FIXED POINTS OF MIZOGUCHI-TAKAHASHI'S TYPE CONTRACTION ON METRIC SPACES WITH A GRAPH

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In this paper, we introduce the notion of Mizoguchi-Takahashi's type G-contraction and using this new notion we prove a fixed point theorem to generalize a recent fixed point theorem by Asrifa and Ventrivel. Some examples are also been constructed to demonstrate generality of our result. We also obtain Mizoguchi-Takahashi's type, and Hicks and Rhoades type fixed point theorems on ε-chainable metric space.

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1. Preliminaries and Introduction

Let (X,d) be a metric space. For each $x \in X$ and $A \subseteq X$, $d(x,A) = \inf\{d(x,y) : y \in A\}$. We denote by $K(X)$ the class of all nonempty compact subset of X, by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $CL(X)$ the class of all nonempty closed subsets of X. For every $A, B \in CL(X)$, let

$$H(A,B) = \begin{cases} \max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\}, & \textit{if the maximum exists;} \\ \infty, & \textit{otherwise.} \end{cases}$$

Such a map H is called generalized Hausdorff metric induced by d. A metric space is said to be ε-chainable if for each $x, y \in X$, there exists $N \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^{N}$ in X such that $x = x_0$, $x_N = y$ and $d(x_i, x_{i+1}) < \varepsilon \ \forall\, i = 0,1,2,\cdots,N-1$. A mapping $f : X \to \mathbb{R}$ is lower semi continuous, if for any sequence $\{x_n\}$ in X such that $x_n \to \xi$, then

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$f(\xi) \le \liminf f(x_n)$. Consider a directed graph G such that the set of its vertices $V(G)$ coincides with X (i.e., $V(G) = X$) and the set of its edges $E(G)$ is such that $E(G) \supseteq \Delta$, where $\Delta = \{(x,x) : x \in X\}$. Let us also assume that G has no parallel edges. We can identify G with the pair $(V(G), E(G))$.

The roots of using method of successive approximation to solve initial value problems goes back to the work of pioneer mathematicians such as Cauchy, Liouville and Picard. But it was Banach who first laid these ideas into concrete frame work. Banach showed that every contraction mapping on a complete metric space has a unique fixed point. Moreover, starting from any point of the metric spaces the successive approximation of the mapping always converges to the fixed point of the mapping. Since many nonlinear problems can be converted into the problem of finding fixed point of some appropriate operator in some suitable complete metric space, the Banach fixed point theorem becomes a very power tool for solving nonlinear problems. Many authors generalized Banach fixed point theorems in different directions [1-15]. Nadler [1] extended Banach contraction principle to multi-valued mappings by generalizing the definition of contraction from single-valued to multi-valued mappings as:

Theorem 1.1 *[1] Let (X,d) be a complete metric space and T is a mapping from X into $CB(X)$ such that*

$$H(Tx,Ty) \le \alpha d(x,y), \quad \textit{for each } x, y \in X, \tag{1}$$

where $\alpha \in [0,1)$. Then T has a fixed point.

Reich [15] generalized the above result by replacing inequality (1) with the following inequality.

$$H(Tx,Ty) \le \alpha(d(x,y))d(x,y), \quad \textit{for each } x, y \in X,$$

where α is a function of $(0,\infty)$ into $[0,1)$ such that $\limsup_{r \to t^+} \alpha(r) < 1$, for each $t \in (0,\infty)$. But the range of the mapping T was $K(X)$ instead of $CB(X)$. Reich [15] posed the question that whether the range $K(X)$, of T, can be replaced by $CB(X)$ or $CL(X)$. Mizoguchi and Takahashi [16] gave the positive answer to the conjecture of Reich [15] provided that the domain of the mapping α contains 0 as well. Daffer and Kaneko [17] and Chang [18] also reached at the same conclusion, independently. Meanwhile, Eldred et al. [19] claimed that Mizoguchi and Takahashi [16] fixed point theorem is equivalent to Nadler's fixed point Theorem [1] . However, Suzuki disproved their claim by producing a countar example [20, p. 753] and reaffirmed that Mizoguchi-Takahashi's fixed point theorem indeed generalize Nadler's fixed point theorem. Kamran [21] generalized Mizoguchi and Takahashi's fixed point theorem in the following way.

Theorem 1.2 *[21] Let (X,d) be a complete metric space and $T:X\to CL(X)$ is a mapping satisfying*

$$d(y,Ty)\le \alpha(d(x,y))d(x,y), \quad \textit{for each } x\in X \textit{ and } y\in Tx,$$

where $\alpha:[0,\infty)\to[0,1)$ is such that $\limsup_{r\to t^+}\alpha(r)<1$, for each $t\in[0,\infty)$. Then,

(i) for each $x_0\in X$, there exists an orbit $\{x_n\}$ of T and $\xi\in X$ such that $\lim_n x_n=\xi$;

(ii) ξ is a fixed point of T if and only if the function $h(x):=d(x,Tx)$ is T-orbitally lower semi continuous at ξ.

On the other hand, Jachymaski [4] generalized Banach fixed point theorem by showing that if a mapping satisfies contractive condition on those points of elements from a complete metric space that are end points of an edge in a Graph in the metric space whose vertex set coincide with the metric space then the mapping has a fixed point. He called such mappings as Banach G-contraction. Afterwards, many authors extended Banach G-contraction in different ways, see for examples, [4, 22-33]. Recently Asrifa and Ventrivel [22] introduced the notion of Mizoguchi and Takahashi G-contraction and generalized Mizoguchi and Takahashi fixed point theorem.

Definition 1.3 *[22] A mapping $T:X\to CB(X)$ is said to be Mizoguchi-Takahashi G-contraction if for each $x,y\in X$, $x\neq y$ with $(x,y)\in E(G)$, we have*

(i) $H(Tx,Ty)\le\alpha(d(x,y))d(x,y)$, where $\alpha:(0,\infty)\to[0,1)$ is such that $\limsup_{r\to t^+}\alpha(r)<1$, for each $t\in[0,\infty)$;

(ii) if $u\in Tx$ and $v\in Ty$ is such that $d(u,v)\le d(x,y)$, then $(u,v)\in E(G)$.

In this paper, we introduce the notion of Mizoguchi-Takahashi's type G-contraction to generalize Definition 1.3 and using this notion we generalize Theorem 1.2, analogous to the fixed point theorem by Asrifa and Ventrivel [22]. We also obtain some fixed point results for ε-chainable metric spaces.

We use following lemma in our main result.

Lemma 1.4 *[21] Let (X,d) be a metric space and $B\in CL(X)$. Then for each $x\in X$ and $q>1$, we have $b\in B$ such that $d(x,b)\le qd(x,B)$.*

2. Main results

Throughout this section, (X,d) is a metric space, G is a directed graph such that $V(G)=X$, $\Delta=\{(x,x):x\in X\}\subseteq E(G)$ and G has no parallel edges. We start this section with the definition of Mizoguchi-Takahashi's type G-contraction.

Definition 2.1 *A mapping $T:X\to CL(X)$ is said to be Mizoguchi-Takahashi's type G-contraction if for each $x\in X$ and $y\in Tx$, with $x\neq y$ and there is an edge between x and y, we have*

(i) $d(y,Ty)\leq \alpha(d(x,y))d(x,y)$, where $\alpha:(0,\infty)\to[0,1)$ is such that $\limsup_{r\to t^+}\alpha(r)<1$, for each $t\in[0,\infty)$;

(ii) if $z\in Ty$ is such that the distance between y and z is less then the distance between x and y, then there is an edge between y and z.

Remark 2.2 *Note that Definition 2.1 is general than Definition 1.3 in two aspects. Since for $y\in Tx$, $d(y,Ty)\leq H(Tx,Ty)$, therefore inequality (i) of Definition 2.1 properly contains the corresponding inequality (i) of Definition 1.3. Moreover, condition (ii) of Definition 1.3 says that $d(u,v)\leq d(x,y)$, then $(u,v)\in E(G)$. But symmetry of d also implies that $(v,u)\in E(G)$ as well. On the other hand condition (ii) of Definition 2.1 says if $z\in Ty$ and $y\in Tx$ is such that the distance between y and z is less then the distance between x and y, we have only one of these, $(y,z)\in E(G)$ or $(z,y)\in E(G)$.*

Following example supports our claim.

Example 2.3 *Let $X=[-100,\infty)$ be endowed with the usual metric space and $G_1=(X,E(X))$ where $E(X)=\{(x,y):x,y\in[0,\frac{3}{5}]\text{ with } x\leq y\}\cup\{(x,y):x=y\}$. Define $T:X\to CL(X)$ by*

$$Tx=\begin{cases}[-100,x] & \text{if } x\leq 0\\ [0,x^2] & \text{if } x>0,\end{cases}$$

and $\alpha:(0,\infty)\to[0,1)$ by $\alpha(t)=\frac{24}{25}$ for each $t\geq 0$. It is easy to see that for each $x\in X$ and $y\in Tx$, with $x\neq y$ and $(y,x)\in E(G_1)$, we have

$$d(y,Ty)\leq |x^2-y^2|=|x+y||x-y|\leq\frac{24}{25}|x-y|\leq\alpha(d(x,y))d(x,y),$$

and for each $z\in Ty$ with the distance between y and z is less then the distance between x and y, we have $(z,y)\in E(G)$, since $0\leq z\leq y\leq x$.

Remark 2.4 *Note that T is Mizoguchi-Takahashi's type G_1-contraction but not Mizoguchi-Takahashi G_1-contraction. To see that*

1. for $x = 0$ and $y = \frac{3}{5}$, we have $(x, y) \in E(G_1)$, but $H(Tx, Ty) = 100$, which is not less than $\alpha(d(x, y))d(x, y)$;

2. Furthermore, we have $u = \frac{-1}{2} \in Tx$ and $v = 0 \in Ty$ such that $d(u,v) < d(x, y)$ but $(u,v) \notin E(G_1)$ and $(v,u) \notin E(G_1)$.

Theorem 2.5 *Let (X, d) be a complete metric space endowed with the graph G and let $T : X \to CL(X)$ be a Mizoguchi-Takahashi's type G-contraction. Assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ with an edge between x_0 and x_1. Then, T has a fixed point in X, provided $f(x) = d(x,Tx)$ is lower semi continuous.*

Proof. By hypothesis of theorem, we have $x_0 \in X$ and $x_1 \in Tx_0$ with an edge between x_0 and x_1. If $x_0 = x_1$, then x_0 is a fixed point of T. If $\alpha(d(x_0, x_1)) = 0$, then by Definition 2.1-(i), we have $x_1 \in Tx_1$, that is, x_1 is a fixed point of T. Thus, we assume that $x_0 \neq x_1$ and $\alpha(d(x_0, x_1)) > 0$. Now by taking $q = \frac{1}{\sqrt{\alpha(d(x_0, x_1))}} > 1$, it follows by Lemma 1.4, that there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \le \frac{1}{\sqrt{\alpha(d(x_0, x_1))}} d(x_1, Tx_1).$$

By using definition of Mizoguchi-Takahashi's type G-contraction, we have

$$d(x_1, x_2) \le \frac{1}{\sqrt{\alpha(d(x_0, x_1))}} d(x_1, Tx_1) \le \sqrt{\alpha(d(x_0, x_1))} d(x_0, x_1) < d(x_0, x_1).$$

Thus without loss of generality we take $(x_1, x_2) \in E(G)$. Repeating the same argument we obtain a sequence $\{x_n\}_{n\ge2}$ in X such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$, $(x_{n-1}, x_n) \in E(G)$ and

$$\begin{aligned} d(x_n, x_{n+1}) &\le \frac{1}{\sqrt{\alpha(d(x_{n-1}, x_n))}} d(x_n, Tx_n) \\ &\le \sqrt{\alpha(d(x_{n-1}, x_n))} d(x_{n-1}, x_n) \qquad (2) \\ &< d(x_{n-1}, x_n), \end{aligned}$$

for each $n \in \mathbb{N}$. Hence $\{d(x_n, x_{n+1})\}$ is a decreasing sequence, it converges to some nonnegative real number b. We claim that $b = 0$, for otherwise, by taking limit in (2), we get

$$b \leq \sqrt{\lim_{n\to\infty} \alpha(d(x_{n-1}, x_n))}\, b < b.$$

Which is a contradiction to our assumption. From (2), we get

$$d(x_n, x_{n+1}) \leq (\sqrt{\alpha(d(x_{n-1}, x_n))} \cdots \sqrt{\alpha(d(x_0, x_1))}\,)\, d(x_0, x_1). \tag{3}$$

By the definition of α, it follows that we may choose $\varepsilon > 0$ and $a \in (0,1)$ such that

$$\alpha(t) < a^2 \quad for\, each\, t \in (0, \varepsilon).$$

Let N be such that

$$d(x_{n-1}, x_n) < \varepsilon \quad for\, each\, n \geq N.$$

From (3), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq a^{n-(N-1)} (\sqrt{\alpha(d(x_{N-2}, x_{N-1}))} \cdots \sqrt{\alpha(d(x_0, x_1))}\,)\, d(x_0, x_1) \\ &< a^{n-(N-1)} d(x_0, x_1). \end{aligned} \tag{4}$$

Therefore for any $m \in N$, we have

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + \ldots + d(x_{n+m-1}, x_{n+m}); \\ &\leq a^{n-(N-1)} (1 + a + a^2 + \ldots + a^{m-1}) d(x_0, x_1); \\ &\leq \frac{a^{n-(N-1)}}{1-a} d(x_0, x_1). \end{aligned}$$

Which shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $\xi \in X$ such that $x_n \to \xi$ as $n \to \infty$. As $x_{n+1} \in Tx_n$ and $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, by definition of Mizoguchi-Takahashi's type G-contraction, we have

$$\begin{aligned} d(x_n, Tx_n) &\leq \alpha(d(x_{n-1}, x_n)) d(x_{n-1}, x_n); \\ &< d(x_{n-1}, x_n). \end{aligned} \tag{5}$$

Letting $n \to \infty$ in (5), we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Since $f(x) = d(x, Tx)$ is lower semi continuous, then

$$d(\xi, T\xi) = f(\xi) \leq \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

Hence by the closedness of T, we have $\xi \in T\xi$.

Example 2.6 *Let* $X = \mathbb{R}$ *be endowed with the usual metric space and*

$G_2 = (\mathbb{R}, E(\mathbb{R}))$, *where* $E(\mathbb{R}) = \{(x, y) : x, y \in [0, \frac{3}{5}]$ *with* $x \geq y\} \cup \{(x, y) : x = y\}$. *Define* $T : X \to CL(X)$ *by*

$$Tx = \begin{cases} (-\infty, x] & if \ x \leq 0 \\ [0, x^2] & if \ x > 0, \end{cases}$$

and $\alpha : (0, \infty) \to [0,1)$ by $\alpha(t) = \frac{24}{25}$ for each $t \geq 0$. It is easy to see that T is Mizoguchi-Takahashi's type G_2 -contraction. Further, we have $x_0 = \frac{3}{5}$ and $x_1 = \frac{9}{25} \in Tx_0$ such that $(x_0, x_1) \in E(G_2)$. Therefore by Theorem 2.5, T has a fixed point in X.

Remark 2.7 *Note that the mappings in Example (2.2) and Example (2.5) are not Mizoguchi-Takahashi* G *-contractions. Therefore [22, Theorem 3] is not applicable here.*

Following corollary follows immediately form Theorem 2.5.

Corollary 2.8 *Let* (X, d) *be a complete metric space endowed with the graph* G *and let* $T : X \to X$ *be a mapping such that for each* $x \in X$ *with an edge between* x *and* Tx, *we have*

(i) $d(Tx, T^2x) \leq \alpha(d(x, Tx))d(x, Tx)$, where $\alpha : (0, \infty) \to [0,1)$ is such that $\limsup_{r \to t^+} \alpha(r) < 1$, for each $t \in [0, \infty)$;

(ii) if the distance between Tx and T^2x is less then the distance between x and Tx, then there is an edge between Tx and T^2x.

Assume that there exists $x_0 \in X$ with an edge between x_0 and Tx_0. Then, T has a fixed point in X, provided $f(x) = d(x, Tx)$ is lower semi continuous.

3. Consequences

In this section we apply our results to prove fixed point theorems for ε -chainable metric spaces.

Theorem 3.1 *Let* (X, d) *be a complete* ε *-chainable metric space and let* $T : X \to CL(X)$ *be a mapping such that for each* $x \in X$ *and* $y \in Tx$ *with* $d(x, y) < \varepsilon$, *we have*

$$d(y, Ty) \leq \alpha(d(x, y))d(x, y),$$

where $\alpha:(0,\infty)\to[0,1)$ is such that $\limsup_{r\to t^+}\alpha(r)<1$, for each $t\in[0,\infty)$. Assume that there exist $x_0\in X$ and $x_1\in Tx_0$ with $d(x_0,x_1)<\varepsilon$. Then, T has a fixed point in X, provided $f(x)=d(x,Tx)$ is lower semi continuous.

Proof. Consider the graph $G=(X,E(X))$, where

$$E(X)=\{(x,y)\in X\times X : d(x,y)<\varepsilon\}.$$

Then clear we have $x_0\in X$ and $x_1\in Tx_0$ with $(x_0,x_1)\in E(G)$ and T is Mizoguchi-Takahashi's type G-contraction and the conclusion follows from Theorem 2.5.

Following theorem follows immediately above theorem.

Theorem 3.2 *Let (X,d) be a complete ε-chainable metric space and let $T:X\to X$ be a mapping such that for each $x\in X$ with $d(x,Tx)<\varepsilon$, we have*

$$d(Tx,T^2x)\le\alpha(d(x,Tx))d(x,Tx),$$

where $\alpha:(0,\infty)\to[0,1)$ is such that $\limsup_{r\to t^+}\alpha(r)<1$, for each $t\in[0,\infty)$. Assume that there exists $x_0\in X$ with $d(x_0,Tx_0)<\varepsilon$. Then, T has a fixed point in X, provided $f(x)=d(x,Tx)$ is lower semi continuous.

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