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ON THE STRUCTURE OF ULTRA-GROUPS OVER A FINITE GROUP

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In this paper, we introduce the new concept of an ultra-group ${}_HM$ which depends on the group G and its subgroup H. An ultra-group is defined by the use of the transversal in a group. Moreover, the elementary properties of an ultra-group are investigated. Finally, after assembling general properties of an ultra group, we try to classify all ultra-groups of the subgroup over the group.

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1. Introduction

Algebraic structures appear in most branches of mathematics, such as the abstract algebra, universal algebra, varieties and category theory. The identification of algebraic structures is also useful in other fields of science. For instance recognition of algebraic structures in quantum physical systems has been an important tool for their understanding (see[4, 9]). Nowadays, answering to the new questions needs some more new tools. One of the very useful notions in mathematics as well as in computer science is the notion of s-acts (see [3, 5] for more details). We establish a new structure, ultra-group. The concept of an ultra-group is the base of a new branch of studies in algebra and the future researches. In the present work we continue the study of a variant of natural generalization of a notion of transversal in a group to its subgroup (see [1, 7]). In the next section of this article we define the notions which are useful through out this work such as a pair of subsets which are transversal, left and right quotient sets, transversal of a partition and complementary set . After these arrangements, we are ready to introduce an ultra-group. In fact an ultra-group is an algebraic structure whose underlying set is depend on a group and its subgroup. Moreover, an abelian ultra-group

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is equivalence to quasigroup and finite loop (see [8, 10]). Furthermore, we present some general properties of an ultra-group. The subultra-group, normal subultra-group, homomorphism between two ultra-groups and quotient ultra-groups are concepts which are defined in this section. These concepts help us to pose isomorphism theorems for ultra-groups. We observe that the ultra-group isomorphism preserves the order of elements in the ultra-group, and all the ultra-groups of conjugate subgroups of a group are isomorphic. This result helps us to classify the ultra-groups over a group more quickly in the third part. In addition we show that if ${}_HM$ is an ultra-group of H over the group G, then for every subultra-group S of ${}_HM$ there exists a subgroup K of group G such that $H \subseteq K$ and $K = HS$. In other words, there is a one to one correspondence between a chain of subultra-groups of an ultra-group and a chain of subgroups of G. Finally, we generalize the Lagrange Theorem for the ultra-groups. Throughout the paper, all the necessary definitions and preliminary statements may be found in (see[6, 2]).

2. **Preliminaries about ultra-groups**

We always consider G as a finite group and e as its identity element. Suppose M and H are two arbitrary subsets of G. Clearly, $HM \neq MH$ in general. A pair of (A, B) of subsets of a group G is called transversal if the equality $ab = a'b'$ implies $a = a'$ and $b = b'$, where $a, a' \in A$, $b, b' \in B$. This definition can be generalized to subgroups. It is not hard to deduce that a pair (H, M) of subgroups of G is transversal if and only if $H \cap M = \{e\}$. Moreover, for a subgroup H and a subset M of group G we conclude that the pair (H, M) is a transversal if and only if $M \cap Hg$ contains at most one element, for all $g \in G$. We denote the right and left quotient sets by $H \setminus G = \{Hx : x \in G\}$ and $G/H = \{xH : x \in G\}$ such that $H \setminus G$ as well as G/H are partitions of G, where H is a subgroup of G.

Definition 2.1. *A transversal of a partition is a set which contains exactly one element from each part of the partition. If $H \leq G$, then a transversal for the partition $H \setminus G$ (G/H) is called a right (left) transversal for H in G.*

Let H be a subgroup of G and M a subset of G. If $|M \cap Hg| = 1$ for all $g \in G$, then $G = HM$. For proving this equality, it is enough to show $G \subseteq HM$. Suppose $g \in G$, $g \notin H$ and $g \notin M$. If $m \in M \cap Hg$, then $g = h^{-1}m$ which implies $g \in HM$. For the group G which satisfies the above conditions, we have $MH \subseteq G = HM$. Therefore, for every element $mh \in MH$ there exists $h' \in H$ and $m' \in M$ such that $mh = h'm'$.

Definition 2.2. *Let H be a subgroup of a multiplicative group G. A subset M of G is called (right unitary) complementary set with respect to subgroup H, if for any elements $m \in M$ and $h \in H$ there exist the unique elements $h' \in H$ and $m' \in M$ such that $mh = h'm'$ and $e \in M$. We denote h' and m' by ${}^mh$ and m^h, respectively.*

Similarly for any elements $m_1, m_2 \in M$ there exist unique elements $[m_1, m_2] \in M$ and $(m_1, m_2) \in H$ such that $m_1 m_2 = (m_1, m_2)[m_1, m_2]$. For every element $a \in M$, there exists a^{-1} belonging to G. As $G = HM$, there is $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)} a^{[-1]}$. Now we are ready to define an ultra-group.

Definition 2.3. *A (right) ultra-group ${}_HM$ is a complementary set of H over group G with a binary operation $\alpha :_H M \times_H M \to_H M$ and unary operation $\beta_h :_H M \to_H M$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$.*

A (left) ultra-group M_H is defined similarly via (left unitary) complementary set. From now on, unless specified otherwise, ultra-group means right ultra group. An ultra-group M is called abelian, if for all elements a, b in M, $[a, b] = [b, a]$. A non-abelian group may have abelian ultra-groups. Obviously, every group is an ultra-group but the converse does not hold. Although every element in right ultra-groups do not have right inverse, but they have left inverse with respect to the first binary operation α. If H is a subgroup of the finite group G and $[G : H] = 2$, then ultra-group M with binary operation α is a group. The first binary operation of an ultra-group M has the right cancellation. If $[b, a] = [c, a]$ for $a, b, c \in M$, then by the argument after Definition 2.2 we have $(b, a)^{-1} b = (c, a)^{-1} c$ and by transversal property we conclude $b = c$. We observe that we do not have left cancellation for right ultra-groups. Let ${}_HM$ be an ultra-group of H over the group G and α, β its binary and unary operation on it. For every element $a \in {}_HM$ one can define a map $\alpha_a : M \longrightarrow M$ by $\alpha_a(b) = \alpha(b, a)$. By use of right cancellation property of the first binary operation of an ultra-group we deduce that α_a is a bijection. In the following proposition we present some properties of operations of an ultra-group.

Proposition 2.1. *Let M be a ultra-group of subgroup H over the group G. Then we have the following properties:*
(i) $a^{hh'} = (a^h)^{h'}$,
(ii) $[a, b]^h = [a^{({}^bh)}, b^h]$,
(iii) $[[a, b], c] = [a^{(b,c)}, [b, c]]$,
(iv) $e^h = e, \quad a^e = a$,
(v) $[e, a] = a = [a, e]$,
(vi) $[a^{[-1]}, a] = e = [a^{(a^{(-1)})}, a^{[-1]}]$,
for $a, b, c \in M$ and $h, h' \in H$.

Proof. (i) For any $a \in M$ and $h, h' \in H$ we have :

$$a(hh') = {}^a(hh')a^{hh'} \text{ and } (ah)h' = ({}^ah\ a^h)h' = ({}^ah\ {}^{a^h}h')(a^h)^{h'}.$$

By property transversal of complementary set (Definition 2.2) and associativity of group G, we have $a^{hh'} = (a^h)^{h'}$.
(ii) For any $a, b \in M$ and $h \in H$,

$$
\begin{aligned}
(ab)h &= a(bh)\\
((a,b)[a,b])h &= a({}^{b}h\ b^{h})\\
(a,b)([a,b]h) &= (a\ {}^{b}h)b^{h}\\
(a,b)({}^{[a,b]}h[a,b]^{h}) &= ({}^{a}({}^{b}h)a^{{}^{b}h})b^{h}\\
((a,b)^{[a,b]}h)[a,b]^{h} &= ({}^{a}({}^{b}h)(a^{{}^{b}h},b^{h}))[a^{{}^{b}h},b^{h}].
\end{aligned}
$$

By property transversal we deduce that $[a,b]^{h} = [a^{{}^{b}h}, b^{h}]$.
(iii) For any $a,b,c \in M$,

$$
\begin{aligned}
(ab)c &= a(bc)\\
\Big((a,b)[a,b]\Big)c &= a\Big((b,c)[b,c]\Big)\\
(a,b)\Big([a,b]c\Big) &= \Big(a(b,c)\Big)[b,c]\\
(a,b)\Big(([a,b],c)[[a,b],c]\Big) &= \Big({}^{a}(b,c)a^{(b,c)}\Big)[b,c]\\
\Big((a,b)([a,b],c)\Big)[[a,b],c] &= {}^{a}(b,c)\Big(a^{(b,c)}[b,c]\Big)\\
&= {}^{a}(b,c)\Big(a^{(b,c)},[b,c]\Big)[a^{(b,c)},[b,c]].
\end{aligned}
$$

Now the assertion is clear.
(iv), (v) For any $a \in M$, $h \in H$ and unitary element $e \in M$ we have :

$$he = eh = {}^{e}he^{h},\ ea = ae = {}^{a}ea^{e} \text{ and } (e,a)[e,a] = (a,e)[a,e] = ea.$$

Hence we obtained the relations

$$e^{h} = e,\ a^{e} = a \text{ and } [e,a] = [a,e] = a.$$

(vi) Since for each element a in M there exists $a^{-1} \in G$, therefore there exists $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)}a^{[-1]}$. Thus we have :
$e = a^{-1}a = a^{(-1)}a^{[-1]}a = (a^{(-1)}(a^{[-1]},a))[a^{[-1]},a]$, and
$e = aa^{-1} = aa^{(-1)}a^{[-1]} = {}^{a}(a^{(-1)})a^{a^{(-1)}}a^{[-1]} = ({}^{a}(a^{(-1)})(a^{a^{(-1)}},a^{[-1]}))[a^{a^{(-1)}},a^{[-1]}]$.
The assertion follows from uniquely represented e of G. □

Definition 2.4. *Let M be ultra-group of H over G. A subset $S \subseteq M$ which contains e, is a subultra-group of H over G, if S is closed with respect to the operations α and β_h in the Definition* 2.3.

Suppose A, B are two subsets of the ultra-group ${}_{H}M$. We use the notation $[A,B]$ for the set of all $[a,b]$, where $a \in A$ and $b \in B$. If B is a singleton $\{b\}$, then we denote $[A,B]$ by $[A,b]$. Moreover, if A is a subultra-group of ultra-group ${}_{H}M$ and $b \in {}_{H}M$, then the subset $[A,b]$ is called a right coset of A in ${}_{H}M$.

Lemma 2.1. *Let S be a subultra-group of ultra-group ${}_HM$ of H over the group G and $a, b \in {}_HM$. Then the following conditions are equivalent.*
(i) $a \in [S, b]$,
(ii) $[S, a] = [S, b]$,
(iii) $[a^{(b^{(-1)})}, b^{[-1]}] \in S$.

Proof. The assertion follows immediately from the Definition 2.4. □

By Lemma 2.1 $[S, a] = [S, b]$ or $[S, a] \cap [S, b] = \emptyset$, which implies

$$ {}_HM = \cup_{a \in {}_HM} [S, a] \ . $$

In the following we are going to present a one to one correspondence between subultra-groups of a fixed group G and the certain subgroups of the group.

Lemma 2.2. *Let S be a subultra-group of ultra-group ${}_HM$ of subgroup H over the group G. Then HS is a subgroup of group G .*

Proof. Let $a, b \in HS$. Then there exist $h_i \in H$ and $s_i \in S$ for $i = 1, 2$ such that $a = h_1 s_1$ and $b = h_2 s_2$. Therefore we have $ab^{-1} = h_1 s_1 s_2^{(-1)} s_2^{[-1]} h_2^{-1} \in HS$. Now, immediately from the Proposition 2.1 the result follows. □

Theorem 2.1. *Let H be a subgroup of group G. Then for every subgroup K of G which is contained in H, there exists a subultra-group S of ${}_HM$ such that $K = HS$.*

Proof. Let S be an ultra-group of subgroup H over the group K. Then the transversal property of the pair (H, S) of group K implies that $S \cap Hg$ contains at most one element, for all $g \in K$ and $K = HS$. Therefore $H \setminus (G - K)$ is partition of $G - K$. Now by choosing arbitrary one element of every right coset $H \setminus (G - K)$ and add to subultra-group S, we have some ultra-groups ${}_HM$ of subgroup H over group G such that S is subultra-group of ${}_HM$. □

Let ${}_HM$ be an ultra-group of H over the group G, and S_i for $i = 1, 2, 3, ..., n$ be subultra-groups of the ultra-group ${}_HM$ such that $S_i \subseteq S_{i+1}$ for $i = 1, 2, 3, ..., n-1$. Then we have subgroups $K_i = HS_i$ for $i = 1, 2, 3, ..., n$ of group G such that $K_i \subseteq K_{i+1}$ for $i = 1, 2, 3, ..., n-1$.

Definition 2.5. *Suppose ${}_{H_i}M_i$ is an ultra-group of H_i over the group G_i, $i = 1, 2$, and φ is a homomorphism between two subgroups H_1 and H_2. A function $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is an ultra-group homomorphism if for all $m, m_1, m_2 \in {}_{H_1}M_1$ and $h \in H_1$ hold:*
(i) $f([m_1, m_2]) = [f(m_1), f(m_2)]$,
(ii) $(f(m))^{\varphi(h)} = f(m^h)$.

If f is a surjective and injective ultra-group homomorphism, then we call it isomorphism and denote it by ${}_{H_1}M_1 \cong {}_{H_2}M_2$. In the sequel φ is a group homomorphism between two subgroups of the group for which the ultra-groups are defined.

Proposition 2.2. *Let $f :_{H_1} M_1 \to_{H_2} M_2$ be a homomorphism between two ultra-groups.*
(i) All homomorphism between the two ultra-groups preserve the identity and left inverse elements.
(ii) If S is a subultra-group of ${}_{H_1}M_1$ and φ is surjective homomorphism, then $f(S)$ is a subultra-group of ${}_{H_2}M_2$.
(iii) The inverse image of a subultra-group of ${}_{H_2}M_2$ is a subultra-group of ${}_{H_1}M_1$.

Proof. (i) Let e_i be the identity element of ${}_{H_i}M_i$, $i = 1, 2$. Applying f to the equation $[e_1, e_1] = e_1$ in ${}_{H_1}M_1$ gives the following equation :

$$[f(e_1), f(e_1)] = f(e_1) = [e_2, f(e_1)].$$

By right cancellation implies that $f(e_1) = e_2$. Now by Proposition 2.1 and homomorphism definition we have $[f(m^{[-1]}), f(m)] = [(f(m))^{[-1]}, f(m)]$ for any $m \in {}_{H_1}M_1$,. Hence by right cancellation $f(m^{[-1]}) = (f(m))^{[-1]}$.
(ii) Since $e_1 \in S$ we have $e_2 = f(e_1) \in f(S)$. Suppose $m_1, m_2 \in f(S)$. There exist $s_1, s_2 \in S$ such that $f(s_i) = m_i$, $i = 1, 2$. Thus $[m_1, m_2] = [f(s_1), f(s_2)] = f([s_1, s_2]) \in f(S)$. Moreover, for every $m \in f(S)$, there exists $s \in S$ such that $f(s) = m$. Therefore $m^{\varphi(h)} = (f(s))^{\varphi(h)} = f(s^h) \in f(S)$, where $h \in H_1$. Hence $f(S)$ is closed with respect to the operations of ${}_{H_2}M_2$.
(iii) Choose a subultra-group B of ${}_{H_2}M_2$. By $f(e_1) = e_2 \in_{H_2} M_2$, we get $f^{-1}(e_2) = e_1 \in f^{-1}(B)$. Pick any two elements of $f^{-1}(B)$, say b_1 and b_2. As B is a subultra-group of ${}_{H_2}M_2$, $[b_1, b_2] \in f^{-1}(B)$. Moreover, $\beta_{\varphi(h)}(f(b)) = f(b^h)$ which implies $f^{-1}(\beta_{\varphi(h)}(f(b)) = b^h$, for $b \in f^{-1}(B)$. □

Definition 2.6. *Let $f :_{H_1} M_1 \to_{H_2} M_2$ be ultra-groups homomorphism. Then* Ker(f) *is defined by $\{(m_1, m_2) \in_{H_1} M_1 \times_{H_1} M_1 : f(m_1) = f(m_2)\}$. In other words, if $(m_1, m_2) \in$ Ker(f) then $f(m_1) = f(m_2)$. Therefore $f([m_1^{(m_2)^{(-1)}}, m_2^{[-1]}]) = f(e)$ which means $([m_1^{(m_2)^{(-1)}}, m_2^{[-1]}], e) \in$ Ker(f). We can refer* Ker(f) *as the inverse image of the identity element e under the homomorphism,* Ker(f) $= \{m \in {}_{H_1}M_1 : f(m) = e\}$.

Lemma 2.3. *Let f be a homomorphism between two ultra groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$ with kernel K where $K = \{m \in {}_{H_1}M_1 : f(m) = e\}$. Then $f(m_1) = f(m_2)$ if and only if $m_1 = [k, m_2]$ for some $k \in K$.*

Proof. Suppose $f(m_1) = f(m_2)$, so we have $[(f(m_1))^{(f(m_2))^{(-1)}}, (f(m_2))^{[-1]}] = f([m_1^{m_2^{(-1)}}, m_2^{[-1]}]) = e$. This means that $[m_1^{m_2^{(-1)}}, m_2^{[-1]}] \in K$, thus $m_1 = [k, m_2]$ for some $k \in K$. Conversely $f(m_1) = f([k, m_2]) = [f(k), f(m_2)] = f(m_2)$. □

Congruence is a special type of equivalence relation which plays a vital role in the study of quotient structures of different algebraic structures (see [2]).

Theorem 2.2. *Suppose the same notations as in Definition* 2.6. *Then* Ker(f) *is a congruence on* ${}_{H_1}M_1$ *and it is a subultra-group of* ${}_{H_1}M_1$.

Proof. It is obvious that Ker(f) is an equivalence relation. If $(m_1 \text{ Ker}(f)\ m_2)$ and $(m_3 \text{ Ker}(f)\ m_4)$, then $f(m_1) = f(m_2)$ and $f(m_3) = f(m_4)$. Therefore $f([m_1, m_3]) = [f(m_1), f(m_3)] = [f(m_2), f(m_4)] = f([m_2, m_4])$. Furthermore, $f(m_1) = f(m_2)$ implies that $f(m_1)^{\varphi(h)} = f(m_2)^{\varphi(h)}$. Hence $f(m_1^h) = f(m_2^h)$ and $(\beta_h(m_1) \text{ Ker}(f)\ \beta_h(m_2))$. The rest is clear. □

Let f be a homomorphism between two ultra-groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$. Then f is injective if and only if Ker$(f) = \Delta_{{}_{H_1}M_1}$. Now, consider the second definition for the kernel of a homomorphism, Ker$(f) = \{m \in {}_{H_1}M_1 : f(m) = e\}$. The ultra-group homomorphism f is injective if and only if Ker$(f) = \{e_1\}$. Clearly, if f is an isomorphism between two ultra-groups, then f^{-1} is an isomorphism.

Definition 2.7. *Let M be an ultra-group over the subgroup H_1 of the group G and θ be a congruence over M. The set $M/\theta = \{[m]_\theta; m \in M\}$ with the operations α_θ and β_{θ_h},*
(i) $\alpha_\theta([m]_\theta, [m']_\theta) = [\alpha(m, m')]_\theta$
(ii) $\beta_{\theta_h}([m]_\theta) = [\beta_h(m)]_\theta$
is an ultra-group of H_2 over the group G, where $H_1 \leq H_2 \leq G$. This ultra-group is called a quotient ultra-group.

For the ultra-group ${}_{H_1}M_1$ and congruence θ over it, $\pi :_{H_1} M_1 \longrightarrow_{H_1} M_1/\theta$ is the canonical (natural) homomorphism. The following theorem is powerful in applications.

Theorem 2.3. *(First isomorphism theorem for ultra-groups) Let f be a surjective ultra-group homomorphism between two ultra-groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$ and θ a congruence over ${}_{H_1}M_1$ such that $\theta \subseteq$ Ker(f). If $\pi :_{H_1} M_1 \longrightarrow_{H_1} M_1/\theta$, then there exists a homomorphism g satisfying $g\pi = f$.*

Proof. It is enough to define the map $g :_{H_1} M_1/\theta \longrightarrow_{H_2} M_2$ by $g([m]_\theta) = f(m)$. Since $[m_1]_\theta = [m_2]_\theta \Leftrightarrow m_1\ \theta\ m_2$, by the hypothesis $\theta \subseteq Ker(f)$ we have $f(m_1) = f(m_2)$ which implies g is well-defined. The map g preserves the first operation α_θ and the unary operation β_{θ_h}. Thus we can conclude that g is an ultra homomorphism. □

In the following it is established a connection between the congruence on a ultra-group and a normal subultra-group.

Lemma 2.4. *Let S be a subultra-group of ${}_HM$. Then*
(i) $[a^{b(-1)}, b^{[-1]}] \in S$ *if and only if there exists $s \in S$ such that $a = [s, b]$.*
(ii) Let θ be a relation on ${}_HM$ defined by $a\theta b$ if and only if there is $s \in S$ such that $a = [s, b]$. Then θ is an equivalence relation.

Proof. (i) By Proposition 2.1 and right cancellation property we have

$$[a^{b^{(-1)}}, b^{[-1]}] = s \iff [[a^{b^{(-1)}}, b^{[-1]}], b] = [s, b] \iff [[a^{b^{(-1)}(b^{[-1]},b)}, [b^{[-1]}, b]] = [s, b] \iff a = [s, b],$$

where $a, b \in {}_HM$ and $s \in S$. The second part follows immediately. □

From now on we use the notation θ for the equivalence relation which is satisfied in the second part of Lemma 2.4.

Definition 2.8. *A subultra-group N of ${}_HM$ is called normal if $[a, [N, b]] = [N, [a, b]]$, for all $a, b \in {}_HM$.*

For instance if we denote the equivalence class of e with respect to the equivalence relation of θ in the second part of Lemma 2.4 by $[e]_\theta$, then $[e]_\theta = S$ is a normal subultra-group of ${}_HM$. If it is necessary, then we can switch S and θ on some situations, in sequel.

Moreover, $\mathrm{Ker}(f)$ is a normal subultra-group of ${}_{H_1}M_1$, where $f : {}_{H_1}M_1 \to {}_{H_2}M_2$ is an ultra-group homomorphism.

Lemma 2.5. *Let N be a normal subultra-group of ${}_HM$. Then we have the following properties,*
(i) $[a, N] = [N, a]$, for all $a \in_H M$.
(ii) $[[N, a], [N, b]] = [N, [a, b]]$, for all $a, b \in_H M$.
(iii) If $[N, b] = N$, then $b \in N$.
(iv) $[N, S]$ is a subultra-group of ${}_HM$, where S is a subultra-group of ${}_HM$. Moreover, $[N, S]$ is a normal subultra-group of ${}_HM$ if S is also normal subultra-group of ${}_HM$.

Proof. (i) It is enough to put e instead of b in the Definition 2.8.
(ii) By the fifth part of Proposition 2.1 and normality of subultra-group N we deduce $[[N, a], [N, b]] = [N, [a, [N, b]]] = [N, [N, [a, b]]] = [[N, N], [a, b]] = [N, [a, b]]$.
(iii) By hypothesis, there exists $n' \in N$ such that $[n, b] = n'$ for all $n \in N$. Moreover, $[(n^{[-1]})^{(n,b)}, [n, b]] = [(n^{[-1]})^{(n,b)}, n'] \in N$. Therefore, by Proposition 2.1 $[[n^{[-1]}, n], b] \in N$. Hence the assertion follows.
(iv) The subset $[N, S]$ is closed with respect to the first and second operations of ultra-group ${}_HM$ by (ii) and the second part of Proposition 2.1, respectively. By Definition 2.8 we have $[a, [[N, S], b]] = [a, [N, [S, b]]] = [N, [a, [S, b]]] = [N, [S, [a, b]]] = [[N, S], [a, b]]$. □

Theorem 2.4. *Let $f :_{H_1} M_1 \to_{H_2} M_2$ be an ultra-group homomorphism. The inverse image of a normal subultra-group B of ${}_{H_2}M_2$ is a normal subultra-group of ${}_{H_1}M_1$.*

Proof. By Proposition 2.2 we have $f^{-1}(B)$ is a subultra-group of ${}_{H_1}M_1$. It is sufficient to show that $f^{-1}(B)$ is normal. We have $f([a, [f^{-1}(B), b]]) =$

$[f(a),[B,f(b)]] = [B,f([a,b])] = f([f^{-1}(B),[a,b]])$ by homomorphism property, for every elements $a,b \in_{H_1} M_1$. Thus $([a,[f^{-1}(B),b]],[f^{-1}(B),[a,b]]) \in \mathrm{Ker}(f) = \Delta_{{}_{H_1}M_1}$ which implies $f^{-1}(B)$ is normal. □

Theorem 2.5. *If S is a subultra-group of ${}_HM$, then the equivalence relation θ is a congruence if and only if S is a normal subultra-group ${}_HM$.*

Proof. Suppose θ is a congruence. By the argument after Definition 2.8, S is a normal subultra-group. For the converse, let $a\theta b$ and $a'\theta b'$, where $a,b,a',b' \in {}_HM$. Then by the second part of Lemma 2.5 $[a,a'] = [[n,b],[n',b]] = [n'',[b,b']]$. This means $[a,a']\theta[b,b']$, for $n,n',n'' \in N$. For the compatibility of the second operation of the ultra-group, assume $a\theta b$. Thus $a^h = [n',b^h]$ and the definition of θ implies $a^h\theta b^h$, for all $h \in H$. □

Theorem 2.6. *Let γ be a congruence on ${}_HM$ and $[e]_\gamma = S$. Then γ and θ are equivalent.*

Proof. By the congruence definition, Proposition 2.1 and right cancellation property we conclude,

$$\begin{aligned} a\ \gamma\ b &\iff a^{b^{(-1)}}\ \gamma\ b^{b^{(-1)}} \iff [a^{b^{(-1)}},b^{[-1]}]\ \gamma\ [b^{b^{(-1)}},b^{[-1]}] \\ &\iff [a^{b^{(-1)}},b^{[-1]}]\ \gamma\ e \iff [a^{b^{(-1)}},b^{[-1]}] \in S \\ &\iff a\ \theta\ b. \end{aligned}$$

□

Theorem 2.7. *If γ is a congruence on the ultra-group ${}_HM$ and $[e]_\gamma = N$, then ${}_HM/\theta \cong_H M/N$.*

Proof. The elements of ${}_HM/\theta$ are the classes of $[m]_\theta = \{m_1 \in_H M : m_1\theta m\}$ by definition of θ we deduce that $[m]_\theta = \{[n,m] : n \in N\} = [N,m]$ (see Lemma 2.4). We define $\psi :_H M/\theta \longrightarrow_H M/N$ such that $\psi([m]_\theta) = [N,m]$. It is clear that ψ is well defined. If $[m_1]_\theta,[m_2]_\theta \in_H M/\theta$, then by the fact that N is a normal subultra-group of ${}_HM$ and Proposition 2.1 we have $\psi([[m_1]_\theta,[m_2]_\theta]) = [N,[m_1,m_2]] = [N,[m_1,[N,m_2]]] = [[N,m_1],[N,m_2]] = [\psi([m_1]_\theta),\psi([m_2]_\theta)]$. Therefore ψ is an ultra-group isomorphism. □

If γ,θ are congruences on ${}_HM$ and $\theta \subseteq \gamma$, then clearly $\gamma/\theta = \{(a/\theta,b/\theta) \in ({}_HM/\theta)^2 : (a,b) \in \gamma\}$ is a congruence on ${}_HM/\theta$.

Theorem 2.8. *(Second isomorphism theorem of ultra-groups) If N',N are normal subultra-groups of the ultra-group ${}_HM$ such that $N \subseteq N'$, then $\frac{\frac{{}_HM}{N}}{\frac{N'}{N}} \cong \frac{{}_HM}{N'}$.*

Proof. By the argument before the theorem N'/N is a normal subultra-group of ${}_HM/N$. The map $\psi :_H M/N/N'/N \longrightarrow M/N'$ is a homomorphism with $N'/N \subseteq \mathrm{Ker}(\psi)$ so the result follows by the first isomorphism theorem. □

The third isomorphism theorem [2, Theorem 2.6.18] which is valid for any algebra can be translated for ultra-groups. Although, we can be translated prove the third isomorphism theorem for ultra-groups by the same method of the proof of the first isomorphism theorem. We need Lemma 2.6 in order to mimic the proof of Theorem 2.6.18 in [2].

Lemma 2.6. *Let B be a subultra-group of ${}_HM$ and θ a congruence on ${}_HM$. Then*
(i) $B^2 \cap \theta = (B \cap N)^2$,
(ii) $B^\theta = [N, B]$, where N is $[e]_\theta$.

Proof. (i) With some basic properties of the congruence in hand, we have :
$B^2 \cap \theta = \{(b_1, b_2) : b_1, b_2 \in B \text{ and } b_1 \; \theta \; b_2\} = \{(b_1, b_2) : b_1, b_2 \in N\} = B^2 \cap N^2$.
(ii) By [2, Definition 2.6.16] we deduce $B^\theta = \{a \in_H M : \; \exists\, b \in B, a \; \theta \; b\} = \{a \in \; {}_HM : \exists\, b \in B, a = [n, b] \text{ for some } n \in N\} \subseteq [N, B]$. The rest is clear. □

Theorem 2.9. *(Third isomorphism theorem) If B is a subultra-group of ${}_HM$ and N is a normal subultra-group of ${}_HM$, then $\frac{B}{B \cap N} \cong \frac{[N,B]}{N}$.*

Proof. Since $[e]_\theta = B \cap N$ the proof is straightforward. □

3. **Classifying ultra-groups of a subgroup over a group**

We specify classes of ultra-groups of a subgroup over a group up to ultra-groups isomorphism. Let us start with the following definition.

Definition 3.1. *Let a be an element of ultra-group ${}_HM$. The smallest positive integer n such that* $a^n := \underbrace{[[[a, a], a], \cdots, a]}_{n \text{ times } \alpha \text{ operation of } a} = e$ *is called the order of a. We denote it by $o(a)$.*

Theorem 3.1. *If ${}_HM$ is an ultra-group over the group G, then the order of an element a in the ultra-group ${}_HM$ is a divisor of the order of the element a in the group G.*

Proof. Suppose the order of the element a is k and m in the ultra-group ${}_HM$ and the group G, respectively. Therefore $a^{m-k} \in H$. We show that $k \mid m - k$. Suppose in the contrary way, there exist q and r such that $a^{m-k} = a^{kq+r}$ and $0 < r < k$. Therefore $a^r \in H$, which contradicts the fact that $o(a)$ is k in the ultra-group ${}_HM$. □

It is not hard to deduce that the ultra-group isomorphism preserves the order of elements. Lagrange theorem is valid for ultra-groups, the proof is clear by the argument after Lemma 2.1 so we omit it.

Theorem 3.2. *If S is a subultra-group of ${}_HM$, then $|S|$ divides $|{}_HM|$.*

If $a \in Hg$, then for every $h \in H$ there exists $g' \in G$ such that $\beta_h(a) \in Hg'$. We deduce that if $a, b \in Hg$, then $\beta_h(a)$ and $\beta_h(b)$ belong to the same coset, where $H \leq G$.

Let H be a subgroup of G and $|G| = n$, $|H| = m$. Then we can make $m^{\frac{n}{m}-1}$ ultra-groups of H over G such that each of them has n/m elements. If $H \setminus G = \{H, Hg_1, \cdots, Hg_{\frac{n}{m}-1}\}$, then the ultra-groups ${}_HM_i = \{a_{i0} = e, a_{i1}, a_{i2}, \cdots, a_{i(\frac{n}{m}-1)}\}$ can be defined, where $a_{ij} \in Hg_j$, $g_j \in G$, $(0 \leq j \leq (n/m) - 1)$ and $i = 1, 2, \cdots, m^{\frac{n}{m}-1}$. We call ${}_HM_k$ and ${}_HM_s$ are equivalent if they have analogues elements of equal order in ${}_HM_i$ for $i = k, s$ and all $g_j \in G$. We denote it by ${}_HM_k \sim {}_HM_s$.

Theorem 3.3. *If M_1 and M_2 are ultra-groups of H over G such that $M_1 \sim M_2$, then M_1 and M_2 are isomorphic.*

Proof. According to the argument before the theorem consider two equivalent ultra-groups $M_1 = \{a_{10}, a_{11}, a_{12}, \cdots, a_{1k}\}$ and $M_2 = \{a_{20}, a_{21}, a_{22}, \cdots, a_{2k}\}$, where $k = \frac{n}{m} - 1$, $a_{10} = a_{20} = e$. We define the map $f : M_1 \to M_2$ by $f(a_{1j}) = a_{2j}$. As $Hg_1Hg_2 = Hg_1g_2$, the multiplication of two elements in Hg_1 and Hg_2 belongs to Hg_1g_2. Therefore, by the fact that elements of ultra-groups are chosen from distinct cosets and their intersection is the empty set, we conclude that f is an ultra-group isomorphism. $\square$

Theorem 3.4. *Let H_1, H_2 be two subgroups of G such that $H_2 = H_1^g$. Then ultra-group ${}_{H_1}M_1^g$ is an ultra-group of H_2 over G.*

Proof. We denote ${}_{H_1}M_1$ by M_1. Since $G = gH_1M_1g^{-1} = H_2M_1^g$, we conclude that M_1^g is a complementary set of H_2 on G. Now we prove that every element of M_1^g is chosen uniquely from each coset of H_2. Suppose contrary $m_2, m_2' \in H_2a \cap M_1^g$, where $a \in G$. There are $m_1, m_1' \in M_1$ and $h_1, h_1' \in H_1$ such that $m_1^g = h_1^g a$ and $m_1'^g = h_1'^g a$. Hence $m_1, m_1' \in Ha^g$ which is a contradiction. $\square$

Finally we present an interesting example of a group such that its ultra-groups are classified up to ultra-groups isomorphism. Suppose $D_n = \langle a, b : a^n = b^2 = 1, a^b = a^{-1}\rangle$ is the dihedral group of order $2n$. All subgroups of D_n are of the type $H_1 = \langle a^d\rangle$ and $H_2 = \langle a^d, a^{d'}b\rangle$, where $d \mid n$ and $0 \leq d' \leq d - 1$. For the subgroup H_1, the set $H_1 \setminus G$ can be partitioned to two sets $K_1 = \{H_1a^i : 0 \leq i \leq d - 1\}$ and $K_2 = \{H_1a^ib : 0 \leq i \leq d - 1\}$. Every element of H_1a^i is of order d and every element of H_1a^ib is of order 2, $1 \leq i \leq d - 1$. By the argument before the Theorem 3.3 we have $(n/d)^{2d-1}$ ultra-groups of H_1 over G. Fortunately all these ultra-groups are isomorphic to $\{e, a, a^2, \cdots, a^{d-1}, b, ab, \cdots, a^{d-1}b\}$ in the similar way we mentioned in Theorem 3.3. Furthermore, for H_2 the set $H_2 \setminus G$ is $\{H_2a^i : 0 \leq i \leq d - 1\}$. For each H_2a^i we have two possible for order of elements : order d', where $d' \mid d$ and order 2. Therefore, for choosing each element from the coset H_2a^i we have two cases. This means there are 2^{d-1} equivalence classes for ultra-groups. The classifying of these subultra-groups is according to the conjugate subgroups of D_n (see Theorem 3.4 for more details).

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