A STABILITY STUDY OF NON-NEWTONIAN FLUID FLOWS

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Se studiază problema de curgere a unui fluid Oldroyd –B generalizat între două plăci paralele. Ne interesează dacă soluțiile de tip von Karman sunt admisibile pentru acest fluid. Precizăm un cadru al problemei din punct de vedere al stabilității. Prezentăm restricții asupra parametrilor constitutivi, ca rezultat al unei anumite inegalități. Punem în evidență caracterul stabil al unei curgeri de bază pentru fluidul Oldroyd–B cu parametri de material constanți.

The problem of the flow of a generalized Oldroyd –B fluid between two parallel plates is studied. We are interested if a von Karman type solutions are admissible for this fluid. We precise a frame of the problem from the point of stability view. We discuss some restrictions by certain inequality upon constitutive parameters. We determine the stability character of non –trivial base flows for Oldroyd–B fluid with constant material moduli.

Keywords: Oldroyd-B fluid, von Karman's type solutions, stability character.

1. Introduction

By a stability study for an incompressible second grade fluid, from Clausius- Duhem inequality are obtained restrictions for the constitutive moduli of Cauchy stress tensor:

$$\mu \geq 0, \alpha_1 + \alpha_2 = 0, \tag{1.1}$$

and $\alpha_1 \geq 0$ if the free energy is to be minimum in equilibrium(see J.E. Dunn, R.L Fosdick [1]). In the paper of R. L. Fosdick and B. Straughan (see [2]), for instance, was investigated the instability in a fluid of third grade. Employing the Clausius- Duhem inequality and demanding that the free energy be a minimum in equilibrium, Fosdick and Rajagopal [3] have shown that the corresponding constitutive equation for an incompressible fluid of third grade is:

$$\mu \geq 0, \beta \geq 0, -\sqrt{24\mu\beta} \leq \alpha_1 + \alpha_2 \leq \sqrt{24\mu\beta}, \alpha_1 \geq 0. \tag{1.2}$$

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In [2], the authors assume that the inequalities $(1.2)_1$ are strict inequalities. They show that the condition $\alpha_1 < 0$, which is compatible with the Clausius-Duhem inequality but not with the free energy, being a minimum in equilibrium and thus they leads to behavior which may not be physically acceptable.

2. The flow problem

The paper deals with the problem concerning the flow of a generalized Oldroyd –B fluid between two parallel plates. The Cauchy stress tensor is:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T_E}, \frac{\mathbf{DT_E}}{\mathbf{D}t} + \frac{1}{\lambda}\mathbf{T_E} = \mu\mathbf{A_1} + \alpha_2\mathbf{A_1}^{\,2} + \alpha_1\frac{\mathbf{DA_1}}{\mathbf{D}t}, \tag{2.1}$$

where the convective derivative is expressed by (see Fetecau [4], [5]):

$$\frac{\mathbf{DA}}{\mathbf{D}t} = \dot{\mathbf{A}} + \mathbf{AL} + \mathbf{L^T A}. \tag{2.2}$$

In the equation (2.1), $\mathbf{T_E}$ is the extra–stress tensor (effective stress – tensor), $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{L}$ is the velocity gradient, $\mathbf{A_1} = \mathbf{L} + \mathbf{L^T}$ is the first Rivlin – Ericksen tensor, λ is the relaxation time, μ is the dynamic viscosity, α_1 and α_2 are the constant constitutive coefficients. We have the constitutive restrictions (see Tigoiu [6], [7]):

$$\mu \geq 0, \alpha_1 + \alpha_2 = 0. \tag{2.3}$$

The fluid flows between two parallel plates. The upper plate is supposed to be porous and the fluid passes through with constant vertical velocity, meaning:

$$\mathbf{v(x,y)}\big|_{\mathbf{y=d}} = -v_0\,\mathbf{j}, \tag{2.4}$$

and the lower plate moves with the velocity:

$$\mathbf{v(x,y)}\big|_{\mathbf{y=0}} = cx\mathbf{i}, \tag{2.5}$$

where “d” is distance between the two plates, **i** and **j** are the unit vector in the horizontal and respective vertical directions and “c” is a given constant. We remark that the origin is preserved at rest (see Fig. 1).

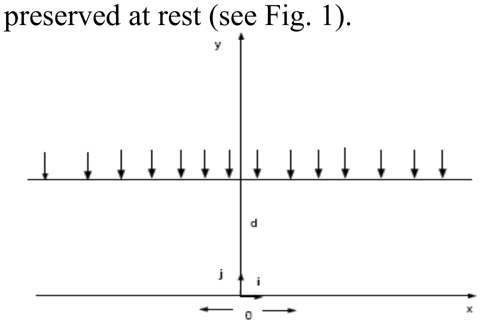

Fig. 1. Flow domain

We shall suppose, like in [8], that the admissible velocity field is of von Karman's type:

$$\mathrm{u} = \mathrm{cxf}'(\eta),\ \mathrm{v} = -\mathrm{cdf}(\eta),\ y \equiv d\eta. \tag{2.6}$$

We study if a generalized Oldroyd -B fluid accept a von Karman's solutions for the flow problem described above.

This flow field satisfies the constraint of incompressibility. Since the velocity field is independent of z, the stress field will also be independent of z. Therefore from the constitutive equation (1.1) we obtain the following system:

$$\begin{aligned}
&2c\mu f' - \alpha_1 \left(2c^2 ff'' + \frac{c^2x^2}{d^2} f''^2\right) = cxf' \frac{\partial T_E{}^{11}}{\partial x} - cf \frac{\partial T_E{}^{11}}{\partial \eta} + 2cf' T_E{}^{11} + \frac{1}{\lambda} T_E{}^{11} \\
&c\mu \frac{x}{d} f'' + \alpha_1 \frac{c^2 x}{d^2} (3f' f'' - f f''') = cxf' \frac{\partial T_E{}^{12}}{\partial x} - cf \frac{\partial T_E{}^{12}}{\partial \eta} + \frac{cx}{d} f'' T_E{}^{11} + \frac{1}{\lambda} T_E{}^{12} \\
&cxf' \frac{\partial T_E{}^{13}}{\partial x} - cf \frac{\partial T_E{}^{13}}{\partial \eta} + cf' T_E{}^{13} + \frac{1}{\lambda} T_E{}^{13} = 0 \\
&-2c\mu f' + \alpha_1 \left(2c^2 f f'' + \frac{c^2 x^2}{d^2} f''^2\right) = cxf' \frac{\partial T_E{}^{22}}{\partial x} - cf \frac{\partial T_E{}^{22}}{\partial \eta} + \\
&\qquad + 2\frac{cx}{d} f'' T_E{}^{12} - 2cf' T_E{}^{22} + \frac{1}{\lambda} T_E{}^{22} \\
&cxf' \frac{\partial T_E{}^{23}}{\partial x} - cf \frac{\partial T_E{}^{23}}{\partial \eta} + \frac{cx}{d} f'' T_E{}^{13} - cf' T_E{}^{23} + \frac{1}{\lambda} T_E{}^{23} = 0 \\
&cxf' \frac{\partial T_E{}^{33}}{\partial x} - cf \frac{\partial T_E{}^{33}}{\partial \eta} + \frac{1}{\lambda} T_E{}^{33} = 0.
\end{aligned} \tag{2.7}$$

The equations of the motion are: $\rho \mathbf{a} = \rho \mathbf{b} + \mathrm{div}\mathbf{T}$. The acceleration is given by: $\mathbf{a} = \mathrm{c}^2 \mathrm{x}(\mathrm{f}'^2 - \mathrm{ff}'')\mathbf{i} + \mathrm{c}^2 \mathrm{d\, ff}'\mathbf{j}$. If we consider $\mathbf{b} = \mathbf{0}$, then the flow equations are:

$$\rho c^2 x (f'^2 - ff'') = \frac{\partial T_E{}^{12}}{\partial \eta} + \frac{dT_E{}^{11}}{\partial x} - \frac{\partial p}{\partial x}, \tag{2.8}$$

$$\rho c^2 d^2 ff' = \frac{\partial T_E{}^{22}}{\partial \eta} + \frac{\partial T_E{}^{12}}{\partial x} - \frac{\partial p}{\partial \eta},\ \frac{\partial T_E{}^{23}}{\partial \eta} + \frac{\partial T_E{}^{13}}{\partial x} = 0.$$

We suppose that the effective stress is of the form:

$$T_E{}^{ij} = \sum_{n=0}^{2} \frac{x^n}{d^n} T_{E\,n}{}^{ij}(\eta). \tag{2.9}$$

Then we will be able to suppose that the pressure has the same type like function T_E^{ij} does:

$$p(x,\eta) = p_0(\eta) + (x/d)p_1(\eta) + (x^2/d^2)p_2(\eta). \tag{2.10}$$

If we use the relations $(2.7)_3, (2.7)_5$, $(2.8)_3$ we can determine the expressions for the components $T_{E\,n}^{13}, T_{E\,n}^{23}$ and the equation for the function $f(\eta)$. We remark that employing the relations: $T_1^{23} \equiv 0$, $T_2^{13} \equiv 0$, we arrive at the following equation for f:

$$\lambda c f'' f^2 + 6\lambda c f'^2 - 3f' = 0. \tag{2.11}$$

Thus the problem is to solve the equation (2.11) under conditions obtained from the described mechanical problem, which are:

$$f(0) = 0, \quad f(1) = \frac{v_0}{cd}, \quad f'(0) = 1, \quad f'(1) = 0. \tag{2.12}$$

The problem is if the equation obtained for $f(\eta)$ has a solution if we consider any two point problem of type (2.12). Using $(2.12)_1$ $(2.12)_3$ we found:

$$c = 1/2\lambda. \tag{2.13}$$

Thus the equation (2.11) becomes:

$$f'' f^2 + 6f'^2 - 6f' = 0. \tag{2.14}$$

For the study of the above problem, we first develop the function f in power series, in order to determine the coefficient of second order in η:

$$f(\eta) = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + ...a_n\eta^n + ... \tag{2.15}$$

From $(2.12)_1$ $(2.12)_3$ we calculate that:

$$a_0 = 0, a_1 = 1, a_2 = 0.5 = v_0/cd. \tag{2.16}$$

For the second approximation (of f) in η we make a change of the function introducing a new function $h(1-\eta)$:

$$f(\eta) = \eta + a_2\eta^2 - h(1-\eta)\eta^2, \tag{2.17}$$

getting for $h(t)$, $t \equiv 1-\eta$, a differential Cauchy problem:

$$(1-t)^5(\,(2-t)h - 3\,)hh'' = (\,1 - 2h + 4(1-t)h'\,)(1-t)^3 h(\,(1-t)h - 3 + t) +$$
$$+ 6\,(2 - t - 2(1-t)h + (1-t)^2 h'\,)^2 - 6\,(2 - t - 2(1-t)h + (1-t^2)h'\,) \tag{2.18}$$
$$h(0) = 1, h'(0) = 0.$$

By a numerical calculus we determine the function h only for $\eta \in [0, 0.62]$, fit the data of function h, and obtain a seventh degree polynom (see Fig.2). The value $\eta = 0.62$ express the first point for which $h(\eta) = 0$. For $\eta \in [\,0.62, 1]$ we shall use the fitted polynom, h being a continuous function.

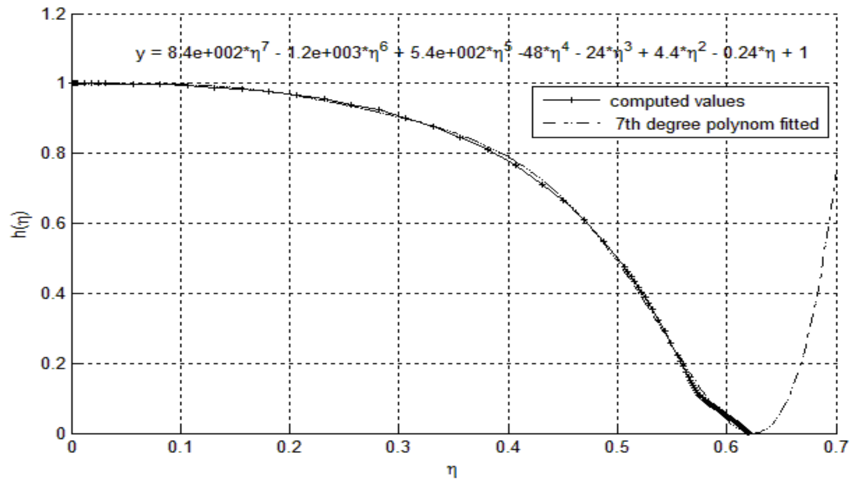

Fig.2 Computed function h and fitted polynom

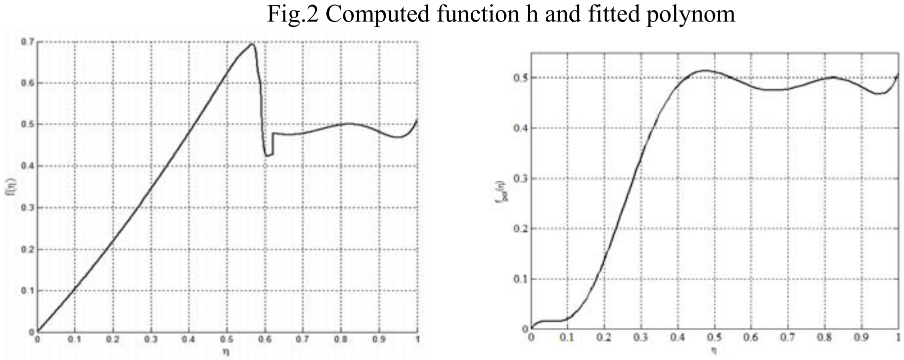

Fig. 3 Function f: numerical computed and using interpolated polynom

The approximation of h over the interval [0, 0.62] is:

$$h(t) \cong 840t^7 - 1200t^6 + 540t^5 - 48t^4 - 24t^3 + 4.4t^2 - 0.24t + 1, \qquad (2.19)$$

and the f approximation by polynom over the same interval is:

$$f \cong \eta + 0.5\eta^2 - \eta^2[840(1-\eta)^7 - 1200(1-\eta)^6 + 540(1-\eta)^5 -$$
$$- 48(1-\eta)^4 - 24(1-\eta)^3 + 4.4(1-\eta)^2 - 0.24(1-\eta) + 1]. \qquad (2.20)$$

3. Stability of the solution by numerical analysis

For the fluid studied, now we consider a small perturbation of the base flow. The perturbed flow is given by the following expressions:

$$\tilde{u} = cx(f'(\eta) + \varepsilon g'(\eta)) = u + \bar{u}, \; \tilde{v} = -cd(f(\eta) + \varepsilon g(\eta)) = v + \bar{v}. \qquad (3.1)$$

Since $\tilde{u}$, $\tilde{v}$ are given by the same equations of motion like u and v does, the small perturbation $\bar{u}$, $\bar{v}$, must satisfies:

$g''(f^2 + 2\varepsilon fg) + 6g'(2f' - 1 + \varepsilon g') + g f''(2f + \varepsilon g) = 0, g(0) = \varepsilon^2, g'(0) = 0$,(3.2)

neglecting $O(\varepsilon^2 g^2)$ in respect with $O(f)$.

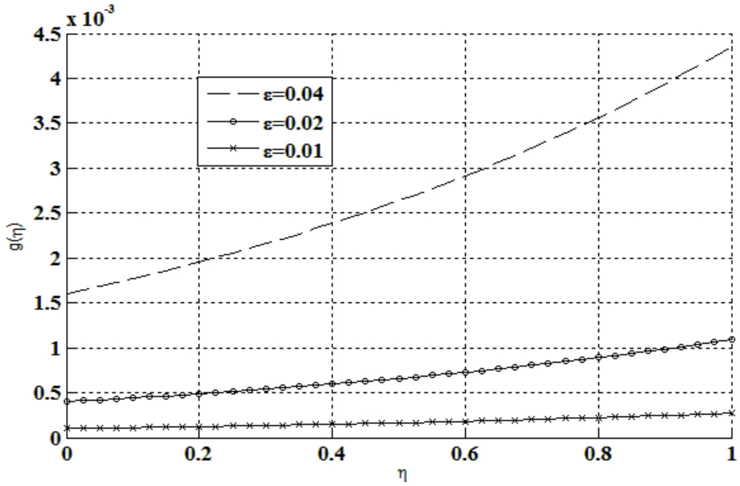

Fig. 4 The small perturbation g.

6. Conclusions

Our study conclude that existence of von Karman type solution for the Oldroyd –B fluid implies a certain constants for obtaining dimensionless values of the velocity: $c = 1/2\lambda$, and $v_0 = 2cd$. Also, we observe that small perturbations of the base flow are numerically stable ($O(g) \approx O(\varepsilon^2)$ as were imposed initially in $\eta = 0$, see Fig. 4).

B I B L I O G R A P H Y

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