$, is coherently oriented. Then, by Theorem 4.3.2 in [15], the affine variational problem has a unique PWA solution:

$$\bar{u}\left(g(x(t))\right)=\Pi_K\left(\left(M_K^{nor}\right)^{-1}\left(-g(x(t))\right)\right)$$

and thus, Lipschitz continuous, as a function of $g(x(t))$. So also the function $x\mapsto v(x)=\bar{u}\left(g(x)\right)$ is Lipschitz continuous. The relay system (1.1) becomes a fractional system without relays

$$\begin{cases} D^{\alpha}x(t)=f(x(t))+B\Pi_K\left(\left(M_K^{nor}\right)^{-1}\left(-g(x(t))\right)\right) \\ y(t)=g(x(t))+D\Pi_K\left(\left(M_K^{nor}\right)^{-1}\left(-g(x(t))\right)\right) \\ x(0)=x_0\in\mathbb{R}^n. \end{cases} \tag{1.10}$$

The right-hand side in the first equation of (1.10) is a Lipschitz continuous function and the state equation in (1.10) is equivalent to the integral equation

$$x(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[f(x(\tau))+B\Pi_K\left(\left(M_K^{nor}\right)^{-1}\left(-g(x(\tau))\right)\right)\right]d\tau .$$

Therefore, we may apply Theorem 3.1. in [14] and we obtain that there exists a unique local solution to (1.10), of class C^1 in time. □

Remark 5. Let us notice that when D is a P-matrix and $G(s)=C(s^{\alpha}I_n-A)^{-1}B+D$ is an invertible P_0-matrix (see Example 1), Theorem 1 gives a characterization of the unique solution, namely the C^1-solution is analytic.

3. Conclusions

In this paper, we introduced a new class of relay fractional-order systems and a sufficient condition concerning the well-posedness of relay feedback systems is worked out along the same lines as in the integer-order case (see [10], [1] for $\alpha=1$). This is done in Theorem 1. Next, under a stronger condition, we prove the local uniqueness of C^1 solutions (Theorem 2). A simple example to illustrate the use of these results is given. We also remark that the condition of $G(s)$ being an invertible P_0-matrix is only sufficient in order to prove the global uniqueness of forward solutions. To the authors' knowledge, relay fractional-order systems have not been considered before.

REFERENCES

[1] *Y.J. Lootsma, A.J. van der Schaft, M.C. Çamlibel,* Uniqueness of solutions of linear relay systems, Automatica, **vol. 35**, no. 3, 1999, pp. 467–478.

[2] *I. Podlubny,* Geometric and physical interpretation of fractional integration and fractional differentiation, Fractional Calculus & Applied Analysis, **vol. 5**, no. 4, 2002, pp. 367–386.

[3] *R. Hilfer, H.J. Seybold,* Computation of the generalized Mittag-Leffler function and its inverse in complex plane, Integral Transforms and Special Functions, **vol. 17**, no. 9, 2006, pp. 637–652.

[4] *F. Mainardi, R. Gorenflo,* On Mittag-Leffler-type functions in fractional evolution processes, J. of Comp. and Appl. Math., **vol. 118**, no. 1-2, 2000, pp. 283–299.

[5] *A.A. Kilbas, S.A. Marzan,* Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, Diff. Eq., **vol. 41**, no. 1, 2005, pp. 82–86.

[6] *M-F. Danca,* Chaotic behavior of a class of discontinuous dynamical systems of fractional-order, Nonlinear Dynamics, **vol. 60**, no. 4, pp. 525-534, 2010.

[7] *K. Diethelm, N.J. Ford, A.D. Freed,* A Predictor-corrector approach for the numerical solutions of fractional differential equations, Nonlinear Dynamics, **vol. 29**, no. 1-4, 2002, pp. 3–22.

[8] *T. Kaczorek,* Reachability of cone fractional continuous-time linear systems, Int. J. Appl. Math. Comput. Science, **vol. 19**, no. 1, 2009, pp. 89–93.

[9] *A.J. Van der Schaft, J.M. Schumacher*, Complementarity modeling of hybrid systems, IEEE Trans. Automat. Control, Special Issue on Hybrid Control Systems, **vol. 43**, no. 4, 1998, pp. 483–490.

[10] *C. Lin, Q-G. Wang,* On uniqueness of solutions to relay feedback systems, Automatica, **vol. 38**, no. 1, 2002, pp. 177–180.

[11] *W.P.M.H. Heemels, J.M. Schumacher, S. Weiland*, The rational complementarity problem, Linear Algebra and its Applications, **vol. 294**, no. 1-3, 1999, pp. 93–135.

[12] *T. Kaczorek,* Fractional positive continuous-time linear systems and their reachability, Int. J. Appl. Math.Comput. Science, **vol. 18**, no. 2, 2008, pp. 223–228.

[13] *P. Ballard*, The dynamics of discrete mechanical systems with perfect unilateral constraints, Arch. Rational Mech. Anal., **vol. 154**, no. 3, 2000, pp. 199–274.

[14] *S. Momani, A. Jameel, S.Al-Azawi,* Local and global uniqueness theorems on fractional integro-differential equations via Bihari's and Gronwall's inequalities. Soochow Journal of Mathematics, **vol. 33**, no.4, 2007, pp. 619–627.

[15] *F. Facchinei, Jong-Shi Pang*, Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer, 2002.