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HERMITE-HADAMARD INEQUALITIES FOR CO-ORDINATED HARMONIC CONVEX FUNCTIONS

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In this paper, we derive some new Hermite-Hadamard type inequality for co-ordinated harmonic convex functions, Some special cases are discussed, which can be viewed as significant refinement of the known results. The ideas and technique of this paper may motivate further research in this dynamic and interesting field.

Keywords: Harmonic convex functions, Convex functions on the co-ordinates, Hermite-Hadamard type inequality.

MSC2010: 26D15, 26D10, 90C23

1. Introduction

Inequalities play an important and significant role in almost all subjects of Mathematical and engineering sciences. Inequalities present a very active and attractive field of research. In recent years, much attention have given to develop various inequalities for several classes of convex functions and their generalizations using novel and new ideas. In recent years, much attention have been given to derive several Hermite-Hadamard type inequalities related to various type of convex functions, using new ideas and innovative techniques, see [8, 9, 15, 17]. Hermite-Hadamard inequalities are used to find the upper and lower bounds of the mean value. For a novel application of Hermite-Hadamard inequalities in the proof of inequality $e < \left(1 + \frac{1}{n}\right)^{n+0.5}$, see Khattri [10].

It is well known that the harmonic means have applications in electrical circuits. The total resistance of a set of parallel resistance is obtained by adding up the reciprocals of the individual resistance values and then considering the reciprocal of their total. For example if R_1 and R_2 are the resistances of two parallel resistors, then the total resistance is obtained by the formula $R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$, which is half the harmonic mean. These harmonic means also played a crucial in the developments of parallel algorithms. Al-Azemi and Colin [1] have considered the applications of harmonic means in the stock markets, which is another novel aspect of the harmonic means. Anderson et al. [2] considered the harmonic functions and investigated various applications of harmonic functions. Iscan [9] derived the Hermite-Hadamard type inequality for the harmonic functions. These integral inequalities are used to find the upper and lower bounds. For recent developments and other aspects of harmonic convex functions, see [8,9,14-19] and the references therein.

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The concept of convex functions on the co-ordinates was introduced by Dragomir [7]. He obtained some new Hermite-Hadamard type inequalities for co-ordinated convex functions. Bakula [3], established the refinement of the Hermite-Hadamard type inequality for co-ordinated convex functions. Chen [6] has obtained some new Hermite-Hadamard type inequalities for co-ordinated convex functions, which can be regarded as refinement of the Hermite-Hadamard inequality obtained by Bakula [3]. For recent results and generalizations concerning Hermite-Hadamard type inequalities for co-ordinated convex functions, see [4, 5, 6, 11, 12, 20, 21] and reference therein. Noor et al. [14] have investigated the coordinated harmonic convex functions and derived some integral inequalities for harmonic functions on the coordinates. It is known that the coordinated harmonic function may not be harmonic function in the ordinary sense.

Motivated and inspired by the ongoing research in this dynami field, we again consider the co-ordinated harmonic convex functions. We first establish an auxiliary results for Lebesgue integrable functions on the rectangle. We use this auxiliary results to derive some new general Hermite-Hadamard type inequalities for the co-ordinated harmonic convex functions. As an applications of our main result, we obtain a new refinement of the known result.

2. **Preliminaries**

In this section, we recall the well known concepts:

Definition 2.1. [23]. *A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic convex set, if*

$$\frac{xy}{tx + (1-t)y} \in I, \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 2.2. [9, 2]. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be a harmonic convex function, if and only if,*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1].$$

Iscan [9] established the following Hermite-Hadamard inequality for harmonic convex function.

Theorem 2.1. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a harmonic convex function, if and only if,*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \qquad x \in [a, b].$$

Dragomir [8] and Noor et al. [18] have obtained the following refinement of Hermite-Hadamard inequality for harmonic convex functions independently.

Theorem 2.2. [8, 18]. *Let $f : I = [a, b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonic convex function on the interval $[a, b]$. Then for any $\lambda \in [0, 1]$, we have*

$$\begin{aligned}
f\left(\frac{2ab}{a+b}\right) &\leq (1-\lambda)f\left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b}\right) + \lambda f\left(\frac{2ab}{(2-\lambda)a + \lambda b}\right) \\
&\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\
&\leq \frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)a + \lambda(b)}\right) + (1-\lambda)f(a) + \lambda f(b)\right] \\
&\leq \frac{f(a)+f(b)}{2}.
\end{aligned}$$

We now consider harmonic convex functions on the coordinates:

Definition 2.3. [14]. *Consider a rectangle $\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \setminus \{0\}$ with $a < b$ and $c < d$. A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to be harmonic convex function on the rectangle Δ, if*

$$f\left(\frac{ab}{\lambda a + (1-\lambda)b}, \frac{cd}{\lambda c + (1-\lambda)d}\right) \leq (1-\lambda)f(a,c) + \lambda(b,d), \quad \forall (a,b),(c,d) \in \Delta, \lambda \in [0,1].$$

These co-ordinated harmonic convex functions may be defined as:

Definition 2.4. [14]. *Consider a rectangle $\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \setminus \{0\}$ with $a < b$ and $c < d$. A function $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to be co-ordinated harmonic convex function on the rectangle Δ, if*

$$\begin{aligned} f\left(\frac{ab}{\lambda a + (1-\lambda)b}, \frac{cd}{tc + (1-t)d}\right) \quad \leq \quad & (1-\lambda)(1-t)f(a,c) + (1-\lambda)tf(a,d) \\ & + \lambda(1-t)f(b,c) + \lambda t(b,d), \end{aligned}$$

for all (a,b), $(c,d) \in \Delta$ and $\lambda, t \in [0,1]$.

We would like to mention that a function $f : \Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is called harmonic on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, defined by $f_y(u) := f(u,y)$, and $f_x : [c,d] \to \mathbb{R}$, defined by $f_x(v) := f(x,v)$, are harmonic convex for all $x \in [a,b]$ and $y \in [c,d]$.

It is known that the co-ordinated harmonic convex functions maay not be harmonic convex functions. For example, One can easily show that the function f defined by $f(x,y) = \frac{1}{xy}$ is co-ordinated harmonic convex function, but it is not harmonic convex function in the ordinary sense.

Noor et al. [14] have established the follwing Hermite-Hadamard inequalities for the co-ordinated harmonic convex functions.

Theorem 2.3. [14]. *Let $f : \Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is co-ordinated harmonic convex function on the rectangle Δ. Then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right) \quad & \leq \quad \frac{ab(cd)}{(b-a)(d-c)} \int_a^b \int_c^d \frac{f(x,y)}{x^2y^2} \mathrm{d}y \mathrm{d}x \\ & \leq \quad \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned} \tag{2.1}$$

The following simple fact plays an important role in the derivation of our main results.

Remark 2.1. *If $I \subset \Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \setminus \{0\}$ and consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \times \left[\frac{1}{d}, \frac{1}{c}\right] \to \mathbb{R}$ defined by $g(s_1, s_2) = f\left(\frac{1}{s_1}, \frac{1}{s_2}\right)$, then f is co-ordinated harmonic on $[a,b] \times [c,d]$, if and only if, g is co-ordinated convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right] \times \left[\frac{1}{d}, \frac{1}{c}\right] \to \mathbb{R}$.*

3. **Main results**

In this section, we obtain a more general Hermite-Hadamard inequality, which is main motivation of this paper. For this purpose, we first prove the following auxiliary result.

Theorem 3.1. *Let $g : \Delta = [a,b] \times [c,d] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a Lebesgue integrable function on the rectangle Δ and $\lambda, t \in [0,1]$, then*

$$\begin{aligned}
&\int_0^1 \int_0^1 g((1-t_1)a + t_1 b, (1-t_2)c + t_2 d)\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \lambda t \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda b], (1-t_2)c + t_2[(1-t)c + td])\mathrm{d}t_1\mathrm{d}t_2 \\
&+\lambda(1-t) \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda b], (1-t_2)[(1-t)c + td] + t_2 d)\mathrm{d}t_1\mathrm{d}t_2 \\
&+t(1-\lambda) \int_0^1 \int_0^1 g((1-t_1)[(1-\lambda)a + \lambda b] + t_1 b, (1-t_2)c + t_2[(1-t)c + td])\mathrm{d}t_1\mathrm{d}t_2 \\
&+(1-t)(1-\lambda) \int_0^1 \int_0^1 g((1-t_1)[(1-\lambda)a + \lambda b] + t_1 b, (1-t_2)[(1-t)c + td] + t_2 d)\mathrm{d}t_1\mathrm{d}t_2.
\end{aligned} \tag{3.1}$$

Proof. For $\lambda = 0, t = 0$ and $\lambda = 1, t = 1$, the inequality (3.1) is obvious.
If $\lambda = 0$ and $t \in [0,1]$, then

$$\begin{aligned}
&\int_0^1 \int_0^1 g((1-t_1)a + t_1 b, (1-t_2)c + t_2 d)\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& t \int_0^1 \int_0^1 g((1-t_1)a + t_1 b, (1-t_2)c + t_2[(1-t)c + td])\mathrm{d}t_1\mathrm{d}t_2 \\
&+(1-t) \int_0^1 \int_0^1 g((1-t_1)a + t_1 b, (1-t_2)[(1-t)c + td] + t_2 d)\mathrm{d}t_1\mathrm{d}t_2.
\end{aligned}$$

If $t = 0$ and $\lambda \in [0,1]$, then

$$\begin{aligned}
&\int_0^1 \int_0^1 g((1-t_1)a + t_1 b, (1-t_2)c + t_2 d)\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \lambda \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda b], (1-t_2)c + t_2 d)\mathrm{d}t_1\mathrm{d}t_2 \\
&+(1-\lambda) \int_0^1 \int_0^1 g((1-t_1)[(1-\lambda)a + \lambda b] + t_1 b, (1-t_2)c + t_2 d)\mathrm{d}t_1\mathrm{d}t_2.
\end{aligned}$$

Also for $\lambda \in (0,1)$ and $t \in (0,1)$, we observe that

$$\begin{aligned}
(i). \quad & \lambda t \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda b], (1-t_2)c + t_2[(1-t)c + td])\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \lambda t \int_0^1 \int_0^1 g(t_1 \lambda b + (1-\lambda t_1)a, t_2 t d + (1-t t_2)c)\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \int_0^\lambda \int_0^t g(ub + (1-u)a, vd + (1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
(ii). \quad & \lambda(1-t) \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda b], (1-t_2)[(1-t)c + td] + t_2 d)\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \lambda(1-t) \int_0^1 \int_0^1 g(t_1 \lambda b + (1-\lambda t_1)a, ((1-t_2)t + t_2)d + (1-t_2)(1-t)c)\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \int_0^\lambda \int_t^1 g(ub + (1-u)a, vd + (1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
(iii). \quad & t(1-\lambda)\int_0^1\int_0^1 g((1-t_1)[(1-\lambda)a+\lambda b]+t_1b,(1-t_2)c+t_2[(1-t)c+td])\mathrm{d}t_1\mathrm{d}t_2\\
= \quad & t(1-\lambda)\int_0^1\int_0^1 g(((1-t_1)\lambda+t_1)b+(1-t_1)(1-\lambda)a,t_2td+(1-tt_2)c)\mathrm{d}t_1\mathrm{d}t_2\\
= \quad & \int_\lambda^1\int_0^t g(ub+(1-u)a,vd+(1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
(iv). \quad & (1-t)(1-\lambda)\int_0^1\int_0^1 g((1-t_1)[(1-\lambda)a+\lambda b]+t_1b,(1-t_2)[(1-t)c+td]+t_2d)\mathrm{d}t_1\mathrm{d}t_2\\
= \quad & (1-t)(1-\lambda)\int_0^1\int_0^1 g(((1-t_1)\lambda+t_1)b+(1-t_1)(1-\lambda)a,((1-t_2)t+t_2)d\\
& +(1-t_2)(1-t)c)\mathrm{d}t_1\mathrm{d}t_2\\
= \quad & \int_\lambda^1\int_t^1 g(ub+(1-u)a,vd+(1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}
$$

Combining the above results, we have

$$
\begin{aligned}
& \int_0^1\int_0^1 g((1-t_1)a+t_1b,(1-t_2)c+t_2d)\mathrm{d}t_1\mathrm{d}t_2\\
= \ & \int_0^\lambda\int_0^t g(ub+(1-u)a,vd+(1-v)c)\mathrm{d}v\mathrm{d}u\\
& +\int_0^\lambda\int_t^1 g(ub+(1-u)a,vd+(1-v)c)\mathrm{d}v\mathrm{d}u\\
& +\int_\lambda^1\int_0^t g(ub+(1-u)a,vd+(1-v)c)\mathrm{d}v\mathrm{d}u\\
& +\int_\lambda^1\int_t^1 g(ub+(1-u)a,vd+(1-v)c)\mathrm{d}v\mathrm{d}u\\
= \ & \int_0^1\int_0^1 g(ub+(1-u)a,vd+(1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}
$$

and the identity (3.1) is proved. □

We now derive a new Hermite-Hadamard inequality for co-ordinated harmonic convex function.

Theorem 3.2. *Let $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ be co-ordinated harmonic convex function on the rectangle Δ. Then for any $\lambda\in[0,1]$ and $t\in[0,1]$, we have*

$$
\begin{aligned}
f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right) &\leq \phi(\lambda,t)\\
&\leq \frac{(ab)(cd)}{(b-a)(d-c)}\int_a^b\int_c^d \frac{f(x,y)}{x^2y^2}\mathrm{d}x\mathrm{d}y\\
&\leq \psi(\lambda,t)\\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (3.2)
\end{aligned}
$$

where

$$\begin{aligned}\phi(\lambda,t) &= t\lambda f\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(2-t)c+td}\Big)\\ &+\lambda(1-t)f\Big(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(1-t)c+(1+t)d}\Big)\\ &+t(1-\lambda)f\Big(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(2-t)c+td}\Big)\\ &+(1-\lambda)(1-t)f\Big(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(1-t)c+(1+t)d}\Big).\end{aligned}$$

and

$$\begin{aligned}\psi(\lambda,t) &= \frac{t\lambda}{4}f(b,d)+\frac{\lambda(1-t)}{4}f(b,c)+\frac{t(1-\lambda)}{4}f(a,d)+\frac{(1-t)(1-\lambda)}{4}f(a,c)\\ &+\frac{1}{4}f\Big(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td}\Big)+\frac{\lambda}{4}f\Big(b,\frac{cd}{(1-t)c+td}\Big)\\ &+\frac{1-\lambda}{4}f\Big(a,\frac{cd}{(1-t)c+td}\Big)+\frac{t}{4}f\Big(\frac{ab}{(1-\lambda)a+\lambda b},d\Big)\\ &+\frac{1-t}{4}f\Big(\frac{ab}{(1-\lambda)a+\lambda b},c\Big).\end{aligned}$$

Proof. Let g be co-ordinated convex function on $\Delta=\big[\frac{1}{b},\frac{1}{a}\big]\times\big[\frac{1}{d},\frac{1}{c}\big]$, defined by $g(s_1,s_2)=f\big(\frac{1}{s_1},\frac{1}{s_2}\big)$, $s_1,s_2\in\big[\frac{1}{b},\frac{1}{a}\big]\times\big[\frac{1}{d},\frac{1}{c}\big]$, then using Remark 2.1 and for $\lambda,t\in[0,1]$, we have

(*i*).
$$\begin{aligned}& g\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(2-t)c+td}{2cd}\Big)\\ =\ & g\Big(\frac{\frac{1}{b}+(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}}{2},\frac{\frac{1}{d}+(1-t)\frac{1}{d}+t\frac{1}{c}}{2}\Big)\\ \le\ & \int_0^1\int_0^1 g\Big((1-t_1)\frac{1}{b}+t_1\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big),(1-t_2)\frac{1}{d}+t_2\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big)\Big)\mathrm{d}t_1\mathrm{d}t_2\\ \le\ & \frac{g\big(\frac{1}{b},\frac{1}{d}\big)+g\big(\frac{1}{b},(1-t)\frac{1}{d}+t\frac{1}{c}\big)+g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},\frac{1}{d}\big)+g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},(1-t)\frac{1}{d}+t\frac{1}{c}\big)}{4}\\ =\ & \frac{g\big(\frac{1}{b},\frac{1}{d}\big)+g\big(\frac{1}{b},\frac{(1-t)c+td}{cd}\big)+g\big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{d}\big)+g\big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c+td}{cd}\big)}{4}\end{aligned} \tag{3.3}$$

(*ii*).
$$\begin{aligned}& g\Big(\frac{(2-\lambda)a+\lambda b}{2ab},\frac{(1-t)c+(1+t)d}{2cd}\Big)\\ =\ & g\Big(\frac{\frac{1}{b}+(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}}{2},\frac{(1-t)\frac{1}{d}+t\frac{1}{c}+\frac{1}{c}}{2}\Big)\\ \le\ & \int_0^1\int_0^1 g\Big((1-t_1)\frac{1}{b}+t_1\Big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\Big),(1-t_2)\Big((1-t)\frac{1}{d}+t\frac{1}{c}\Big)+t_2\frac{1}{c}\Big)\mathrm{d}t_1\mathrm{d}t_2\\ \le\ & \frac{g\big(\frac{1}{b},(1-t)\frac{1}{d}+t\frac{1}{c}\big)+g\big(\frac{1}{b},\frac{1}{c}\big)+g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},(1-t)\frac{1}{d}+t\frac{1}{c}\big)+g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},\frac{1}{c}\big)}{4}\\ =\ & \frac{g\big(\frac{1}{b},\frac{(1-t)c++td}{cd}\big)+g\big(\frac{1}{b},\frac{1}{c}\big)+g\big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{(1-t)c++td}{cd}\big)+g\big(\frac{(1-\lambda)a+\lambda b}{ab},\frac{1}{c}\big)}{4}\end{aligned} \tag{3.4}$$

(iii).
$$\begin{aligned}
& g\left(\frac{(1-\lambda)a+(\lambda+1)b}{2ab}, \frac{(2-t)c+td}{2cd}\right)\\
= & g\left(\frac{(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}+\frac{1}{a}}{2}, \frac{\frac{1}{d}+(1-t)\frac{1}{d}+t\frac{1}{c}}{2}\right)\\
\leq & \int_0^1\int_0^1 g\left((1-t_1)\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+t_1\frac{1}{a}, (1-t_2)\frac{1}{d}+t_2\left((1-t)\frac{1}{d}+t\frac{1}{c}\right)\right)\mathrm{d}t_1\mathrm{d}t_2\\
\leq & \frac{g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}, \frac{1}{d}\big)+g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}, (1-t)\frac{1}{d}+t\frac{1}{c}\big)+g\big(\frac{1}{a}, \frac{1}{d}\big)+g\big(\frac{1}{a}, (1-t)\frac{1}{d}+t\frac{1}{c}\big)}{4}\\
= & \frac{g\big(\frac{(1-\lambda)a+\lambda b}{ab}, \frac{1}{d}\big)+g\big(\frac{(1-\lambda)a+\lambda b}{ab}, \frac{(1-t)c+td}{cd}\big)+g\big(\frac{1}{a}, \frac{1}{d}\big)+g\big(\frac{1}{a}, \frac{(1-t)c+td}{cd}\big)}{4} \qquad (3.5)
\end{aligned}$$

(iv).
$$\begin{aligned}
& g\left(\frac{(1-\lambda)a+(\lambda+1)b}{2ab}, \frac{(1-t)c+(1+t)d}{2cd}\right)\\
= & g\left(\frac{(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}+\frac{1}{a}}{2}, \frac{(1-t)\frac{1}{d}+t\frac{1}{c}+\frac{1}{c}}{2}\right)\\
\leq & \int_0^1\int_0^1 g\left((1-t_1)\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+t_1\frac{1}{a}, (1-t_2)\left((1-t)\frac{1}{d}+t\frac{1}{c}\right)+t_2\frac{1}{c}\right)\mathrm{d}t_1\mathrm{d}t_2\\
\leq & \frac{g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}, (1-t)\frac{1}{d}+t\frac{1}{c}\big)+g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}, \frac{1}{c}\big)+g\big(\frac{1}{a}, (1-t)\frac{1}{d}+t\frac{1}{c}\big)+g\big(\frac{1}{a}, \frac{1}{c}\big)}{4}\\
= & \frac{g\big(\frac{(1-\lambda)a+\lambda b}{ab}, \frac{(1-t)c+td}{cd}\big)+g\big(\frac{(1-\lambda)a+\lambda b}{ab}, \frac{1}{c}\big)+g\big(\frac{1}{a}, \frac{(1-t)c+td}{cd}\big)+g\big(\frac{1}{a}, \frac{1}{c}\big)}{4} \qquad (3.6)
\end{aligned}$$

Multiply (3.3) by λt, (3.4) by $\lambda(1-t)$, (3.5) by $t(1-\lambda)$, (3.6) by $(1-\lambda)(1-t)$, and adding the resultant, we have

$$\begin{aligned}
& t\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}, \frac{2cd}{(2-t)c+td}\right)+\lambda(1-t)f\left(\frac{2ab}{(2-\lambda)a+\lambda b}, \frac{2cd}{(1-t)c+(1+t)d}\right)\\
& +t(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(1+\lambda)b}, \frac{2cd}{(2-t)c+td}\right)\\
& +(1-\lambda)(1-t)f\left(\frac{2ab}{(1-\lambda)a+(1+\lambda)b}, \frac{2cd}{(1-t)c+(1+t)d}\right)\\
= & \phi(\lambda, t)\\
\leq & \int_0^1\int_0^1 f\left(\frac{ab}{(1-t_1)a+t_1b}, \frac{cd}{(1-t_2)c+t_2d}\right)\mathrm{d}t_1\mathrm{d}t_2\\
= & \frac{(ab)(cd)}{(b-a)(d-c)}\int_a^b\int_c^d \frac{f(x,y)}{x^2y^2}\mathrm{d}x\mathrm{d}y\\
\leq & \frac{\lambda t}{4}\left[f\big(b,d\big)+f\left(b, \frac{cd}{(1-t)c+td}\right)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}, d\right)\right.\\
& \left.+f\left(\frac{ab}{(1-\lambda)a+\lambda b}, \frac{cd}{(1-t)c+td}\right)\right]+\frac{\lambda(1-t)}{4}\left[f\left(b, \frac{cd}{(1-t)c++td}\right)+f\big(b,c\big)\right.
\end{aligned}$$

$$
\begin{aligned}
& +f\left(\frac{ab}{(1-\lambda)a+\lambda b}, \frac{cd}{(1-t)c++td}\right)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}, c\right)\Bigg] \\
& +\frac{t(1-\lambda)}{4}\Bigg[f\left(\frac{ab}{(1-\lambda)a+\lambda b}, d\right)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}, \frac{cd}{(1-t)c+td}\right) \\
& +f(a,d)+f\left(a, \frac{cd}{(1-t)c+td}\right)\Bigg]+\frac{(1-\lambda)(1-t)}{4}\Bigg[f\left(\frac{ab}{(1-\lambda)a+\lambda b}, \frac{cd}{(1-t)c+td}\right) \\
& +f\left(\frac{ab}{(1-\lambda)a+\lambda b}, c\right)+f\left(a, \frac{cd}{(1-t)c+td}\right)+f(a,c)\Bigg] \\
= {} & \frac{t\lambda}{4}f(b,d)+\frac{\lambda(1-t)}{4}f(b,c)+\frac{t(1-\lambda)}{4}f(a,d)+\frac{(1-t)(1-\lambda)}{4}f(a,c) \\
& +\frac{1}{4}f\left(\frac{ab}{(1-\lambda)a+\lambda b}, \frac{cd}{(1-t)c+td}\right)+\frac{\lambda}{4}f\left(b, \frac{cd}{(1-t)c+td}\right)+\frac{1-\lambda}{4}f\left(a, \frac{cd}{(1-t)c+td}\right) \\
& +\frac{t}{4}f\left(\frac{ab}{(1-\lambda)a+\lambda b}, d\right)+\frac{1-t}{4}f\left(\frac{ab}{(1-\lambda)a+\lambda b}, c\right) \\
= {} & \psi(\lambda,t). \qquad (3.7)
\end{aligned}
$$

Also

$$
\begin{aligned}
\phi(\lambda,t) = {} & t\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}, \frac{2cd}{(2-t)c+td}\right) \\
& +\lambda(1-t)f\left(\frac{2ab}{(2-\lambda)a+\lambda b}, \frac{2cd}{(1-t)c+(1+t)d}\right) \\
& +t(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(1+\lambda)b}, \frac{2cd}{(2-t)c+td}\right) \\
& +(1-\lambda)(1-t)f\left(\frac{2ab}{(1-\lambda)a+(1+\lambda)b}, \frac{2cd}{(1-t)c+(1+t)d}\right) \\
= {} & t\lambda g\left(\frac{(2-\lambda)a+\lambda b}{2ab}, \frac{(2-t)c+td}{2cd}\right) \\
& +\lambda(1-t)g\left(\frac{(2-\lambda)a+\lambda b}{2ab}, \frac{(1-t)c+(1+t)d}{2cd}\right) \\
& +t(1-\lambda)g\left(\frac{(1-\lambda)a+(1+\lambda)b}{2ab}, \frac{(2-t)c+td}{2cd}\right) \\
& +(1-\lambda)(1-t)g\left(\frac{(1-\lambda)a+(1+\lambda)b}{2ab}, \frac{(1-t)c+(1+t)d}{2cd}\right) \\
\geq {} & g\Bigg((1-\lambda)\left(\frac{(1-\lambda)a+(1+\lambda)b}{2ab}\right)+\lambda\left(\frac{(2-\lambda)a+\lambda b}{2ab}\right), \\
& (1-t)\left(\frac{(1-t)c+(1+t)d}{2cd}\right)+t\left(\frac{(2-t)c+td}{2cd}\right)\Bigg) \\
= {} & g\left(\frac{a+b}{2ab}, \frac{c+d}{2cd}\right)=f\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right) \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi(\lambda,t) \leq {} & \frac{t\lambda}{4}f(b,d)+\frac{\lambda(1-t)}{4}f(b,c)+\frac{t(1-\lambda)}{4}f(a,d)+\frac{(1-t)(1-\lambda)}{4}f(a,c) \\
& +\frac{(1-\lambda)(1-t)}{4}f(b,d)+\frac{\lambda t}{4}f(a,c)+\frac{\lambda(1-t)}{4}f(a,d)+\frac{t(1-\lambda)}{4}f(b,c) \\
& +\frac{\lambda t}{4}f(b,c)+\frac{\lambda(1-t)}{4}f(b,d)+\frac{\lambda t}{4}f(a,d)+\frac{t(1-\lambda)}{4}f(b,d)
\end{aligned}
$$

$$+\frac{(1-t)(1-\lambda)}{4}f(b,c)+\frac{(1-t)\lambda}{4}f(a,c)+\frac{(1-\lambda)(1-t)}{4}f(a,d)$$
$$+\frac{(1-\lambda)t}{4}f(a,c)$$
$$=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \tag{3.9}$$

From (3.7)-(3.9), we obtained the desired inequality (3.2). □

We now consider an application of our main result(Theorem 3.2). For $\lambda = \frac{1}{2}$ and $t = \frac{1}{2}$, Theorem 3.2 reduces to the following result.

Theorem 3.3. *Let* $f : \Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ *is co-ordinated harmonic convex function on the rectangle* Δ*. Then*

$$\begin{aligned}
f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right) &\le \frac{1}{4}\left[f\left(\frac{4ab}{3a+b},\frac{4cd}{3c+d}\right)+f\left(\frac{4ab}{3a+b},\frac{4cd}{c+3d}\right)\right.\\
&\quad \left.+f\left(\frac{4ab}{a+3b},\frac{4cd}{3c+d}\right)+f\left(\frac{4ab}{a+3b},\frac{4cd}{c+3d}\right)\right]\\
&\le \frac{(ab)(cd)}{(b-a)(d-c)}\int_a^b\int_c^d \frac{f(x,y)}{x^2y^2}\mathrm{d}x\mathrm{d}y\\
&\le \frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{16}+\frac{1}{4}f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)\\
&\quad +\frac{f(b,\frac{2cd}{c+d})+f(a,\frac{2cd}{c+d})+f(\frac{2ab}{a+b},d)+f(\frac{2ab}{a+b},c)}{8}\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (3.10)
\end{aligned}$$

It is clear that the inequality represents a significant refinement of the inequality (2.1), which was obtained by Noor et al [14].

Conclusion: In this paper, we have proved an auxiliary result for Lebesgue integrable functions on the rectangle. Using this auxiliary results, we derived some new general Hermite-Hadamard inequalities for co-ordinated harmonic convex functions. Some special cases are discussed, which can be viewed as significant refinement of the known results.To the best of knowledge, this is new field, which required further investigation. It is our hope that the interested readers may find novel and innovative applications of the coordinated harmonic convex functions in various fields of pure and applied sciences.

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