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NONHOLONOMIC GEOMETRY OF GIBBS CONTACT STRUCTURE

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Lucrarea face legătura dintre teoria congruenţelor a lui Vrânceanu şi teoria autorilor referitoare la sisteme termodinamice neolonome de dimensiune impară. Pentru a construi metrica riemanniană, autorii folosesc congruenţe cu semnificaţie termodinamică. Pe baza acestei metrici sunt construiţi invarianţii diferenţiali ai spaţiului neolonom Gibbs-Vrânceanu-Riemann. Mai departe, se demonstrează că coeficienţii covarianţilor biliniari şi coeficienţii Ricci (cu trei şi patru indici) sunt signoame, iar vectorii tangenţi la geodezice sunt funcţii raţionale. Ca noutate, se introduc şi se studiază subvarietatea coeficienţilor covarianţilor biliniari, subvarietatea coeficienţilor de rotaţie Ricci şi subvarietatea coeficienţilor Ricci cu patru indici.

The paper connects the Vrânceanu congruence theory with our theory of odd-dimensional nonholonomic thermodynamic systems. To build the Riemannian metric, the authors use certain congruences with thermodynamical meaning. Based on this metric, the differential invariants of the Gibbs-Vrânceanu-Riemann nonholonomic space are built. Further, it is proved that the coefficients of bilinear covariants and the coefficients of Ricci (with three and four indexes) are signomials, and the tangent vectors to the geodesics are rational functions. As a novelty, one introduces and studies also the submanifold of coefficients of the bilinear covariants, the submanifold of Ricci rotation coefficients and the submanifold of Ricci coefficients with four indexes.

Keywords: Vrânceanu congruences, nonholonomic thermodynamic system, contact structure, geodesics, Ricci coefficients.
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1. Introduction

Gibbs ([6]), Caratheodory ([2]), Hermann ([8]) and later Mrugala ([11], [12]) and Udrişte ([5], [16], [24], [25]) studied the thermodynamics from the

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differential geometry perspective, based on contact structure of thermodynamic state space. The coordinates of this $(2n+1)$-dimensional space are defined by n extensive variables, n intensive variables and one thermodynamical potential. Using suitable differential forms, the first law of thermodynamics is naturally incorporated into an original approach [20], [21]. Similar ideas has been extensively used to study the properties of the structure generated by the Weinhold metric ([4], [7]), the thermodynamic length ([18]) and the associated Riemannian structure ([9], [17]). Rupeiner ([17]) introduced a metric conformally-equivalent with the Weinhold metric, which later led to applications in black holes thermodynamics ([19],[27],[28]). Recently, to incorporate the concept of Legendre invariance into the geometric description of thermodynamics, a geometro-thermodynamic's formalism has been developed ([14], [15]).

A system whose state depends on the path taken to achieve it is called *nonholonomic*. Such a system is characterized by a set of parameters subject to Pfaff differential constraints, so that when the system evolves continuously along a path in its parameter space, but finally returns to the original set of values at the start of the path, the system itself may not have returned to its original state. More exactly, a nonholonomic system is one in which (1) there is a continuous closed circuit of the governing parameters, by which the system may be transformed from any given state to any other state; (2) any two points in the parameter space can be connected by a path (Carathéodory Theorem).

2. Gibbs contact structure in nonholonomic description

In this paper we will use the Vrânceanu congruences method ([3], [28], [29]) to the nonholonomic study of Gibbs contact structure $(\mathbf{R^5}, \delta_{\mathbf{ab}}, \theta)$, $\mathbf{R}^5 = \{(U, T, S, P, V)\}$, $\theta = dU - TdS + PdV$, where we preserve the names U - internal energy, S - entropy, T - temperature, V - volume and P - pressure for the independent variables, but neither is restricted to positive values as in thermodynamics.

For mathematical convenience, we denote the coordinates by

$$x^1 = U,\ x^2 = T,\ x^3 = S,\ x^4 = P,\ x^5 = V. \tag{1}$$

Then the differential 1-form θ rewrites

$$\theta^5 = \lambda^5_i dx^i,\ i = \overline{1,5}, \tag{2}$$

where

$$\lambda^5_1 = 1,\ \lambda^5_2 = \lambda^5_4 = 0,\ \lambda^5_3 = -x^2, \lambda^5_5 = x^4.$$

To this 1-form we add another four linearly independent Pfaff forms

$$\theta^1 = x^2 dx^3,$$

$$\theta^2 = x^4 dx^5,$$

$$\theta^3 = dx^1 + x^5 dx^4 + x^4 dx^5,$$

$$\theta^4 = dx^1 - x^3 dx^2 - x^2 dx^3,$$

which can be written as

$$\theta^a = \lambda^a_i dx^i,\ a = \overline{1,4}. \tag{3}$$

The Pfaff forms (2)+(3) determine a *system of congruences* (*moving coframe*), with the moments λ^a_i ([3], [28], [29]). Along the curves we can use the Vrânceanu notations $ds^a = \theta^a$, $a = \overline{1,5}$. The congruences (2)+(3) are not orthogonal in the Euclidean space $(\mathbf{R^5}, \delta_{\mathbf{ab}})$. The physical meaning of their restrictions is

- $ds^1 = TdS = dQ$, the *elementary heat of the system*;
- $ds^2 = PdV = dW$, the *elementary expansion work of the system*;
- $ds^3 = dU + PdV + VdP = dH$, the *elementary enthalpy*;
- $ds^4 = dU - TdS - SdT = dA$, the *elementary Helmholtz energy*;
- $ds^5 = dU - TdS + PdV$, the *Gibbs-Pfaff contact* 1*-form.*

To prevent contradictory discussions, we accept the dimensionless expression of the 1-forms. As example, the dimensionless form of θ requires the dimensionless ratio $\frac{P_0V_0}{T_0S_0} = \lambda = 1$.

On the region where the matrix of moments (λ_i^a), $a=\overline{1,5}$; $i=\overline{1,5}$ is nondegenerate, we introduce its inverse matrix (μ_i^a). Explicitly,

$$(\lambda_i^a)=\begin{pmatrix} 0 & 0 & x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x^4 \\ 1 & 0 & 0 & x^5 & x^4 \\ 1 & -x^3 & -x^2 & 0 & 0 \\ 1 & 0 & -x^2 & 0 & x^4 \end{pmatrix},$$

$$\left(\mu_a^i\right)=\begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & -\frac{1}{x^3} & 0 & -\frac{1}{x^3} & \frac{1}{x^3} \\ \frac{1}{x^2} & 0 & 0 & 0 & 0 \\ -\frac{1}{x^5} & 0 & \frac{1}{x^5} & 0 & -\frac{1}{x^5} \\ 0 & \frac{1}{x^4} & 0 & 0 & 0 \end{pmatrix},$$

$$\text{on}\, D:\ \det(\lambda_i^a)=-x^2x^3x^4x^5\neq 0,\ D\subset R^5.$$

The components of these matrices are related by

$$\begin{aligned} \lambda_i^a\mu_b^i &= \delta_b^a,\ a,b,i=\overline{1,5} \\ \lambda_a^i\mu_j^a &= \delta_j^i,\ i,j,a=\overline{1,5}. \end{aligned}$$

Also we remark that

$$\theta^a=\lambda_i^a dx^i,\ a,i=\overline{1,5},$$

is equivalent to

$$dx^i=\mu_a^i\theta^a,\ i,a=\overline{1,5}.$$

Given the system of independent congruences θ^a, $a=\overline{1,5}$, on $(D\subset\mathbf{R^5},\delta_{\mathbf{ab}})$, one can associate only one Riemannian manifold $(D\subset\mathbf{R^5},\mathbf{g_{ij}},\theta^{\mathbf{a}})$ (with a positive definite metric and a moving co-frame, *framed manifold*), in which the system of congruences is orthonormal ([29], p. 260). The Riemannian metric g_{ij} is given by the square of arc element

$$ds^2=\delta_{ab}ds^a ds^b=g_{ij}dx^i dx^j,\ a,b=\overline{1,5};\ i,j=\overline{1,5},$$

or by the matrix

$$(g_{ij}) = (\delta_{ab}\lambda_i^a\lambda_j^b) = \begin{pmatrix} 3 & -x^3 & -2x^2 & x^5 & 2x^4 \\ -x^3 & x^{3^2} & x^2x^3 & 0 & 0 \\ -2x^2 & x^2x^3 & 3x^{2^2} & 0 & -x^2x^4 \\ x^5 & 0 & 0 & x^{5^2} & x^4x^5 \\ 2x^4 & 0 & -x^2x^4 & x^4x^5 & 3x^{4^2} \end{pmatrix}. \qquad (4)$$

In Riemannian mechanics, the components g_{ij} of the metric are called *gravitational potentials*. These determine the *Levi-Civita connection*, the *curvature tensor field*, the *Ricci tensor field*, the *scalar curvature*, etc.

3. The coefficients of the bilinear covariants of the Gibbs-Vrânceanu-Riemann nonholonomic space

For a system of orthonormal congruences, the coefficients of the *bilinear covariants* are

$$w_{bc}^a = \left(\frac{\partial\lambda_i^a}{\partial x^j} - \frac{\partial\lambda_j^a}{\partial x^i}\right)\mu_b^i\mu_c^j.$$

They verify the relation $w_{bc}^a = -w_{cb}^a$ (skew-symmetric in the indexes b and c). Also, the relations $w_{bc}^a = 0$, $a =$ fixed, indicate that the Pfaff 1-form θ^a is closed.

Now we apply this theory to the co-framed manifold $(D \subset \mathbf{R^5}, \delta_{\mathbf{ab}}, \theta^{\mathbf{a}})$. Taking a, b, $c = \overline{1,5}$; $i, j = \overline{1,5}$, we find

Theorem 1 *The coefficients of the bilinear covariants are zero excepting the following signomials:*

$$w_{12}^1 = -w_{21}^1 = -\tfrac{1}{x^2x^3},\ w_{14}^1 = -w_{41}^1 = -\tfrac{1}{x^2x^3},\ w_{15}^1 = -w_{51}^1 = \tfrac{1}{x^2x^3},$$

$$w_{12}^2 = -w_{21}^2 = \tfrac{1}{x^4x^5},\ w_{23}^2 = -w_{32}^2 = \tfrac{1}{x^4x^5},\ w_{25}^2 = -w_{52}^2 = -\tfrac{1}{x^4x^5},$$

$$w_{12}^5 = -w_{21}^5 = \tfrac{1}{x^2x^3} + \tfrac{1}{x^4x^5},\ w_{14}^5 = -w_{41}^5 = \tfrac{1}{x^2x^3},\ w_{15}^5 = -w_{51}^5 = -\tfrac{1}{x^2x^3},$$

$$w_{23}^5 = -w_{32}^5 = \tfrac{1}{x^4x^5},\ w_{25}^5 = -w_{52}^5 = -\tfrac{1}{x^4x^5},\ w_{15}^5 = -w_{51}^5 = -\tfrac{1}{x^2x^3}.$$

Obviously, the 1-forms θ^3, θ^4 are closed.

Returning to the thermodynamic restrictions, the non-zero coefficients of the bilinear covariants rewrites:

$$w^1_{12} = -w^1_{21} = -\frac{1}{TS},\ w^1_{14} = -w^1_{41} = -\frac{1}{TS},\ w^1_{15} = -w^1_{51} = \frac{1}{TS},$$

$$w^2_{12} = -w^2_{21} = \frac{1}{PV},\ w^2_{23} = -w^2_{32} = \frac{1}{PV},\ w^2_{25} = -w^2_{52} = -\frac{1}{PV},$$

$$w^5_{12} = -w^5_{21} = \frac{1}{TS} + \frac{1}{PV},\ w^5_{14} = -w^5_{41} = \frac{1}{TS},\ w^5_{15} = -w^5_{51} = -\frac{1}{TS},$$

$$w^5_{23} = -w^5_{32} = \frac{1}{PV},\ w^5_{25} = -w^5_{52} = -\frac{1}{PV},\ w^5_{15} = -w^5_{51} = -\frac{1}{TS}.$$

In other words, the coefficients of nonholonomicity depend only on two thermodynamic potentials (parameters with dimension of energy): *energy from environment* TS and *expansion work* PV.

Remark. To show that the phenomenological thermodynamics equilibrium is a superposition of symplectic structure of the phase space and of latice structure of thermodynamical potentials, the paper [10] used the potentials TS and PV. From our point of view, the appearence of these potentials in the nonholonomic geometry of Gibbs-Pfaff structure gives strength to the Vrănceanu point of view.

4. The Ricci rotation coefficients of the Gibbs-Vrânceanu-Riemann nonholonomic space

The components of the *Levi-Civita connection* Γ^i_{jk} are determined by the *Riemannian metric* g_{ij}. The associated components on the considered system of orthonormal congruences, are the *Ricci rotation coefficients* of these congruences ([28], p. 267):

$$\gamma^a_{bc} = \frac{1}{2}\left(w^a_{bc} + w^b_{ca} + w^c_{ba}\right), \tag{5}$$

where w^a_{bc} are the coefficients of the bilinear covariants.

The Ricci rotation coefficients of the 1-forms (2)+(3) verify the relation

$$\gamma^a_{bc} - \gamma^a_{cb} - w^a_{bc} = 0, \tag{6}$$

which means that the Riemannian connection closes the infinitesimal parallelograms. Moreover,

$$\gamma^a_{bc} + \gamma^b_{ac} = 0 \Rightarrow \gamma^a_{ac} = 0,\ \gamma^a_{bc} = -\gamma^b_{ac},\ (a \neq b),$$

i.e., the Ricci rotation coefficients are skew-symmetric in the indexes a and b.

Applying this theory to the framed manifold $(D \subset \mathbf{R^5}, \delta_{\mathbf{ab}}, \theta^{\mathbf{a}})$, we find:

Theorem 2 *The Ricci rotation coefficients associated to the orthogonal congruences (2)+(3) are zero, excepting the following signomials:*

$$\gamma^1_{21} = -\gamma^2_{11} = \tfrac{1}{x^2x^3},\ \gamma^1_{22} = -\gamma^2_{12} = -\tfrac{1}{x^4x^5},\ \gamma^1_{25} = -\gamma^2_{15} = -\tfrac{1}{2x^2x^3} - \tfrac{1}{2x^4x^5},$$

$$\gamma^1_{41} = -\gamma^4_{11} = \tfrac{1}{x^2x^3},\ \gamma^1_{45} = -\gamma^4_{15} = -\tfrac{1}{2x^2x^3},\ \gamma^1_{51} = -\gamma^5_{11} = -\tfrac{1}{x^2x^3},$$

$$\gamma^1_{52} = -\gamma^5_{12} = -\tfrac{1}{2x^2x^3} - \tfrac{1}{2x^4x^5},\ \gamma^1_{54} = -\gamma^5_{14} = -\tfrac{1}{2x^2x^3},\ \gamma^1_{55} = -\gamma^5_{15} = \tfrac{1}{x^2x^3},$$

$$\gamma^2_{32} = -\gamma^3_{22} = -\tfrac{1}{x^4x^5},\ \gamma^2_{35} = -\gamma^3_{25} = -\tfrac{1}{2x^4x^5},\ \gamma^2_{51} = -\gamma^5_{21} = \tfrac{1}{2x^2x^3} + \tfrac{1}{2x^4x^5},$$

$$\gamma^2_{52} = -\gamma^5_{22} = \tfrac{1}{x^4x^5},\ \gamma^2_{53} = -\gamma^5_{23} = -\tfrac{1}{2x^4x^5},\ \gamma^2_{55} = -\gamma^5_{25} = \tfrac{1}{x^4x^5},$$

$$\gamma^3_{52} = -\gamma^5_{32} = \tfrac{1}{2x^4x^5},\ \gamma^4_{51} = -\gamma^5_{41} = \tfrac{1}{2x^2x^3}.$$

Remark. This is an important example of Ricci rotation coefficients which are linear in two parameters $u = \frac{1}{x^2x^3}$, $v = \frac{1}{x^4x^5}$ (the first example of the theory in [12], [22]). From thermodynamic point of view, the Ricci rotation coefficients depend only on the products with dimension of energy TS, PV.

5. The geodesics of the Gibbs-Vrânceanu-Riemann nonholonomic space

Generally, it is known that on a Riemannian manifold, the *autoparallel curves* of the Levi-Civita connection are *geodesics*, and conversely. Taking s as the arc of autoparallel curves ($s = t\sqrt{c} + t_0$, c being a positive constant), the equations of geodesics can be written as follows ([28], p. 274):

$$\frac{dx^i}{ds} = \mu^i_a u^a,\ \frac{du^a}{ds} = \gamma^a_{bc} u^b u^c,$$

where $u^a = \dfrac{ds^a}{ds}$ are the consinuses of the angles which the tangent to the curve makes with the five orthogonal congruences, and γ^a_{bc} are the previous Ricci rotation coefficients (signomials of variables $\dfrac{1}{x^2x^3}, \dfrac{1}{x^4x^5}$). The tangent vectors to geodesics are rational functions.

6. The Ricci coefficients with four indexes of the Gibbs-Vrânceanu-Riemann nonholonomic space

The components of the *curvature tensor field* on the previous orthogonal congruences are the *coefficients of Ricci with four indexes*,

$$\gamma^a_{bcd} = \frac{\partial \gamma^a_{bc}}{\partial s^d} - \frac{\partial \gamma^a_{bd}}{\partial s^c} + \gamma^a_{jc}\gamma^j_{bd} - \gamma^a_{jd}\gamma^j_{bc} + \gamma^a_{bj} w^j_{cd}, \qquad (7)$$

where $\frac{\partial}{\partial s^a} = \mu^i_a \frac{\partial}{\partial x^i}$, and γ^a_{bc} are the Ricci rotation coefficients (5). The Ricci coefficients with four indexes satisfy identically

$$\gamma^a_{bcd} + \gamma^b_{acd} = 0,\ \gamma^a_{bcd} + \gamma^a_{bdc} = 0,$$

$$\gamma^a_{bcd} + \gamma^a_{cdb} + \gamma^a_{dbc} = 0$$

and consequently $\gamma^a_{bcd} = \gamma^c_{dab}$. Now we apply this theory to the framed manifold $(D \subset \mathbf{R^5}, \delta_{\mathbf{ab}}, \theta^{\mathbf{a}})$ and we find:

Theorem 3 *The coefficients of Ricci with four indexes associated to the orthogonal congruences (2)+(3) are zero, excepting the following signomials:*

$$\gamma^1_{212} = -\gamma^1_{221} = -\gamma^2_{112} = \gamma^2_{121} = -\tfrac{1}{2x^2x^3x^4x^5} - \tfrac{3}{4x^{2^2}x^{3^2}} - \tfrac{3}{4x^{4^2}x^{5^2}},$$

$$\gamma^1_{213} = -\gamma^1_{231} = -\gamma^2_{113} = \gamma^2_{131} = -\tfrac{1}{2x^2x^3x^4x^5},$$

$$\gamma^1_{214} = -\gamma^1_{241} = -\gamma^2_{114} = \gamma^2_{141} = -\tfrac{3}{4x^2x^3x^4x^5} - \tfrac{3}{4x^{2^2}x^{3^2}},$$

$$\gamma^1_{215} = -\gamma^1_{251} = -\gamma^2_{115} = \gamma^2_{151} = \tfrac{2}{x^2x^3x^4x^5} + \tfrac{1}{2x^{2^2}x^{3^2}} + \tfrac{1}{2x^{4^2}x^{5^2}},$$

$$\gamma^1_{223} = -\gamma^1_{232} = -\gamma^2_{123} = \gamma^2_{132} = -\tfrac{3}{4x^2x^3x^4x^5} - \tfrac{3}{4x^{4^2}x^{5^2}},$$

$$\gamma^1_{224} = -\gamma^1_{242} = -\gamma^2_{124} = \gamma^2_{142} = -\tfrac{1}{2x^2x^3x^4x^5},$$

$$\gamma^1_{225} = -\gamma^1_{252} = -\gamma^2_{125} = \gamma^2_{152} = \tfrac{2}{x^2x^3x^4x^5} + \tfrac{1}{2x^{2^2}x^{3^2}} + \tfrac{1}{2x^{4^2}x^{5^2}},$$

$$\gamma^1_{234} = -\gamma^1_{243} = -\gamma^2_{134} = \gamma^2_{143} = \tfrac{1}{4x^2x^3x^4x^5},$$

$$\gamma^1_{235} = -\gamma^1_{253} = -\gamma^2_{135} = \gamma^2_{153} = -\tfrac{1}{2x^2x^3x^4x^5} - \tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^1_{245} = -\gamma^1_{254} = -\gamma^2_{145} = \gamma^2_{154} = \tfrac{1}{2x^2x^3x^4x^5} + \tfrac{1}{2{x^2}^2{x^3}^2},$$

$$\gamma^1_{312} = -\gamma^1_{321} = -\gamma^3_{112} = \gamma^3_{121} = -\tfrac{1}{2x^2x^3x^4x^5},$$

$$\gamma^1_{315} = -\gamma^1_{351} = -\gamma^3_{115} = \gamma^3_{151} = -\tfrac{1}{2x^2x^3x^4x^5},$$

$$\gamma^1_{324} = -\gamma^1_{342} = -\gamma^3_{124} = \gamma^3_{142} = -\tfrac{1}{4x^2x^3x^4x^5},$$

$$\gamma^1_{412} = -\gamma^1_{421} = -\gamma^4_{112} = \gamma^4_{121} = -\tfrac{3}{4x^2x^3x^4x^5} - \tfrac{3}{4{x^2}^2{x^3}^2},$$

$$\gamma^1_{414} = -\gamma^1_{441} = -\gamma^4_{114} = \gamma^4_{141} = -\tfrac{3}{4{x^2}^2{x^3}^2},$$

$$\gamma^1_{415} = -\gamma^1_{451} = -\gamma^4_{115} = \gamma^4_{151} = \tfrac{1}{2{x^2}^2{x^3}^2},$$

$$\gamma^1_{423} = -\gamma^1_{432} = -\gamma^4_{123} = \gamma^4_{132} = -\tfrac{1}{2x^2x^3x^4x^5},$$

$$\gamma^1_{425} = -\gamma^1_{452} = -\gamma^4_{125} = \gamma^4_{152} = \tfrac{1}{2x^2x^3x^4x^5} + \tfrac{1}{2{x^2}^2{x^3}^2},$$

$$\gamma^1_{324} = -\gamma^1_{342} = -\gamma^3_{124} = \gamma^3_{142} = -\tfrac{1}{4x^2x^3x^4x^5},$$

$$\gamma^1_{445} = -\gamma^1_{454} = -\gamma^4_{145} = \gamma^4_{154} = \tfrac{1}{2{x^2}^2{x^3}^2},$$

$$\gamma^1_{512} = -\gamma^1_{521} = -\gamma^5_{112} = \gamma^5_{121} = \tfrac{2}{x^2x^3x^4x^5} + \tfrac{1}{2{x^2}^2{x^3}^2} + \tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^1_{513} = -\gamma^1_{531} = -\gamma^5_{113} = \gamma^5_{131} = -\tfrac{1}{2x^2x^3x^4x^5},$$

$$\gamma^1_{514} = -\gamma^1_{541} = -\gamma^5_{114} = \gamma^5_{141} = \frac{1}{2{x^2}^2{x^3}^2},$$

$$\gamma^1_{515} = -\gamma^1_{551} = -\gamma^5_{115} = \gamma^5_{151} = \frac{3}{2x^2x^3x^4x^5} + \frac{1}{2{x^2}^2{x^3}^2} + \frac{1}{4{x^4}^2{x^5}^2},$$

$$\gamma^1_{523} = -\gamma^1_{532} = -\gamma^5_{123} = \gamma^5_{132} = \frac{1}{2x^2x^3x^4x^5} + \frac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^1_{525} = -\gamma^1_{552} = -\gamma^5_{125} = \gamma^5_{152} = -\frac{1}{2{x^2}^2{x^3}^2} - \frac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^1_{535} = -\gamma^1_{553} = -\gamma^5_{135} = \gamma^5_{153} = -\frac{1}{4x^2x^3x^4x^5} - \frac{1}{4{x^4}^2{x^5}^2},$$

$$\gamma^1_{545} = -\gamma^1_{554} = -\gamma^5_{145} = \gamma^5_{154} = -\frac{1}{2{x^2}^2{x^3}^2},$$

$$\gamma^2_{312} = -\gamma^2_{321} = -\gamma^3_{212} = \gamma^3_{221} = -\frac{3}{4x^2x^3x^4x^5} - \frac{3}{4{x^4}^2{x^5}^2},$$

$$\gamma^2_{314} = -\gamma^2_{341} = -\gamma^3_{214} = \gamma^3_{241} = -\frac{1}{2x^2x^3x^4x^5},$$

$$\gamma^2_{315} = -\gamma^2_{351} = -\gamma^3_{215} = \gamma^3_{251} = \frac{1}{2x^2x^3x^4x^5} + \frac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^2_{323} = -\gamma^2_{332} = -\gamma^3_{223} = \gamma^3_{232} = -\frac{3}{4{x^4}^2{x^5}^2},$$

$$\gamma^2_{325} = -\gamma^2_{352} = -\gamma^3_{225} = \gamma^3_{252} = \frac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^2_{335} = -\gamma^2_{353} = -\gamma^3_{235} = \gamma^3_{253} = -\frac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^2_{412} = -\gamma^2_{421} = -\gamma^4_{212} = \gamma^4_{221} = -\frac{1}{2x^2x^3x^4x^5},$$

$$\gamma^2_{413} = -\gamma^2_{431} = -\gamma^4_{213} = \gamma^4_{231} = -\frac{1}{4x^2x^3x^4x^5},$$

$$\gamma^3_{412} = -\gamma^3_{421} = -\gamma^4_{312} = \gamma^4_{321} = \frac{1}{4x^2x^3x^4x^5},$$

$$\gamma^2_{413} = -\gamma^2_{431} = -\gamma^4_{213} = \gamma^4_{231} = -\tfrac{1}{4x^2x^3x^4x^5},$$

$$\gamma^2_{425} = -\gamma^2_{452} = -\gamma^4_{225} = \gamma^4_{252} = -\tfrac{1}{2x^2x^3x^4x^5},$$

$$\gamma^2_{512} = -\gamma^2_{521} = -\gamma^5_{212} = \gamma^5_{221} = \tfrac{2}{x^2x^3x^4x^5} + \tfrac{1}{2{x^2}^2{x^3}^2} + \tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^2_{514} = -\gamma^2_{541} = -\gamma^5_{214} = \gamma^5_{241} = \tfrac{1}{2x^2x^3x^4x^5} + \tfrac{1}{2{x^2}^2{x^3}^2},$$

$$\gamma^2_{515} = -\gamma^2_{551} = -\gamma^5_{215} = \gamma^5_{251} = -\tfrac{1}{2{x^2}^2{x^3}^2} - \tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^2_{523} = -\gamma^2_{532} = -\gamma^5_{223} = \gamma^5_{232} = \tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^2_{524} = -\gamma^2_{542} = -\gamma^5_{224} = \gamma^5_{242} = -\tfrac{1}{2x^2x^3x^4x^5},$$

$$\gamma^2_{525} = -\gamma^2_{552} = -\gamma^5_{225} = \gamma^5_{252} = \tfrac{3}{2x^2x^3x^4x^5} + \tfrac{1}{4{x^2}^2{x^3}^2} + \tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^2_{535} = -\gamma^2_{553} = -\gamma^5_{235} = \gamma^5_{253} = \tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^2_{545} = -\gamma^2_{554} = -\gamma^5_{245} = \gamma^5_{254} = \tfrac{1}{4{x^2}^2{x^3}^2} + \tfrac{1}{4x^2x^3x^4x^5},$$

$$\gamma^3_{512} = -\gamma^3_{521} = -\gamma^5_{312} = \gamma^5_{321} = -\tfrac{1}{2{x^4}^2{x^5}^2} - \tfrac{1}{2x^2x^3x^4x^5},$$

$$\gamma^3_{515} = -\gamma^3_{551} = -\gamma^5_{315} = \gamma^5_{351} = -\tfrac{1}{4{x^4}^2{x^5}^2} - \tfrac{1}{4x^2x^3x^4x^5},$$

$$\gamma^3_{523} = -\gamma^3_{532} = -\gamma^5_{323} = \gamma^5_{332} = -\tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^3_{525} = -\gamma^3_{552} = -\gamma^5_{325} = \gamma^5_{352} = \tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma^3_{535} = -\gamma^3_{553} = -\gamma^5_{335} = \gamma^5_{353} = \tfrac{1}{4{x^4}^2{x^5}^2},$$

$$\gamma^4_{512} = -\gamma^4_{521} = -\gamma^5_{412} = \gamma^5_{421} = \frac{1}{2x^{2^2}x^{3^2}} + \frac{1}{2x^2x^3x^4x^5},$$

$$\gamma^4_{514} = -\gamma^4_{541} = -\gamma^5_{414} = \gamma^5_{441} = \frac{1}{2x^{2^2}x^{3^2}},$$

$$\gamma^4_{515} = -\gamma^4_{551} = -\gamma^5_{415} = \gamma^5_{451} = -\frac{1}{2x^{2^2}x^{3^2}},$$

$$\gamma^4_{525} = -\gamma^4_{552} = -\gamma^5_{425} = \gamma^5_{452} = \frac{1}{4x^{2^2}x^{3^2}} + \frac{1}{4x^2x^3x^4x^5},$$

$$\gamma^4_{545} = -\gamma^4_{554} = -\gamma^5_{445} = \gamma^5_{454} = \frac{1}{4x^{2^2}x^{3^2}}.$$

Remark. This is an important example of Ricci coefficients with four indexes which are square functions in two parameters $u = \frac{1}{x^2x^3}$, $v = \frac{1}{x^4x^5}$ (see [13], [23]). From thermodynamic point of view, it can be observed that the Ricci coefficients with four indexes depend only on the products with dimension of energy TS, PV, and their product $TSPV$.

7. The Ricci tensor and scalar curvature of the Gibbs-Vrânceanu-Riemann nonholonomic space

The *Ricci tensor field* is the trace ([28], p. 294):

$$\gamma_{bd} = \gamma^a_{bad}.$$

Using the Theorem 3, we find:

Theorem 4 *The nonzero components of the Ricci tensor field, associated to the orthogonal congruences (2)+(3), are the following signomials:*

$$\gamma_{11} = \frac{2x^2x^3x^4x^5 - 2x^{4^2}x^{5^2} - x^{2^2}x^{3^2}}{2x^{2^2}x^{3^2}x^{4^2}x^{5^2}}, \ \gamma_{12} = \gamma_{21} = -\frac{x^{2^2}x^{3^2} + x^{4^2}x^{5^2}}{2x^{2^2}x^{3^2}x^{4^2}x^{5^2}},$$

$$\gamma_{13} = \gamma_{31} = \frac{x^2x^3 + x^4x^5}{2x^2x^3x^{4^2}x^{5^2}}, \ \gamma_{14} = \gamma_{41} = \frac{x^2x^3 - x^4x^5}{2x^{2^2}x^{3^2}x^4x^5},$$

$$\gamma_{15} = \gamma_{51} = -\frac{x^{2^2}x^{3^2} + 2x^{4^2}x^{5^2} + 4x^2x^3x^4x^5}{2x^{2^2}x^{3^2}x^{4^2}x^{5^2}}, \ \gamma_{22} = \frac{2x^2x^3x^4x^5 - x^{4^2}x^{5^2} - 2x^{2^2}x^{3^2}}{4x^{2^2}x^{3^2}x^{4^2}x^{5^2}},$$

$$\gamma_{23} = \gamma_{32} = \frac{x^2x^3 - x^4x^5}{2x^2x^3x^{4^2}x^{5^2}}, \ \gamma_{24} = \gamma_{42} = -\frac{x^2x^3 + x^4x^5}{2x^{2^2}x^{3^2}x^4x^5},$$

$$\gamma_{25} = \gamma_{52} = \tfrac{2{x^2}^2{x^3}^2+{x^4}^2{x^5}^2+4x^2x^3x^4x^5}{2{x^2}^2{x^3}^2{x^4}^2{x^5}^2},\ \gamma_{33} = -\tfrac{1}{2{x^4}^2{x^5}^2},$$

$$\gamma_{35} = \gamma_{53} = \tfrac{x^2x^3-x^4x^5}{2x^2x^3{x^4}^2{x^5}^2},\ \gamma_{44} = -\tfrac{1}{2{x^2}^2{x^3}^2},$$

$$\gamma_{45} = \gamma_{54} = \tfrac{x^4x^5-x^2x^3}{2{x^2}^2{x^3}^2x^4x^5},\ \gamma_{55} = \tfrac{{x^2}^2{x^3}^2+{x^4}^2{x^5}^2+3x^2x^3x^4x^5}{{x^2}^2{x^3}^2{x^4}^2{x^5}^2}.$$

Returning to the notations (1), the nonzero coefficients of Ricci tensor rewrite

$$\gamma_{11} = \tfrac{2TSPV-2P^2V^2-T^2S^2}{2T^2S^2P^2V^2},\ \gamma_{12} = \gamma_{21} = -\tfrac{T^2S^2+P^2V^2}{2T^2S^2P^2V^2},$$

$$\gamma_{13} = \gamma_{31} = \tfrac{TS+PV}{2TSP^2V^2},\ \gamma_{14} = \gamma_{41} = \tfrac{TS-PV}{2T^2S^2PV},$$

$$\gamma_{15} = \gamma_{51} = -\tfrac{T^2S^2+2P^2V^2+4TSPV}{2T^2S^2P^2V^2},\ \gamma_{22} = \tfrac{2TSPV-P^2V^2-2T^2S^2}{4T^2S^2P^2V^2},$$

$$\gamma_{23} = \gamma_{32} = \tfrac{TS-PV}{2TSP^2V^2},\ \gamma_{24} = \gamma_{42} = -\tfrac{TS+PV}{2T^2S^2PV},$$

$$\gamma_{25} = \gamma_{52} = \tfrac{2T^2S^2+P^2V^2+4TSPV}{2T^2S^2P^2V^2},\ \gamma_{33} = -\tfrac{1}{2P^2V^2},$$

$$\gamma_{35} = \gamma_{53} = \tfrac{TS-PV}{2TSP^2V^2},\ \gamma_{44} = -\tfrac{1}{2T^2S^2},$$

$$\gamma_{45} = \gamma_{54} = \tfrac{PV-TS}{2T^2S^2PV},\ \gamma_{55} = \tfrac{T^2S^2+P^2V^2+3TSPV}{T^2S^2P^2V^2}.$$

Remark. This is an important example of Ricci tensor coefficients which are square functions in two parameters $u = \frac{1}{x^2x^3}$, $v = \frac{1}{x^4x^5}$ (see [13], [23]). From thermodynamic point of view, it can be observed that the Ricci tensor coefficients depend only on the square of the products with dimension of energy TS, PV, and their product $TSPV$.

The *scalar curvature* $c = \delta^{bd}\gamma_{bd}$ is given by

$$c = \frac{5x^2x^3x^4x^5 - {x^2}^2{x^3}^2 - {x^4}^2{x^5}^2}{{x^2}^2{x^3}^2{x^4}^2{x^5}^2}.$$

This is an important example of scalar curvature which is a square function in two parameters $u = \frac{1}{x^2x^3}$, $v = \frac{1}{x^4x^5}$ (see [12], [22]).

From thermodynamical point of view, it can be observed that the scalar curvature depends only on the energies TS, PV and their product $TSPV$. On the other hand, the scalar curvature

$$c = \frac{5}{TSPV} - \frac{1}{P^2V^2} - \frac{1}{T^2S^2}$$

reflects the saddle shape (behavior) in $\mathbf{R}^3 = \left\{u = \frac{1}{TS},\ v = \frac{1}{PV},\ c\right\}$ via the submersion $\pi : \mathbf{R}^5 \to \mathbf{R}^2,\ \pi(U,T,S,P,V) = (u,v),\ u = \frac{1}{TS}, v = \frac{1}{PV}$.

8. The submanifold of the coefficients of bilinear covariants: what is it, and from where did it come?

Let us introduce an Euclidean space $\mathbf{R}^{125}$ whose points are of the form (w^a_{bc}), a, b, $c = \overline{1,5}$. Then the Cartesian implicit equations

$$w^a_{bc} + w^a_{cb} = 0$$

describe a vector space (hyperplane) H of dimension 50.

Theorem 5 *Suppose we are in the conditions of Section 3. Then, the submanifold of the coefficients of bilinear covariants in H is 2-dimensional.*

Proof. The parametric equations in Theorem 1, Section 3, depend linearly on two parameters

$$u = \frac{1}{x^2x^3},\ v = \frac{1}{x^4x^5}.$$

The rank of the associated Jacobi matrix is 2.

What properties of the previous submanifold in $\mathbf{R}^{125}$ have connection with the geometry of 1-forms θ^a, $a = \overline{1,5}$ in $\mathbf{R}^5$?

9. The submanifold of Ricci rotation coefficients: what is it, and from where did it come?

Let us introduce an Euclidean space $\mathbf{R}^{125}$ whose points are of the form (γ^a_{bc}), a, b, $c = \overline{1,5}$. Then the Cartesian implicit equations

$$\gamma^a_{bc} + \gamma^b_{ac} = 0$$

describe a vector space (hyperplane) H of dimension 50.

Theorem 6 *Suppose we are in the conditions of Section 4. Then, the submanifold of Ricci rotation coefficients in H is 2-dimensional.*

Proof. The parametric equations in the Theorem 2 of Section 4, depend linearly on two parameters

$$u = \frac{1}{x^2x^3},\ v = \frac{1}{x^4x^5}.$$

The rank of the associated Jacobi matrix is 2.

The results in this section are related to the theory of connections whose components are polynomials (see also [13] and [23]).

What properties of the previous submanifold in $\mathbf{R}^{125}$ have connection with the geometry of 1-forms θ^a, $a = \overline{1,5}$ in $\mathbf{R}^5$?

10. The submanifold of Ricci coefficients with four indexes: what is it, and from where did it come?

Let us introduce an Euclidean space $\mathbf{R}^{625}$ whose points are of the form $(\gamma^a_{bcd}), a, b, c, d = \overline{1,5}$. Then the Cartesian implicit equations (a complete list of symmetries)

$$\gamma^a_{bcd} + \gamma^b_{acd} = 0,\ \gamma^a_{bcd} + \gamma^a_{bdc} = 0,\ \gamma^a_{bcd} + \gamma^a_{cdb} + \gamma^a_{dbc} = 0$$

describe a vector space (hyperplane) H of dimension 50.

Now we define a *generalized Steiner hypersurface* as the image of a rational parametrization of two variables (for Steiner surfaces, see [22]).

Theorem 7 *Suppose we are in the conditions of Section 6. Then, the submanifold of Ricci coefficients with four indexes in H is a generalized Steiner hypersurface of dimension 2.*

Proof. The parametric equations in the Theorem 3, Section 6, depend quadratically on two parameters

$$u = \frac{1}{x^2x^3},\ v = \frac{1}{x^4x^5}.$$

The rank of the associated Jacobi matrix is 2.

Can we create a geometry of the previous generalized Steiner hypersurface in connection with the 1-forms θ^a, $a = \overline{1,5}$ in $\mathbf{R}^5$?

11. Conclusions

A new approach to the thermodynamical systems based on the Gibbs-Pfaff differential 1-form, including the congruences theory of Vrânceanu combined with the Udrişte's theory concerning odd-dimensional nonholonomic equations, has been presented. The physical meanings of the used congruences are underlined and a Riemannian metric of orthonormalization is built. The proofs that the coefficients of bilinear covariants and the coefficients of Ricci (with three and four indexes) are signomials and the tangent vectors to geodesics are rational functions are also given. As a novelty, the authors introduce and study the submanifold of the coefficients of bilinear covariants, the submanifold of Ricci rotation coefficients and the submanifold of Ricci coefficients with four indexes.

R E F E R E N C E S

1. *J. Aman, I. Bengtsson, N. Pidokrajt*, "Geometry of black hole thermodynamics", Gen. Rel. Grav. 35, 1733, 2003.
2. *C. Caratheodory*, "Undersuchungen uber die Grunlagen der Thermodynamik", Math. Ann. 67, 355, 1909.
3. *A. Dobrescu*, "Differential Geometry", Didactical and Pedagogical Editorial House, Bucharest, 1963, (in Romanian).
4. *T. Feldman, B. Andersen, A. Qi, P. Salamon*, "Thermodynamic lengths and intrinsic time scales in molecular relaxation", DChem. Phys. 83, 5849, 1985.
5. *M. Ferrara, C. Udrişte*, "Area Conditions Associated to Thermodynamic and Economic Systems", Proceedings of The 2-nd International Colloquium Mathematics in Engineering and Numerical Physics, April 22-24 (2002), University Politehnica Bucharest, Romania, BSG Proceedings 8, pp. 60-68, Geometry Balkan Press; presented at The International Workshop on Complex Systems in Natural and Social Sciences, September 26-29 (2002), Matrafured, Hungary.
6. *J.W. Gibbs*, "Collected Works", Vol. I, *"Thermodynamics"*, Yale University Press, New Haven, 1948.
7. *R. Gilmore*, "Thermodynamic partial derivatives", J. Chem. Phys. 75, 5964, 1981.
8. *R. Hermann*, "Geometry, Physics, and Systems", Marcel Dekker Inc., New York, 1973.
9. *G. Hernández, E.A. Lacomba*, "Contact Riemannian geometry and thermodynamics", Diff. Geom. and Appl. 8, 205, 1998.
10. *J. Kocik*, On geometry of phenomenological thermodynamics, Symmetries in Science II, B. Gruber Ed., New York, 1986.

11. *R. Mrugala*, "Geometrical formulation of equilibrium phenomenological thermodynamics", Rep. Math. Phys. 14, 419, 1978.
12. *R. Mrugala*, "Submanifolds in the thermodynamic phase space", Rep. Math. Phys. 21, 197, 1985.
13. *T. Postelnicu*, "Espaces A_2 à connexion linéaire localement euclidiens", Revue of Bucharest University 4-5, pp. 101-131, 1954.
14. *H. Quevedo, A. Vasques* , "The geometry of Thermodynamics", J. Math. Phys. 48, 013506, 2007.
15. *H. Quevedo*, "Geometrothermodynamics", AIP Conf. Proc. March 6 (2008), Volume 977, pp. 165-172, (arXiv:0712.0868v1 [math-ph].
16. *V. Radcenko, C. Udrişte, D. Udrişte*, "Thermodynamic systems and their interaction", 22-nd Conference on Differential Geometry and Topology, Applications in Physics and Technics, Sept. 9-13 (1991), Bucharest; Scientific Bulletin, Polytechnic Institute of Bucharest, Electrical Engineering, vol. 53, 3-4, pp. 285-294, 1991.
17. *G. Ruppeiner*, "Thermodynamics: A Riemannian geometric model", Phys. Rev. A 20, 1608, 1979.
18. *P. Salamon, B. Andersen, P.D. Gait, R.S. Berry*, "The significance of Weinhold's length", J. Chem. Phys. 73, 1001, 1980.
19. *J. Shen, R. Cai, B. Wang, R. Su*, "Thermodynamic geometry and critical behavior of black holes", (arXiv: gr-qc/0512035), 2005.
20. *C. Stamin, C. Udrişte*, "The Geometry of Gibbs-Duhem-Pfaff Thermochemical System", Proceedings of the 7th WSEAS International Conference on Instrumentation, Measurement, Circuits and Systems (IMCAS'08), Electrical and Computer Engineering Series, pp. 77-89, Hangzhou, China, April 6-8, 2008.
21. *C. Stamin, C. Udrişte*, "Subriemannian Geometry of an Open Thermodynamical System", Proceedings of the 2nd WSEAS International Conference on Multivariate Analysis and Its Application in Science and Engineering (MAASE'09), pp. 60-67, Istanbul, Turkey, May 30-June 1, 2009.
22. "Steiner Surfaces", www.ipfw.edu/math/Coffman /steinersurface.html, INTERNET 2009.
23. *I. Teodorescu*, "On the spaces A_2 with polynomial affine connection, locally Euclidean", Reports of the Romanian Academy 13, pp. 405-411, 1963 (in Romanian).
24. *C. Udrişte*, "Geometric Dynamics", Kluwer Academic Publishers, 2000.
25. *C. Udrişte*, "Thermodynamics versus Economics", University Politehnica of Bucharest, Scientific Bulletin, Series A, 69, 3, pp. 89-91, 2007.
26. *C. Udrişte, O. Dogaru, I. Ţevy*, "Extrema with Nonholonomic Constraints", Selected papers, Monographs and Textbooks 4, Geometry Balkan Press, Bucharest, 2002.

27. *C. Udrişte, M. Ferrara*, "Black hole models in economics", The 4-th International Colloquium of Mathematics in Engineering and Numerical Physics, Univ. Politehnica of Bucharest, 6-8 Oct., (2006); Tensor, N.S., vol. 70, 1, pp. 53-62, (2008).
28. *G. Vrânceanu*, "Lectures of Differential Geometry", Didactical and Pedagogical Editorial House, Bucharest, Vol. I, 1962; Vol. II, 1964, (in Romanian).
29. *G. Vrânceanu*, "Leçons de géométrie différentielle", Vol. I, Académie Roumaine, Bucharest, 1957.