MORE RESULTS ON VAGUE GRAPHS

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The main purpose of this paper is to show the rationality of some operations, defined or to be defined, on vague graphs. Three kinds of new product operations (called direct product, lexicographic product, and strong product) of vague graphs are defined, and rationality of these notions and some defined important notions on vague graphs, such as vague graph, vague complete graph, cartesian product of vague graphs and union of vague graphs are demonstrated by characterizing these notions by their level counterparts graphs.

Keywords: Rationality, vague graphs, direct product, vague complete graphs.
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1. Introduction

The major role of graph theory in computer applications is the development of graph algorithms. A number of algorithms are used to solve problems that are modeled in the form of graphs. These algorithms are used to solve the corresponding computer science application problems. In 1965, Zadeh [26] first proposed the theory of fuzzy sets. The most important feature of a fuzzy set is that a fuzzy set A is a class of objects that satisfy a certain (or several) property. Gau and Buehrer [9] proposed the concept of vague set in 1993, by replacing the value of an element in a set with a subinterval of $[0, 1]$. Namely, a true-membership function $t_v(x)$ and a false-membership function $f_v(x)$ are used to describe the boundaries of the membership degree. These two boundaries form a subinterval $[t_v(x), 1 - f_v(x)]$ of $[0, 1]$. The vague set theory improves the description of the objective real world becoming a promising tool to deal with inexact, uncertain or vague knowledge.

The initial definition given by Kaufmann [12] of a fuzzy graph was based on the fuzzy relation proposed by Zadeh [26]. Later Rosenfeld [15] introduced the fuzzy analogue of several basic graph-theoretic concepts. Mordeson and Nair [13] defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. Akram et al. [1, 2, 3, 4, 5, 6] introduced bipolar fuzzy graphs, interval-valued fuzzy line graphs, strong intuitionistic fuzzy

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graphs, certain types of vague graphs, regularity in vague intersection graphs and vague line graphs, and vague hypergraphs. Talebi and Rashmanlou investigated isomorphism on vague graphs [22]. Ramakrishna [19] introduced the concept of vague graphs and studied some of their properties. Pal and Rashmanlou [14] studied irregular interval-valued fuzzy graphs. Likewise, they defined antipodal interval valued fuzzy graphs [16], balanced interval-valued fuzzy graphs [17]. Rashmanlou and Jun [18] introduced complete interval-valued fuzzy graphs. In this paper, we defined three kinds of new product operations (called direct product, lexicographic product and strong product) of vague graphs and the rationality of these notions and some defined important notions on vague graphs are demonstrated.

2. Preliminaries

In this section, we define three kinds of new product operations (called direct product, lexicographic product, and strong product) of vague graphs and show that direct product, lexicographic product and strong product of two vague graphs is a vague graph also.

Definition 2.1. [26, 27] *A fuzzy subset μ on a set X is a map $\mu : X \to [0,1]$. A fuzzy binary relation on X is a fuzzy subset μ on $X \times X$.*

Definition 2.2. [9] *A vague set on an ordinary finite non-empty set X is a pair (t_A, f_A), where $t_A : X \to [0,1]$, $f_A : X \to [0,1]$ are true and false membership functions, respectively such that $0 \leq t_A(x) + f_A(x) \leq 1$, for all $x \in X$.*

In the above definition, $t_A(x)$ is considered as the lower bound for degree of membership of x in A (based on evidence), and $f_A(x)$ is the lower bound for negation of membership of x in A (based on evidence against). So, the degree of membership of x in the vague set A is characterized by the interval $[t_A(x), 1-f_A(x)]$. Therefore, a vague set is a special case of interval valued sets studied by many mathematicians and applied in many branches of mathematics (see for example [21, 23]*). The interval $[t_A(x), 1-f_A(x)]$ is called the vague value of x in A, and is denoted by $V_A(x)$. We denote zero vague and unit vague value by $0 = [0,0]$ and $1 = [1,1]$, respectively.*

Definition 2.3. *Let X and Y be ordinary finite non-empty sets. We call a vague relation to be a vague subset of $X \times Y$, that is, an expression R defined by:*

$$R = \{\langle (x,y), t_R(x,y), f_R(x,y)\rangle \mid x \in X, y \in Y\}$$

where $t_R : X \times Y \to [0,1]$, $f_R : X \times Y \to [0,1]$, which satisfies the condition $0 \leq t_R(x,y) + f_R(x,y) \leq 1$, for all $(x,y) \in X \times Y$. A vague relation R on X is called reflexive if $t_R(x,x) = 1$ and $f_R(x,x) = 0$, for all $x \in X$. A vague relation R is symmetric if $t_R(x,y) = t_R(y,x)$ and $f_R(x,y) = f_R(y,x)$, for all $x, y \in X$.

A vague set, as well as an intuitionistic fuzzy set [5]*, is a further generalization of a fuzzy set. In the literature, the notions of intuitionistic fuzzy sets*

and vague sets are regarded as equivalent, in the sense that an intuitionistic fuzzy set is isomorphic to a vague set [6].
Throughout this paper, G^* *will be a crisp graph* (V, E)*, and* G *a vague graph* (A, B)*. Since an edge* $xy \in E$ *is identified with an ordered pair* $(x, y) \in V \times V$*, a vague relation on* E *can be identified with a vague set on* E*. This gives a possibility to define a vague graph as a pair of vague sets.*

Definition 2.4. [19] *Let* $G^* = (V, E)$ *be a crisp graph. A pair* $G = (A, B)$ *is called a vague graph on a crisp graph* $G^* = (V, E)$*, where* $A = (t_A, f_A)$ *is a vague set on* V *and* $B = (t_B, f_B)$ *is a vague set on* $E \subseteq V \times V$ *such that*

$$t_B(xy) \leq \min(t_A(x), t_A(y)) \text{ and } f_B(xy) \geq \max(f_A(x), f_A(y))$$

for each edge $xy \in E$.

Definition 2.5. [19] *A vague graph* G *is called complete if*

$$t_B(xy) = \min(t_A(x), t_A(y)) \text{ and } f_B(xy) = \max(f_A(x), f_A(y))$$

for each edge $xy \in E$.

Example 2.1. *Consider a vague graph* G *such that* $V = \{x, y, z\}$, $E = \{xy, yz, xz\}$.

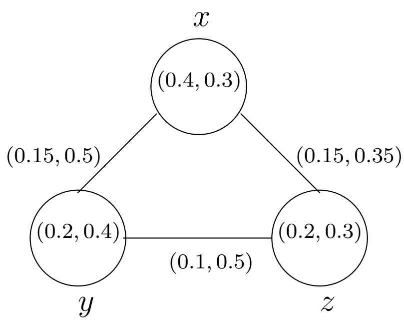

Figure 1: Vague graph G

By routine computations, it is easy to show that G *is a vague graph.*

Definition 2.6. *A homomorphism* $h : G_1 \to G_2$ *is a mapping* $h : V_1 \to V_2$ *which satisfies the following conditions:*
(a) $t_{A_1}(x_1) \leq t_{A_2}(h(x_1))$, $f_{A_1}(x_1) \geq f_{A_2}(h(x_1))$,
(b) $t_{B_1}(x_1y_1) \leq t_{B_2}(h(x_1)h(y_1))$, $f_{B_1}(x_1y_1) \geq f_{B_2}(h(x_1)h(y_1))$,
for all $x_1 \in V_1$, $x_1y_1 \in E_1$.

Definition 2.7. *Let* G_1 *and* G_2 *be vague graphs. An isomorphism* $h : G_1 \to G_2$ *is a bijective mapping* $h : V_1 \to V_2$ *which satisfies the following conditions:*
(c) $t_{A_1}(x_1) = t_{A_2}(h(x_1))$, $f_{A_1}(x_1) = f_{A_2}(h(x_1))$,

(d) $t_{B_1}(x_1y_1) = t_{B_2}(h(x_1)h(y_1))$, $f_{B_1}(x_1y_1) = f_{B_2}(h(x_1)h(y_1))$,
for all $x_1 \in V_1$, $x_1y_1 \in E_1$.

Definition 2.8. *Let G_1 and G_2 be vague graphs. A weak isomorphism $h : G_1 \to G_2$ is a bijective mapping $h : V_1 \to V_2$ which satisfies the following conditions:*
(e) h is homomorphism,
(f) $t_{A_1}(x_1) = t_{A_2}(h(x_1))$, $f_{A_1}(x_1) = f_{A_2}(h(x_1))$,
for all $x_1 \in V_1$. Thus, a weak isomorphism preserves the weights of the nodes but not necessarily the weights of the arcs.

Definition 2.9. *Let G_1 and G_2 be vague graphs. A co weak isomorphism $h : G_1 \to G_2$ is a bijective mapping $h : V_1 \to V_2$ which satisfies:*
(g) h is homomorphism,
(h) $t_{B_1}(x_1y_1) = t_{B_2}(h(x_1)h(y_1))$, $f_{B_1}(x_1y_1) = f_{B_2}(h(x_1)h(y_1))$,
for all $x_1y_1 \in E_1$. Thus a co-weak isomorphism preserves the weights of the arcs but not necessarily the weights of the nodes.

Definition 2.10. *A vague graph G is called strong if*

$$t_B(xy) = \min(t_A(x), t_A(y)) \text{ and } f_B(xy) = \max(f_A(x), f_A(y))$$

, for all $xy \in V$.

Definition 2.11. *Let $A : X \to \Pi$ be a vague set on X where $\Pi = \{[b, c] \mid 0 \leq b \leq c \leq 1\}$ (i.e., the set of all closed intervals in $[0, 1]$), then $A_{[b,c]} = \{x \in X \mid t_A(x) \geq b, f_A(x) \geq c\}$ is called a $[b, c]-$level set of A, for all $[b, c] \in \Pi$.*

Definition 2.12. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs.*
(1) ([20, 21]) The graph $G_1^* \times G_2^* = (V, E)$ is called the cartesian product of G_1^* and G_2^* where $V = V_1 \times V_2$ and

$$E = \{(x, x_2)(x, y_2) \mid x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \mid z \in V_2, x_1y_1 \in E_1\}.$$

(2) ([24]) The graph $G_1^* * G_2^* = (V, E)$ is called the direct product of G_1^* and G_2^*, where $V = V_1 \times V_2$ and

$$E = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1, x_2y_2 \in E_2\}.$$

(3) ([11]) The graph $G_1^* \bullet G_2^* = (V, E)$ is called the lexicographic product of G_1^* and G_2^* where $V = V_1 \times V_2$ and

$$E = \{(x, x_2)(x, y_2) \mid x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1, x_2y_2 \in E_2\}.$$

(4) ([21]) The graph $G_1^* \boxtimes G_2^* = (V, E)$ is called the strong product of G_1^* and G_2^*, where $V = V_1 \times V_2$ and

$$E = \{(x, x_2)(x, y_2) \mid x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \mid z \in V_2, x_1y_1 \in E_1\}$$

$$\cup \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1, x_2y_2 \in E_2\}.$$

(5) ([10]) The graph $G_1^* \cup G_2^* = (V, E)$ *is called the union of* G_1^* *and* G_2^**, where* $V = V_1 \cup V_2$ *and* $E = E_1 \cup E_2$*.*

Definition 2.13. *Let* $G_1 = (A_1, B_1)$ *(resp.,* $G_2 = (A_2, B_2)$*) be a vague graph of* $G_1^* = (V_1, E_1)$ *(resp.,* $G_2^* = (V_2, E_2)$*).*
(1) The cartesian product $G_1 \times G_2$ *of* G_1 *and* G_2 *is defined as a pair* (A, B)*, where* $A = (t_A, f_A)$ *and* $B = (t_B, f_B)$ *are vague sets on* $V = V_1 \times V_2$ *and*

$$E = \{(x, x_2)(x, y_2) \mid x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \mid z \in V_2, x_1y_2 \in E_1\}$$

respectively which satisfies the following:

$$(i) \begin{cases} t_A(x_1, x_2) = \min(t_{A_1}(x_1), t_{A_2}(x_2)) \\ f_A(x_1, x_2) = \max(f_{A_1}(x_1), f_{A_2}(x_2)) \end{cases} ((x_1, x_2) \in V_1 \times V_2),$$

$$(ii) \begin{cases} t_B((x, x_2)(x, y_2)) = \min(t_{A_1}(x), t_{B_2}(x_2y_2)) \\ f_B((x, x_2)(x, y_2)) = \max(f_{A_1}(x), f_{B_2}(x_2y_2)) \end{cases} (x \in V_1, x_2y_2 \in E_2),$$

$$(iii) \begin{cases} t_B((x_1, z)(y_1, z)) = \min(t_{B_1}(x_1y_1), t_{A_2}(z)) \\ f_B((x_1, z)(y_1, z)) = \max(f_{B_1}(x_1y_1), f_{A_2}(z)) \end{cases} (z \in V_2, x_1y_1 \in E_1).$$

(2) The union $G_1 \cup G_2$ *of* G_1 *and* G_2 *is defined as a pair* (A, B)*, where* $A = (t_A, f_A)$ *and* $B = (t_B, f_B)$ *are vague sets on* $V = V_1 \cup V_2$ *and* $E = E_1 \cup E_2$ *respectively which satisfies the following:*

$$(i) \begin{cases} t_A(x) = t_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ t_A(x) = t_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ t_A(x) = \max(t_{A_1}(x), t_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$$

$$(ii) \begin{cases} f_A(x) = f_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2, \\ f_A(x) = f_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1, \\ f_A(x) = \min(f_{A_1}(x), f_{A_2}(x)) & \text{if } x \in V_1 \cap V_2. \end{cases}$$

$$(iii) \begin{cases} t_B(xy) = t_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ t_B(xy) = t_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ t_B(xy) = \max(t_{B_1}(xy), t_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

$$(iv) \begin{cases} f_B(xy) = f_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2, \\ f_B(xy) = f_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1, \\ f_B(xy) = \min(f_{B_1}(xy), f_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

Finally, we define three kinds of new operations (called direct product, lexicographic product, and strong product) on vague graphs, which can be looked as a generalization of their counterparts in Definition 2.12.

Definition 2.14. *The direct product* $G_1 * G_2$ *of two vague graphs* $G_1 = (A_1, B_1)$ *and* $G_2 = (A_2, B_2)$ *of* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$ *respectively is defined as a pair* (A, B)*, where* $A = (t_A, f_A)$ *and* $B = (t_B, f_B)$ *are vague sets on* $V = V_1 \times V_2$ *and* $E = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1, x_2y_2 \in E_2\}$ *respectively which satisfies the following:*

$$(i) \begin{cases} t_A(x_1, x_2) = \min(t_{A_1}(x_1), t_{A_2}(x_2)) \\ f_A(x_1, x_2) = \max(f_{A_1}(x_1), f_{A_2}(x_2)) \end{cases} ((x_1, x_2) \in V_1 \times V_2)$$

$$(ii)\ \begin{cases} t_B((x_1,x_2)(y_1,y_2)) = \min(t_{B_1}(x_1y_1), t_{B_2}(x_2y_2)) \\ f_B((x_1,x_2)(y_1,y_2)) = \max(f_{B_1}(x_1y_1), f_{B_2}(x_2y_2)) \end{cases} \quad (x_1y_1 \in E_1, x_2y_2 \in E_2).$$

Theorem 2.1. *The direct product $G_1 * G_2$ of two vague graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ is a vague graph also.*

Proof. Let $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$, then we have

$$\begin{aligned}(t_{B_1} * t_{B_2})\left((x_1,x_2)(y_1,y_2)\right) &= \min\left(t_{B_1}(x_1y_1), t_{B_2}(x_2y_2)\right) \\ &\le \min\left(\min\left(t_{A_1}(x_1), t_{A_1}(y_1)\right), \min\left(t_{A_2}(x_2), t_{A_2}(y_2)\right)\right) \\ &= \min\left(\min\left(t_{A_1}(x_1), t_{A_2}(x_2)\right), \min\left(t_{A_1}(y_1), t_{A_2}(y_2)\right)\right) \\ &= \min\left((t_{A_1} * t_{A_2})(x_1,x_2), (t_{A_1} * t_{A_2})(y_1,y_2)\right)\end{aligned}$$

$$\begin{aligned}(f_{B_1} * f_{B_2})\left((x_1,x_2)(y_1,y_2)\right) &= \max\left(f_{B_1}(x_1y_1), f_{B_2}(x_2y_2)\right) \\ &\ge \max\left(\max\left(f_{A_1}(x_1), f_{A_1}(y_1)\right), \max\left(f_{A_2}(x_2), f_{A_2}(y_2)\right)\right) \\ &= \max(\max(f_{A_1}(x_1), f_{A_2}(x_2)), \max(f_{A_1}(y_1), f_{A_2}(y_2))) \\ &= \max((f_{A_1} * f_{A_2})(x_1,x_2), (f_{A_1} * f_{A_2})(y_1,y_2)).\end{aligned}$$

□

Definition 2.15. *The lexicographic product $G_1 \bullet G_2$ of two vague graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is defined as a pair (A, B), where $A = (t_A, f_A)$ and $B = (t_B, f_B)$ are vague sets on $V = V_1 \times V_2$ and $E = \{(x, x_2)(x, y_2) \mid x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1, x_2y_2 \in E_2\}$ respectively which satisfies the following:*

$$(i)\ \begin{cases} t_A(x_1,x_2) = \min(t_{A_1}(x_1), t_{A_2}(x_2)) \\ f_A(x_1,x_2) = \max(f_{A_1}(x_1), f_{A_2}(x_2)) \end{cases} \quad ((x_1,x_2) \in V_1 \times V_2),$$

$$(ii)\ \begin{cases} t_B((x,x_2)(x,y_2)) = \min(t_{A_1}(x), t_{B_2}(x_2y_2)) \\ f_B((x,x_2)(x,y_2)) = \max(f_{A_1}(x), f_{B_2}(x_2y_2)) \end{cases} \quad (x \in V_1, x_2y_2 \in E_2),$$

$$(iii)\ \begin{cases} t_B((x_1,x_2)(y_1,y_2)) = \min(t_{B_1}(x_1y_1), t_{B_2}(x_2y_2)) \\ f_B((x_1,x_2)(y_1,y_2)) = \max(f_{B_1}(x_1y_1), f_{B_2}(x_2y_2)) \end{cases} \quad (x_1y_1 \in E_1, x_2y_2 \in E_2).$$

Theorem 2.2. *The lexicographic product $G_1 \bullet G_2$ of two vague graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ is a vague graph also.*

Proof. If $x \in V_1$ and $x_2y_2 \in E_2$, we have

$$\begin{aligned}(t_{B_1} \bullet t_{B_2})\left((x,x_2)(x,y_2)\right) &= \min\left(t_{A_1}(x), t_{B_2}(x_2y_2)\right) \\ &\le \min\left((t_{A_1}(x), \min(t_{A_2}(x_2), t_{A_2}(y_2))\right) \\ &= \min\left(\min\left(t_{A_1}(x), t_{A_2}(x_2)\right), \min\left(t_{A_1}(x), t_{A_2}(y_2)\right)\right) \\ &= \min\left((t_{A_1} \bullet t_{A_2})(x,x_2), (t_{A_1} \bullet t_{A_2})(x,y_2)\right),\end{aligned}$$

$$\begin{aligned}(f_{B_1} \bullet f_{B_2})\left((x,x_2)(x,y_2)\right) &= \max\left(f_{A_1}(x), f_{B_2}(x_2y_2)\right)\\ &\geq \max\left(f_{A_1}(x), \max(f_{A_2}(x_2), f_{A_2}(y_2))\right)\\ &= \max\left(\max\left(f_{A_1}(x), f_{A_2}(x_2)\right), \max\left(f_{A_1}(x), f_{A_2}(y_2)\right)\right)\\ &= \max\left((f_{A_1} \bullet f_{A_2})(x,x_2), (f_{A_1} \bullet f_{A_2})(x,y_2)\right).\end{aligned}$$

If $x_1y_1 \in E$ and $x_2y_2 \in E_2$, then

$$\begin{aligned}(t_{B_1} \bullet t_{B_2})\left((x_1,x_2)(y_1,y_2)\right) &= \min\left(t_{B_1}(x_1y_1), t_{B_2}(x_2y_2)\right)\\ &\leq \min\left(\min(t_{A_1}(x_1), t_{A_1}(y_1)), \min(t_{A_2}(x_2), t_{A_2}(y_2))\right)\\ &= \min\left(\min\left(t_{A_1}(x_1), t_{A_2}(x_2)\right), \min\left(t_{A_1}(y_1), t_{A_2}(y_2)\right)\right)\\ &= \min\left((t_{A_1} \bullet t_{A_2})(x_1,x_2), (t_{A_1} \bullet t_{A_2})(y_1,y_2)\right),\end{aligned}$$

$$\begin{aligned}(f_{B_1} \bullet f_{B_2})\left((x_1,x_2)(y_1,y_2)\right) &= \max\left(f_{B_1}(x_1y_1), f_{B_2}(x_2y_2)\right)\\ &\geq \max\left(\max(f_{A_1}(x_1), f_{A_1}(y_1)), \max(f_{A_2}(x_2), f_{A_2}(y_2))\right)\\ &= \max\left(\max\left(f_{A_1}(x_1), f_{A_2}(x_2)\right), \max\left(f_{A_1}(y_1), f_{A_2}(y_2)\right)\right)\\ &\max\left((f_{A_1} \bullet f_{A_2})(x_1,x_2), (f_{A_1} \bullet f_{A_2})(y_1,y_2)\right).\end{aligned}$$

□

Definition 2.16. *The strong product $G_1 \boxtimes G_2$ of two vague graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively is defined as a pair (A, B), where $A = (t_A, f_A)$ and $B = (t_B, f_B)$ are vague sets on $V = V_1 \times V_2$ and*
$E = \{(x,x_2)(x,y_2) \mid x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1,z)(y_1,z) \mid z \in V_2, x_1y_1 \in E_1\} \cup \{(x_1,x_2)(y_1,y_2) \mid x_1y_1 \in E_1, x_2y_2 \in E_2\}$ respectively which satisfies the following:

(i) $\begin{cases} t_A(x_1,x_2) = \min(t_{A_1}(x_1), t_{A_2}(x_2)) \\ f_A(x_1,x_2) = \max(f_{A_1}(x_1), f_{A_2}(x_2)) \end{cases}$ $((x_1,x_2) \in V_1 \times V_2)$,

(ii) $\begin{cases} t_B((x,x_2)(x,y_2)) = \min(t_{A_1}(x), t_{B_2}(x_2y_2)) \\ f_B((x,x_2)(x,y_2)) = \max(f_{A_1}(x), f_{B_2}(x_2y_2)) \end{cases}$ $(x \in V_1, x_2y_2 \in E_2)$,

(iii) $\begin{cases} t_B((x_1,z)(y_1,z)) = \min(t_{B_1}(x_1y_1), t_{A_2}(z)) \\ f_B((x_1,z)(y_1,z)) = \max(f_{B_1}(x_1y_1), f_{A_2}(z)) \end{cases}$ $(z \in V_2, x_1y_1 \in E_1)$,

(iv) $\begin{cases} t_B((x_1,x_2)(y_1,y_2)) = \min(t_{B_1}(x_1y_1), t_{B_2}(x_2y_2)) \\ f_B((x_1,x_2)(y_1,y_2)) = \max(f_{B_1}(x_1y_1), f_{B_2}(x_2y_2)) \end{cases}$ $(x_1y_1 \in E_1, x_2y_2 \in E_2)$.

Theorem 2.3. *The strong product $G_1 \boxtimes G_2$ of two vague graphs $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ is a vague graph also.*

Proof. It is similar to Theorem (2.1) and Theorem (2.2). □

3. Rationality of some defined notions on vague graphs

In this section, we demonstrate the rationality of some notions (i.e. vague complete graph, cartesian product of vague graphs, direct product of vague graphs, lexicographic product of vague graphs, strong product of vague graphs and union of vague graphs) on vague graphs by characterizing them by their level counterpart graphs. As a result, we obtain a kind of representation of vague graphs (respectively, vague complete graphs). We firstly show the rationality of vague graphs and vague complete graphs.

Theorem 3.1. *Let V be a set, and $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be vague sets on V and $E \subseteq V \times V$ respectively. Then $G = (A, B)$ is a vague graph (respectively, vague complete graph) if and only if $(A_{[a,b]}, B_{[a,b]})$ (called $[a, b]-$level graph of $G = (A, B)$) is a graph (respectively, a complete graph) for each $[a, b] \in \Pi$.*

Proof. We only show the case of vague graph.
Necessity. Suppose that $G = (A, B)$ is a vague graph. For each $[a, b] \in \Pi$ and each $xy \in B_{[a,b]}$, we have $t_B(xy) \geq a$, $f_B(xy) \geq b$, $t_B(xy) \leq \min(t_A(x), t_A(y))$, and $f_B(xy) \geq \max(f_A(x), f_A(y))$ since $G = (A, B)$ is a vague graph. It follows that $t_A(x) \geq a$, $f_A(x) \geq b$, $t_A(y) \geq a$ and $f_A(y) \geq b$ which means $x, y \in A_{[a,b]}$. Therefore, $(A_{[a,b]}, B_{[a,b]})$ is a graph ($\forall\ [a, b] \in \Pi$).

Sufficiency. Assume that $(A_{[a,b]}, B_{[a,b]})$ is a graph ($\forall\ [a, b] \in \Pi$). For each $xy \in E$, Let $t_B(xy) = a$ and $f_B(xy) = b$, then $xy \in B_{[a,b]}$. Since $(A_{[a,b]}, B_{[a,b]})$ is a graph for each $[a, b] \in \Pi$, we have $x, y \in A_{[a,b]}$, which means $t_A(x) \geq a$ and $f_A(x) \geq b$, $t_A(y) \geq a$ and $f_A(y) \geq b$. Therefore, $t_B(xy) = a \leq \min(t_A(x), t_A(y))$, $f_B(xy) = b \geq \max(f_A(x), f_A(y))$, i.e., $G = (A, B)$ is a vague graph.

Next we show the rationality of the defined four kinds of product operations on vague graphs. □

Theorem 3.2. *Let $G_1 = (A_1, B_1)$ (respectively, $G_2 = (A_2, B_2)$) be a vague graph of $G_1^* = (V_1, E_1)$ (respectively, $G_2^* = (V_2, E_2)$). Then $G_1 \times G_2 = (A, B)$ is the cartesian product of G_1 and G_2 if and only if $(A_{[a,b]}, B_{[a,b]})$ is the cartesian product of $((A_1)_{[a,b]}, (B_1)_{[a,b]})$ and $((A_2)_{[a,b]}, (B_2)_{[a,b]})$ for each $[a, b] \in \Pi$.*

Proof. Necessity. Suppose that $G_1 \times G_2 = (A, B)$ is the cartesian product of G_1 and G_2. Firstly, we show $A_{[a,b]} = (A_1)_{[a,b]} \times (A_2)_{[a,b]}$ for each $[a, b] \in \Pi$. Actually, for every $x, y \in A_{[a,b]}$, we have $\min(t_{A_1}(x), t_{A_2}(y)) = t_A(x, y) \geq a$ and $\max(f_{A_1}(x), f_{A_2}(y)) = f_A(x, y) \geq b$ since (A, B) is the cartesian product of G_1 and G_2. It follows that $x \in (A_1)_{[a,b]}$ and $y \in (A_2)_{[a,b]}$ (i.e., $(x, y) \in (A_1)_{[a,b]} \times (A_2)_{[a,b]}$).
Therefore, $A_{[a,b]} \subseteq (A_1)_{[a,b]} \times (A_2)_{[a,b]}$. Conversely, for every $(x, y) \in (A_1)_{[a,b]} \times (A_2)_{[a,b]}$, we have $x \in (A_1)_{[a,b]}$ and $y \in (A_2)_{[a,b]}$ which implies $\min(t_{A_1}(x), t_{A_2}(y)) \geq a$ and $\max(f_{A_1}(x), f_{A_2}(y)) \geq b$. Thus we have $t_A(x, y) \geq a$ and $f_A(x, y) \geq b$ since (A, B) is the cartesian product of G_1 and G_2. Therefore, $(A_1)_{[a,b]} \times$

$(A_2)_{[a,b]} \subseteq A_{[a,b]}$. Secondly, we prove $B_{[a,b]} = E(a,b)$ for each $[a,b] \in \Pi$, where $E(a,b) = \{(x,x_2)(x,y_2) \mid x \in (A_1)_{[a,b]}, x_2y_2 \in (B_2)_{[a,b]}\} \cup \{(x_1,z)(y_1,z) \mid z \in (A_2)_{[a,b]}, x_1y_1 \in (B_1)_{[a,b]}\}$. For every $(x_1,x_2)(y_1,y_2) \in B_{[a,b]}$ (which means $t_B((x_1,x_2)(y_1,y_2)) \geq a$ and $f_B((x_1,x_2)(y_1,y_2)) \geq b$), either $x_1 = y_1$ and $x_2y_2 \in E_2$ hold or $x_2 = y_2$ and $x_1y_1 \in E_1$ hold since (A,B) is the cartesian product of G_1 and G_2. For the first case, we have

$$t_B((x_1,x_2)(y_1,y_2)) = \min(t_{A_1}(x_1), t_{B_2}(x_2y_2)) \geq a$$

and

$$f_B((x_1,x_2)(y_1,y_2)) = \max(f_{A_1}(x_1), f_{B_2}(x_2y_2)) \geq b,$$

which implies $t_{A_1}(x_1) = a$, $f_{A_1}(x_1) \geq b$, $t_{B_2}(x_2y_2) \geq a$ and $f_{B_2}(x_2y_2) \geq b$. Therefore, $x_1 = y_1 \in (A_1)_{[a,b]}$, $x_2y_2 \in (B_2)_{[a,b]}$, i.e., $(x_1,x_2)(y_1,y_2) \in E(a,b)$. Analogously, for the second case, we have $(x_1,x_2)(y_1,y_2) \in E(a,b)$. Conversely, for every $(x,x_2)(x,y_2) \in E(a,b)$ (i.e., $t_{A_1}(x) \geq a$, $f_{A_1}(x) \geq b$, $t_{B_2}(x_2y_2) \geq a$ and $f_{B_2}(x_2y_2) \geq b$), as (A,B) is the cartesian product of G_1 and G_2, we have

$$t_B((x,x_2)(x,y_2)) = \min(t_{A_1}(x), t_{B_2}(x_2y_2)) \geq a$$

and

$$f_B((x,x_2)(x,y_2)) = \max(f_{A_1}(x), f_{B_2}(x_2y_2)) \geq b,$$

which implies $(x,x_2)(x,y_2) \in B_{[a,b]}$. Analogously, for every $(x_1,z)(y_1,z) \in E(a,b)$, we have $(x_1,z)(y_1,z) \in B_{[a,b]}$. Sufficiency. Suppose that $(A_{[a,b]}, B_{[a,b]})$ is the cartesian product of $((A_1)_{[a,b]}, (B_1)_{[a,b]})$ and $((A_2)_{[a,b]}, (B_2)_{[a,b]})$ $(\forall\ [a,b] \in \Pi)$. For each $(x_1,x_2) \in V_1 \times V_2$, let $\min(t_{A_1}(x_1), t_{A_2}(x_2)) = a$ and $\max(f_{A_1}(x_1), f_{A_2}(x_2)) = b$ (which implies $x_1 \in (A_1)_{[a,b]}$ and $x_2 \in (A_2)_{[a,b]}$), then $(x_1,x_2) \in A_{[a,b]}$ since $(A_{[a,b]}, B_{[a,b]})$ is the cartesian product of $((A_1)_{[a,b]}, (B_1)_{[a,b]})$ and $((A_2)_{[a,b]}, (B_2)_{[a,b]})$ thus $t_A(x_1,x_2) \geq a = \min(t_{A_1}(x_1), t_{A_2}(x_2))$ and $f_A(x_1,x_2) \geq b = \max(f_{A_1}(x_1), f_{A_2}(x_2))$. Again, let $t_A(x_1,x_2) = c$ and $f_A(x_1,x_2) = d$ (which implies $(x_1,x_2) \in A_{[c,d]}$), then $x_1 \in (A_1)_{[c,d]}$ and $x_2 \in (A_2)_{[c,d]}$ since $(A_{[c,d]}, B_{[c,d]})$ is the cartesian product of $((A_1)_{[c,d]}, (B_1)_{[c,d]})$ and $((A_2)_{[c,d]}, (B_2)_{[c,d]})$, thus

$$\begin{aligned} t_{A_1}(x_1) &\geq c = t_A(x_1,x_2),\ f_{A_1}(x_1) \geq d = f_A(x_1,x_2), \\ t_{A_2}(x_2) &\geq c = t_A(x_1,x_2),\ f_{A_2}(x_2) \geq d = f_A(x_1,x_2), \end{aligned}$$

which implies $\min(t_{A_1}(x_1), t_{A_2}(x_2)) \geq t_A(x_1,x_2)$ and $\max(f_{A_1}(x_1), f_{A_2}(x_2)) \geq f_A(x_1,x_2)$. It follows that

$$(i)\ \begin{cases} t_A(x_1,x_2) = \min(t_{A_1}(x_1), t_{A_2}(x_2)) \\ f_A(x_1,x_2) = \max(f_{A_1}(x_1), f_{A_2}(x_2)) \end{cases} \quad ((x_1,x_2) \in V_1 \times V_2).$$

Analogously, for each $x \in V_1$ and each $x_2y_2 \in E_2$, let $\min(t_{A_1}(x), t_{B_2}(x_2y_2)) = a$, $\max(f_{A_1}(x), f_{B_2}(x_2y_2)) = b$, $t_B((x,x_2)(x,y_2)) = c$ and $f_B((x,x_2)(x,y_2)) = d$, then

$$(ii)\ \begin{cases} t_B((x,x_2)(x,y_2)) = \min(t_{A_1}(x), t_{B_2}(x_2y_2)) \\ f_B((x,x_2)(x,y_2)) = \max(f_{A_1}(x), f_{B_2}(x_2y_2)) \end{cases} \quad ((x \in V_1, x_2y_2 \in E_2).$$

For each $z \in V_2$ and each $x_1y_1 \in E_1$, let $\min(t_{A_2}(z), t_{B_1}(x_1y_1)) = a$, $\max(f_{A_2}(z),$

$f_{B_1}(x_1y_1)) = b$, $t_B((x_1,z)(y_1,z)) = c$ and $f_B((x_1,z)(y_1,z)) = d$, then

$$(iii) \begin{cases} t_B((x_1,z)(y_1,z)) = \min(t_{B_1}(x_1y_1), t_{A_2}(z)) \\ f_B((x_1,z)(y_1,z)) = \max(f_{B_1}(x_1y_1), f_{A_2}(z)) \end{cases} (z \in V_2, x_1y_1 \in E_1). \qquad \square$$

Theorem 3.3. *Let $G_1 = (A_1, B_1)$ (respectively, $G_2 = (A_2, B_2)$) be a vague graph of $G_1^* = (V_1, E_1)$ (respectively, $G_2^* = (V_2, E_2)$). Then $G_1 * G_2 = (A, B)$ is the direct product of G_1 and G_2 if and only if $(A_{[a,b]}, B_{[a,b]})$ is the direct product of $((A_1)_{[a,b]}, (B_1)_{[a,b]})$ and $((A_2)_{[a,b]}, (B_2)_{[a,b]})$ for each $[a,b] \in \Pi$.*

Proof. Necessity. Suppose that $G_1 \times G_2 = (A, B)$ is the direct product of G_1 and G_2. Firstly, we show $A_{[a,b]} = (A_1)_{[a,b]} \times (A_2)_{[a,b]}$ for each $[a,b] \in \Pi$. Actually, for every $(x,y) \in A_{[a,b]}$, we have $\min(t_{A_1}(x), t_{A_2}(y)) = t_A(x,y) \geq a$ and $\max(f_{A_1}(x), f_{A_2}(y)) = f_A(x,y) \geq b$ since (A,B) is the direct product of G_1 and G_2. It follows that $x \in (A_1)_{[a,b]}$ and $y \in (A_2)_{[a,b]}$ (i.e., $(x,y) \in (A_1)_{[a,b]} \times (A_2)_{[a,b]}$).
Therefore, $A_{[a,b]} \subseteq (A_1)_{[a,b]} \times (A_2)_{[a,b]}$. Conversely, for every $(x,y) \in (A_1)_{[a,b]} \times (A_2)_{[a,b]}$, we have $x \in (A_1)_{[a,b]}$ and $y \in (A_2)_{[a,b]}$ which implies $\min(t_{A_1}(x), t_{A_2}(y)) \geq a$ and $\max(f_{A_1}(x), f_{A_2}(y)) \geq b$. Thus we have $t_A(x,y) \geq a$ and $f_A(x,y) \geq b$ since (A,B) is the direct product of G_1 and G_2. Therefore, $(A_1)_{[a,b]} \times (A_2)_{[a,b]} \subseteq A_{[a,b]}$. Secondly, we prove $B_{[a,b]} = E(a,b)$ for each $[a,b] \in \Pi$, where

$$E(a,b) = \{(x_1,x_2)(y_1,y_2) \mid x_1y_1 \in (B_1)_{[a,b]}, x_2y_2 \in (B_2)_{[a,b]}\}.$$

For every $(x_1,x_2)(y_1,y_1) \in B_{[a,b]}$ (which means $t_B((x_1,x_2)(y_1,y_2)) \geq a$ and $f_B((x_1,x_2)(y_1,y_2)) \geq b$) then $x_1y_1 \in (B_1)_{[a,b]}$ and $x_2y_2 \in (B_2)_{[a,b]}$ hold since (A,B) is the direct product of G_1 and G_2. This implies $(x_1,x_2)(y_1,y_2) \in E(a,b)$.
Conversely, for every $(x_1,x_2)(y_1,y_2) \in E(a,b)$ (i.e., $t_{B_1}(x_1y_1) \geq a$, $f_{B_1}(x_1y_1) \geq b$, $t_{B_2}(x_2y_2) \geq a$ and $f_{B_2}(x_2y_2) \geq b$) as (A,B) is the direct product of G_1 and G_2, we have

$$t_B((x_1,x_2)(y_1,y_2)) = \min(t_{B_1}(x_1y_1), t_{B_2}(x_2y_2)) \geq a$$

and

$$f_B((x_1,x_2)(y_1,y_2)) = \max(f_{B_1}(x_1y_1), f_{B_2}(x_2y_2)) \geq b,$$

which implies $(x_1,x_2)(y_1,y_2) \in B_{[a,b]}$.
Sufficiency. Suppose that $(A_{[a,b]}, B_{[a,b]})$ is the direct product of $((A_1)_{[a,b]}, (B_1)_{[a,b]})$ and $((A_2)_{[a,b]}, (B_2)_{[a,b]})$ $(\forall\, [a,b] \in \Pi)$. For each $(x_1,x_2) \in V_1 \times V_2$, let $\min(t_{A_1}(x_1), t_{A_2}(x_2)) = a$ and $\max(f_{A_1}(x_1), f_{A_2}(x_2)) = b$ (which implies $x_1 \in (A_1)_{[a,b]}$ and $x_2 \in (A_2)_{[a,b]}$), then $(x_1,x_2) \in A_{[a,b]}$ since $(A_{[a,b]}, B_{[a,b]})$ is the direct product of $((A_1)_{[a,b]}, (B_1)_{[a,b]})$ and $((A_2)_{[a,b]}, (B_2)_{[a,b]})$, thus $t_A(x_1,x_2) \geq a = \min(t_{A_1}(x_1), t_{A_2}(x_2))$ and $f_A(x_1,x_2) \geq b = \max(f_{A_1}(x_1), f_{A_2}(x_2))$. Again, let $t_A(x_1,x_2) = c$ and $f_A(x_1,x_2) = d$ (which implies $(x_1,x_2) \in A_{[c,d]}$), then $x_1 \in (A_1)_{[c,d]}$ and $x_2 \in (A_2)_{[c,d]}$ since $(A_{[c,d]}, B_{[c,d]})$ is the direct product of

$((A_1)_{[c,d]}, (B_1)_{[c,d]})$ and $((A_2)_{[c,d]}, (B_2)_{[c,d]})$, thus

$$\begin{aligned} t_{A_1}(x_1) &\geq c = t_A(x_1, x_2), f_{A_1}(x_1) \geq d = f_A(x_1, x_2), \\ t_{A_2}(x_2) &\geq c = t_A(x_1, x_2), f_{A_2}(x_2) \geq d = f_A(x_1, x_2), \end{aligned}$$

which implies $\min(t_{A_1}(x_1), t_{A_2}(x_2)) \geq t_A(x_1, x_2)$ and $\max(f_{A_1}(x_1), f_{A_2}(x_2))$ $\geq f_A(x_1, x_2)$. It follows that

$$(i) \begin{cases} t_A(x_1, x_2) = \min(t_{A_1}(x_1), t_{A_2}(x_2)) \\ f_A(x_1, x_2) = \max(f_{A_1}(x_1), f_{A_2}(x_2)) \end{cases} ((x_1, x_2) \in V_1 \times V_2).$$

Analogously, for each $x_1y_1 \in E_1$ and each $x_2y_2 \in E_2$, let $\min(t_{B_1}(x_1y_1),$ $t_{B_2}(x_2y_2)) = a$, $\max(f_{B_1}(x_1y_1), f_{B_2}(x_2y_2)) = b$, $t_B((x_1, x_2)(y_1, y_2)) = c$ and $f_B((x_1, x_2)(y_1, y_2))$
$= d$, then

$$(ii) \begin{cases} t_B((x_1, x_2)(y_1, y_2)) = \min(t_{B_1}(x_1y_1), t_{B_2}(x_2y_2)) \\ f_B((x_1, x_2)(y_1, y_2)) = \max(f_{B_1}(x_1y_1), f_{B_2}(x_2y_2)) \end{cases} (x_1y_1 \in E_1, x_2y_2 \in E_2).$$

□

Theorem 3.4. *Let $G_1 = (A_1, B_1)$ (respectively, $G_2 = (A_2, B_2)$) be a vague graph of $G_1^* = (V_1, E_1)$ (respectively, $G_2^* = (V_2, E_2)$). Then, $G_1 \bullet G_2 = (A, B)$ is the lexicographic product of G_1 and G_2 if and only if $(A_{[a,b]}, B_{[a,b]})$ is the lexicographic product of $((A_1)_{[a,b]}, (B_1)_{[a,b]})$ and $((A_2)_{[a,b]}, (B_2)_{[a,b]})$ for each $[a, b] \in \Pi$.*

Proof. Necessity. Suppose that $G_1 \bullet G_2 = (A, B)$ is the lexicographic product of G_1 and G_2. Firstly, we show $A_{[a,b]} = (A_1)_{[a,b]} \times (A_2)_{[a,b]}$ for each $[a, b] \in \Pi$ by definition of lexicographic product and the proof of Theorem 3.3. Secondly, we proof $B_{[a,b]} = E(a, b) \cup F(a, b)$ for each $[a, b] \in \Pi$, where $E(a, b)$ is as that defined in Theorem 3.3 and $F(a, b) = \{(x, x_2)(x, y_2) \mid x \in (A_1)_{[a,b]}, x_2y_2 \in (B_2)_{[a,b]}\}$. By the proof of Theorem 3.3, we have $E(a, b) \subseteq B_{[a,b]}$. For every $(x, x_2)(x, y_2) \in F(a, b)$ (i.e., $t_{A_1}(x) \geq a$, $f_{A_1}(x) \geq b$, $t_{B_2}(x_2y_2) \geq a$, $f_{B_2}(x_2y_2) \geq b$), as $G_1 \bullet G_2 = (A, B)$ is the lexicographic product of G_1 and G_2, we have $t_B((x, x_2)(x, y_2)) \geq a$ and $f_B((x, x_2)(x, y_2)) \geq b$, which implies $(x, x_2)(x, y_2) \in B_{[a,b]}$. Therefore, $E(a, b) \cup F(a, b) \subseteq B_{[a,b]}$. Conversely, for every $(x_1, x_2)(y_1, y_2) \in B_{[a,b]}$ (i.e., $t_B((x_1, x_2)(y_1, y_2)) \geq a$ and $f_B((x_1, x_2)(y_1, y_2)) \geq b$) as $G_1 \bullet G_2 = (A, B)$ is the lexicographic product of G_1 and G_2, we have $(x_1x_2)(y_1, y_2) \in E \cup F$, where $E = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1, x_2y_2 \in E_2\}$, $F = \{(x, x_2)(x, y_2) \mid x \in V_1, x_2y_2 \in E_2\}$. If $(x_1, x_2)(y_1, y_2) \in E$, then $(x_1, x_2)(y_1, y_2) \in E(a, b)$ by the proof of Theorem 3.3. If $(x_1, x_2)(y_1, y_2) \in F$, i.e., $x_1 = y_1$, $x_2y_2 \in E_2$, then

$$\min(t_{A_1}(x_1), t_{B_2}(x_2y_2)) = t_B((x_1, x_2)(y_1, y_2)) \geq a$$

and

$$\max(f_{A_1}(x_1), f_{B_2}(x_2y_2)) = f_B((x_1, x_2)(y_1, y_2)) \geq b,$$

which implies $x_1, y_1 \in (A_1)_{[a,b]}$, and $x_2, y_2 \in (B_2)_{[a,b]}$. Therefore, $(x_1, x_2)(y_1, y_2)$ $\in F(a, b)$. It follows that $B_{[a,b]} \subseteq E(a, b) \cup F(a, b)$.
Sufficiency. Assume that $(A_{[a,b]}, B_{[a,b]})$ is the lexicographic product of $((A_1)_{[a,b]},$

$(B_1)_{[a,b]})$ and $((A_2)_{[a,b]}, (B_2)_{[a,b]})$ $(\forall\ [a,b] \in \Pi)$. By the proof of Theorem 3.3, we know

$$(i) \begin{cases} t_A(x_1,x_2) = \min(t_{A_1}(x_1), t_{A_2}(x_2)) \\ f_A(x_1,x_2) = \max(f_{A_1}(x_1), f_{A_2}(x_2)) \end{cases} ((x_1,x_2) \in V_1 \times V_2),$$

$$(ii) \begin{cases} t_B((x_1,x_2)(y_1,y_2)) = \min(t_{B_1}(x_1y_1), t_{B_2}(x_2y_2)) \\ f_B((x_1,x_2)(y_1,y_2)) = \max(f_{B_1}(x_1y_1), f_{B_2}(x_2y_2)) \end{cases} (x_1y_1 \in E_1, x_2y_2 \in E_2).$$

For each $x \in V_1$ and each $x_2y_2 \in E_2$, let $\min(t_{A_1}(x), t_{B_2}(x_2y_2)) = a$, $\max(f_{A_1}(x), f_{B_2}(x_2y_2)) = b$, $t_B((x,x_2)(x,y_2)) = c$ and $f_B((x,x_2)(x,y_2)) = d$, then

$$(iii) \begin{cases} t_B((x,x_2)(x,y_2)) = \min(t_{A_1}(x), t_{B_2}(x_2y_2)) \\ f_B((x,x_2)(x,y_2)) = \max(f_{A_1}(x), f_{B_2}(x_2y_2)) \end{cases} (x \in V_1, x_2y_2 \in E_2). \qquad \square$$

Remark 3.1. *Let* $G_1 = (A_1, B_1)$ *(respectively,* $G_2 = (A_2, B_2)$*) be a vague graph of* $G_1^* = (V_1, E_1)$ *(respectively,* $G_2^* = (V_2, E_2)$*). Then,* $G_1 \boxtimes G_2 = (A, B)$ *is the strong product of* G_1 *and* G_2 *if and only if* $(A_{[a,b]}, B_{[a,b]})$ *is the strong product of* $((A_1)_{[a,b]}, (B_1)_{[a,b]})$ *and* $((A_2)_{[a,b]}, (B_2)_{[a,b]})$ *for each* $[a,b] \in \Pi$.

Remark 3.2. *Let* $G_1 = (A_1, B_1)$ *(respectively,* $G_2 = (A_2, B_2)$*) be a vague graph of* $G_1^* = (V_1, E_1)$ *(respectively,* $G_2^* = (V_2, E_2)$*) and* $V_1 \cap V_2 = \emptyset$*. Then* $G_1 \cup G_2 = (A, B)$ *is the union of* G_1 *and* G_2 *if and only if* $(A_{[a,b]}, B_{[a,b]})$ *is the union of* $((A_1)_{[a,b]}, (B_1)_{[a,b]})$ *and* $((A_2)_{[a,b]}, (B_2)_{[a,b]})$ *for each* $[a,b] \in \Pi$.

4. Conclusion

It is well known that graphs are among the most ubiquitous models of both natural and human-made structure. They can be used to model many types of relations and process dynamics in computer science, physical, biological and social systems. Many problems of practical interest can be represented by graphs. In general graphs theory has a wide range of applications in diverse fields. In this paper, we defined three kinds of new product operations (call directed product, lexicographic product and strong product) of vague graphs and rationality of these notions and some defined important notions on vague graphs, such as vague graph, vague complete graph, cartesian product of vague graphs and union of vague graphs are demonstrated by characterizing theses notions by their level counterparts graphs.

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