U.P.B. Sci. Bull., Series A, Vol. 82, Iss. 4, 2020 ISSN 1223-7027

VARIOUS TYPES OF SHADOWING AND SPECIFICATION PROPERTIES OF SEMIGROUP ACTIONS

Zahra Shabani[1], Haleh Tajadodi[2]

In this paper, we introduce the notions of $\underline{d}$-shadowing and weak specification properties for finitely generated free semigroup actions on the compact metric spaces and investigate their relations with average shadowing and ergodic shadowing properties. Moreover, we define a new notion for the ergodic shadowing property on the non-compact metric spaces, which is a dynamical property and equivalent to its definition in the case of semigroups on the compact metric spaces.

Keywords: semigroup actions, ergodic shadowing, $\underline{d}$-shadowing, specification property.
MSC2020: 37B05 37C50.

1. Introduction

The concept of shadowing was originated from the Anosov closing lemma and because of its rich consequences, shadowing plays an important role in the general qualitative theory of dynamical systems. It is considerably developed in recent years and many authors have studied several kinds of shadowing including ergodic shadowing [7], $\underline{d}$-shadowing [5], and average shadowing [10], which have the common motivation of studying the behavior of a dynamical system by using the closeness of approximate orbits and true orbit. Moreover, there are extensive connections between these variants of shadowing and the chaos and stability; see [7, 11, 13].

Another variant of shadowing is the specification property in which one can approximate distinct finite pieces of orbits by an actual orbit with a certain uniformity. It was first introduced by Bowen[4] to study the ergodic property of Axiom A diffeomorphisms. It has been shown that every mapping with the specification property is chaotic in the sense of Devaney; see [1]. The authors in [7, 10], respectively, defined some kind of specification such as weak specification and pseudo-orbital specification properties for a continuous map on a compact metric space and studied their relations with other dynamical properties.

Recently, Osipov and Tikhomirov [15] introduced the notion of shadowing property for the finitely generated group action. The continuous actions associated with finitely generated semigroups on compact metric spaces are also called iterated function systems(IFSs). IFSs are widely used for the construction of deterministic fractals; see [3]. Over the last ten years or so, many research works have been devoted to the applications of IFSs in image processing theory [6], in the theory of stochastic growth models [8], and in the theory of random dynamical systems [14]. The shadowing property of IFSs was explored in [2, 9, 20]. Bahabadi [2] introduced the notions of shadowing and average shadowing properties for free semigroup actions(IFSs). He obtained that a semigroup with average shadowing property

[1]Assistant professor, Department of Mathematics, Faculty of Mathematics, Statistics and Computer Science, University of Sistan and Baluchestan, Zahedan, Iran, e-mail: zshabani@math.usb.ac.ir

[2]Assistant professor, Department of Mathematics, Faculty of Mathematics, Statistics and Computer Science, University of Sistan and Baluchestan, Zahedan, Iran.

is chain transitive. Wu, Wang, and Liang [20] improved this result and showed that the average shadowing property for a semigroup actions implies chain mixing.

Ergodic shadowing and pseudo orbital specification properties for finitely generated semigroup actions were introduced in [17], and it was proved that these properties are equivalent to the semigroup being topologically mixing and having the ordinary shadowing property. In [21], the definition of shadowing property, average shadowing property, and topological ergodicity for iterated function system, were introduced and relation between average shadowing and topological ergodicity was studied. But these definitions of shadowing and average shadowing properties differ from the definitions given in [2]. In this paper, we introduce the definition of $\underline{d}$-shadowing and weak specification properties for finitely generated semigroup actions and study the connection between these properties with ergodic shadowing and average shadowing properties (in a similar way defined in [2] and [17], respectively).

The main aim of this paper is to prove the following result.

Theorem 1.1. *Let G be a semigroup generated by the family $\{id, g_1, \dots, g_m\}$ of continuous maps on the compact metric space X where g_i is surjective for some $i \in \{1, \dots, m\}$. Then the following properties on G are equivalent:*

(1) *ergodic shadowing,*
(2) *$\underline{d}$-shadowing and ordinary shadowing,*
(3) *chain mixing and ordinary shadowing,*
(4) *topologically mixing and ordinary shadowing,*
(5) *average shadowing property and ordinary shadowing,*
(6) *weak specification and ordinary shadowing,*
(7) *pseudo-orbital specification.*

Notice that the definitions of shadowing and ergodic shadowing properties of continuous map f and also for the semigroup actions on a compact metric space depend on the metrics on non-compact metric spaces (see [12] and Example 6.1). In [12], the authors introduced the new notions of ϵ-chain and the shadowing property for homeomorphisms on non-compact metric spaces, which are dynamical properties, that is, conjugacy invariant and equivalent to the classical definitions in case of compact metric spaces. The authors in [18], extended the notions of ϵ-chain and shadowing property defined in [12] to the case of finitely generated semigroup actions on non-compact metric spaces and showed that the new notions of ϵ-chain and shadowing on a non-compact metric space X are independent of metrics on X. Here, we introduce the definition of ergodic shadowing property of semigroup actions on non-compact metric spaces which is conjugacy invariant, and we obtain some results.

This paper is organized as follows. Sec. 2 contains the main definitions and background information. In particular, we introduce the notions of the $\underline{d}$-shadowing and weak specification properties for finitely generated semigroup actions. In Sec. 3, we show that any semigroup with $\underline{d}$-shadowing property is chain mixing. Moreover, we build a semigroup with $\underline{d}$-shadowing that does not have the shadowing property and so it does not have the ergodic shadowing property. In sec. 4, we show that for a semigroup with shadowing property, the notions of average shadowing and topologically mixing properties are equivalent. The relation between weak specification and pseudo orbital specification properties, is discussed in sec. 5, and the proof of Theorem 1.1 is given in this section. In sec. 6, we present an example that shows that the shadowing and ergodic shadowing properties of semigroup actions on a compact metric space depend on the metrics on non-compact metric spaces and then, we define a new notion for the ergodic shadowing property of semigroup actions on non-compact metric spaces which is a dynamical property.

2. **Preliminaries**

In this section, we describe the free semigroup actions and state some notation and definitions. Throughout this paper, let $\mathbb{N} = \{1, 2, 3, \dots\}$ and let $\mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}$. Given a finitely generated semigroup G with a finite set of generators $G_1 = \{id, g_1, g_2, \dots, g_m\}$, where $g_i : X \to X, i \in \mathcal{P} = \{1, \dots, m\}$, is a continuous self-map on the compact metric space (X, d). We write $G = \bigcup_{n\in\mathbb{Z}^+} G_n$, where $G_0 = id$ and

$$G_n = \{g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1} : g_{i_j} \in G_1\}. \tag{1}$$

Indeed, G_n consists of elements that are concatenations of at the most n elements of G_1.

Let $\mathbb{F}_m$ be the free semigroup with generators $\{1, \dots, m\}$. One way to interpret this statement is to consider the itinerary map $\iota : \mathbb{F}_m \to G$ given by

$$w = w_1 w_2 \dots w_n \to g_w^n = g_{w_1} \circ \cdots \circ g_{w_n}$$

and to regard concatenations on G as images by ι of paths on $\mathbb{F}_m$.

Let $G_1 = \{id, g_1, \dots, g_m\}$ be a finite collection of continuous maps on the compact metric space X. Th symbolic dynamic is a way to display the elements of semigroup G associated with this family. Let Σ^m be the space of infinite sequences of m symbols $\{1, \dots, m\}$, that is, $\Sigma^m = \{\omega = \omega_0\omega_1\omega_2 \dots : \omega_i \in \{1, \dots, m\}\}$. For any sequence $\omega = \omega_0\omega_1\omega_2 \dots \in \Sigma^m$, take $g_\omega^0 := id$ and for any $n > 0$, $g_\omega^n(x) := g_{\omega_{n-1}} \circ \cdots \circ g_{\omega_0}(x)$. Let $\mathcal{A}^m$ be a set of finite words of symbols $\{1, \dots, m\}$, that is, if $w \in \mathcal{A}^m$, then $w = w_0 \dots w_{l-1}$, where $w_i \in \{1, \dots, m\}$ for all $i = 0, \dots, l-1$. Also, for $0 \le i \le l-1$, we denote $g_w^i := g_{w_{i-1}} \circ \cdots \circ g_{w_0}$.

Let (X, d) be a compact metric space and let G be a semigroup associated with the finite family $\{id, g_1, \dots, g_m\}$ of continuous self maps on X. Given $w = w_0 \dots w_{n-1} \in \mathcal{A}^m$ and $\epsilon > 0$. An (ϵ, w)-*chain* of semigroup G from x to y is a finite sequence $x_0 = x, x_1, \dots, x_n = y$ such that

$$d(g_{w_i}(x_i), x_{i+1}) < \epsilon \quad \text{for any } i = 0, \dots, n-1.$$

We say that G is *chain transitive* if for any $x, y \in X$ and any $\epsilon > 0$, there exists an ϵ-chain from x to y. Also G is called *chain mixing* if for any two points $x, y \in X$ and any $\epsilon > 0$, there is a positive integer N such that for any integer $n \ge N$, there is an ϵ-chain from x to y of length n. We say that the semigroup G is *topologically mixing*, if for any two open subsets U and V of X, there exists an integer $N \in \mathbb{N}$, such that for any $n \ge N$, $g_\omega^n(U) \cap V \ne \emptyset$, for some $\omega \in \Sigma^m$.

For any $A \subset \mathbb{Z}^+$, the *density* of A is defined by

$$d(A) = \lim_{n\to\infty} \frac{1}{n} |A \cap \{0, 1, \dots, n-1\}|. \tag{2}$$

Replacing lim with lim inf in (2) gives the definition of $\underline{d}(A)$, the lower density of A.

For a sequence $\xi = \{x_i\}_{i\ge0}$, $\delta > 0$, and $\omega = \omega_0\omega_1\omega_2 \dots \in \Sigma^m$, put

$$Np(\xi, G, \omega, \delta) = \{i \in \mathbb{Z}^+ : d(g_{\omega_i}(x_i), x_{i+1}) \ge \delta\},$$
$$Np^c(\xi, G, \omega, \delta) = \mathbb{Z}^+ \setminus Np(\xi, G, \omega, \delta),$$

and

$$Np_n(\xi, G, \omega, \delta) = Np(\xi, G, \omega, \delta) \cap \{0, \dots, n-1\}.$$

To simplify the notation, we denote them by $Np(\xi, \omega, \delta)$ and $Np_n(\xi, \omega, \delta)$, respectively. Given a sequence $\xi = \{x_i\}_{i\ge0}$ and a point $z \in X$, consider

$$Ns(\xi, \omega, z, \delta) = \{i \in \mathbb{Z}^+ : d(g_\omega^i(z), x_i) \ge \epsilon\},$$
$$Ns^c(\xi, \omega, z, \delta) = \mathbb{Z}^+ \setminus Ns(\xi, \omega, z, \delta),$$

and

$$Ns_n(\xi, \omega, z, \delta) = Ns(\xi, \omega, z, \delta) \cap \{0, \dots, n-1\}.$$

Definition 2.1. *Let $\delta > 0$ and let $\xi = \{x_i\}_{i\geq 0} \subset X$. We have the following concepts:*

(1) *ξ is a (δ, ω)-*pseudo orbit *of G for some $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$, if for any $i \in \mathbb{Z}^+$, $d(g_{\omega_i}(x_i), x_{i+1}) < \delta$; see* [2].

(2) *ξ is a (δ, ω)-*ergodic pseudo orbit *of G for some $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$ provided that the set $Np(\xi, \omega, \delta)$ has zero density (see* [17]*), that is,*

$$\lim_{n\to\infty} \frac{|Np_n(\xi, \omega, \delta)|}{n} = 0.$$

(3) [2] *ξ is a (δ, ω)-average pseudo orbit of G for some $\omega = \omega_0\omega_1\omega_2 \ldots \in \Sigma^m$, if there is $N \in \mathbb{N}$ such that for any $n \geq N$,*

$$\frac{1}{n} \sum_{i=0}^{n-1} d(g_{\omega_i}(x_i), x_{i+1}) < \delta.$$

Remark 2.1. Clearly, every orbit $\{g^n_\omega(x)\}_{n\geq 0}$ is a (δ, ω)-pseudo orbit, and every (δ, ω)-pseudo orbit is a (δ, ω)-ergodic pseudo orbit. Moreover a (δ, ω)-ergodic pseudo orbit may be represented as

$$\xi = \{x_0, x_1, x_2, \ldots x_{m_1}; x_{m_1+1}, x_{m_1+2}, \ldots, x_{m_2}; x_{m_2+1}, x_{m_2+2}, \ldots\}$$

where, $\{x_{m_i+1}, x_{m_i+2}, \ldots, x_{m_{i+1}}\}, i \in \mathbb{Z}^+$ are finite (δ, w^i)-chains with $w^i = \omega_{m_i+1}\omega_{m_i+2} \ldots \omega_{m_{i+1}-1} \in \mathcal{A}^m$, $m_0 = -1$, and $\{m_i\}_{i\in\mathbb{Z}^+}$ has zero density.

Now, we use the above notions of approximate trajectories to define the various types of shadowing properties of this paper.

Definition 2.2. (1) [2] *A semigroup G has the* shadowing property*, provided that for every $\epsilon > 0$, there exists $\delta > 0$ such that, for any (δ, ω)-pseudo orbit ξ of G, there is a point $z \in X$ such that for any $i \in \mathbb{Z}^+$,*

$$d(g^i_\omega(z), x_i) < \epsilon.$$

(2) [17] *A semigroup G has the* ergodic shadowing property *if for each $\epsilon > 0$ there exists $\delta > 0$ such that every (δ, ω)-ergodic pseudo orbit ξ of G can be ϵ-ergodic shadowed by some point z in X, that is, there exists $\varphi \in \Sigma^m$ with $\varphi_i = \omega_i$ for $i \in Np^c(\xi, \omega, \delta)$, such that*

$$\lim_{n\to\infty} \frac{|Ns_n(\xi, \varphi, z, \epsilon)|}{n} = 0.$$

(3) *A semigroup G has the* $\underline{d}$-shadowing property *if for each $\epsilon > 0$ there exists $\delta > 0$ such that every (δ, ω)-ergodic pseudo orbit ξ of G can be $\underline{d}$-ϵ-shadowed by some point z in X, that is, there exists $\varphi \in \Sigma^m$ with $\varphi_i = \omega_i$ for $i \in Np^c(\xi, \omega, \delta)$ such that*

$$\underline{d}(Ns(\xi, \varphi, z, \epsilon)) > 0.$$

(4) [2] *A semigroup G has the average shadowing property if for any $\epsilon > 0$, there is $\delta > 0$ such that every (δ, ω)-average pseudo orbit ξ of G is ϵ-shadowed on average by a point $z \in X$, that is, there is $\varphi \in \Sigma^m$ such that*

$$\frac{1}{n} \sum_{i=0}^{n-1} d(g^i_\varphi(z), x_i) < \epsilon.$$

Now, we introduce the notion of weak specification property for the context of semigroup actions.

Definition 2.3. *We say that the semigroup G has weak specification property if for any $\epsilon > 0$, there exists $N(\epsilon) > 0$ such that for any set $\{x_1, \ldots, x_k\}$ of points of X, any sequence of nonnegative integers $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$ with $a_{j+1} - b_j \geq N(\epsilon)$, and any*

$w^j = w_{a_j} \dots w_{b_j - 1} \in \mathcal{A}^m$, $(1 \leq j \leq k)$, *there exist a point* $z \in X$ *and* $\omega \in \Sigma^m$ *with* $\omega_i = w_i$ *for any* $a_j \leq i \leq b_j - 1$ *such that*

$$d(g^i_\omega(z), g^{i-a_j}_{w^j}(x_j)) < \epsilon, \quad \textit{for any} \quad a_j \leq i \leq b_j, \quad 1 \leq j \leq k.$$

In the following, we recall the stronger notion of specification for the semigroup G, which is called pseudo-orbital specification property and is equivalent to ergodic shadowing property (see [17]).

Definition 2.4. *We say that a semigroup* G *has the pseudo-orbital specification property if for any* $\epsilon > 0$, *there exist* $\delta = \delta(\epsilon) > 0$ *and* $N(\epsilon) > 0$ *such that for any nonnegative integers* $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$ *with* $a_{j+1} - b_j \geq N(\epsilon)$ *and* (δ, w^j)*-pseudo orbits* ξ_j *with* $\xi_j = \{x_{(j,i)}\}, a_j \leq i \leq b_j$, $1 \leq j \leq k$, *and* $w^j = w^j_{a_j} \dots w^j_{b_j-1} \in \mathcal{A}^m$, *there exist a point* $z \in X$ *and* $\omega \in \Sigma^m$ *with* $\omega_i = w^j_i$ *for* $a_j \leq i \leq b_j - 1$ *and* $1 \leq j \leq k$, *such that*

$$d(g^i_\omega(z), x_{(j,i)}) < \epsilon, \quad \textit{for any} \quad a_j \leq i \leq b_j, \quad 1 \leq j \leq k.$$

It is clear from the definition that the pseudo-orbital specification property implies the weak specification property.

3. $\underline{d}$-shadowing property of semigroup actions

Let X be a compact metric space and let G be a finitely generated semigroup associated with $\{id, g_1, \dots, g_m\}$, of continuous maps on X. The aim of this section is to show that every semigroup with $\underline{d}$-shadowing property is chain mixing. Moreover, the connection between the $\underline{d}$-shadowing and ergodic shadowing properties is studied.

First, we show that the $\underline{d}$-shadowing property of semigroups implies chain transitivity.

Theorem 3.1. *Suppose that* $g_1, \dots, g_m$ *are continuous maps on a compact metric space* X *such that one of them is surjective. If the semigroup* G *generated by these maps has the* $\underline{d}$*-shadowing property, then it is chain transitive.*

Proof. Let $x, y \in X$ and $\epsilon > 0$ be given. Let $\delta > 0$ be as in the definition of the $\underline{d}$-shadowing property. Set $m_0 = -1$ and consider $M = \{m_i\}_{i \in \mathbb{N}}$ an increasing subsequence of natural numbers with $d(M) = 0$ such that $\mathbb{N}$ partitions into two subsets M_1 and M_2 satisfying

$$M_1 = \{0, 1, 2, \dots, m_1\} \cup \{m_2 + 1, m_2 + 2, \dots, m_3\} \cup \{m_4 + 1, m_4 + 2, \dots, m_5\} \cup \dots$$

and $M_2 = \mathbb{N} \setminus M_1$ with $\overline{d}(M_1) = \overline{d}(M_2) = 1$. Fix a sequence $\omega = \omega_0 \omega_1 \ldots \in \Sigma^m$. Suppose that $g_\ell, \ell \in \{1, \dots, m\}$ is surjective. For each $j \geq 0$, consider the sequence

$$y_i := \begin{cases} g_\omega^{i - m_{2j} - 1}(x), & m_{2j} + 1 \leq i \leq m_{2j+1}, \\ g_\ell^{i - m_{2j+2}}(y), & m_{2j+1} + 1 \leq i \leq m_{2j+2}, \end{cases}$$

which is the orbit of x on $[m_{2j} + 1, m_{2j+1}]$ and a piece of backward orbit of y on $[m_{2j+1} + 1, m_{2j+2}]$. Then $\{y_i\}_{i \geq 0}$ is a (δ, η)-ergodic pseudo orbit of G with $\eta = \eta_0 \eta_1 \eta_2 \ldots \in \Sigma^m$,

$$\eta_i = \begin{cases} \omega_i, & m_{2j} + 1 \leq i < m_{2j+1}, \\ t, & i \in \{m_i\}_{i \in \mathbb{N}} \\ \ell, & m_{2j+1} + 1 \leq i < m_{2j+2}, \end{cases}$$

for some $t \in \{1, \dots, m\}$. Since the semigroup G has $\underline{d}$-shadowing property, there are a sequence $\gamma \in \Sigma^m$ and a point $z \in X$ such that $Ns(\{y_i\}_{i \geq 0}, \gamma, z, \epsilon)$ has positive lower density, which implies that $Ns(\{y_i\}_{i \geq 0}, \gamma, z, \epsilon) \cap M_i \neq \emptyset$, for $i = 1, 2$. Thus we can find positive integers i, j, r, s with $r < s$ such that

$$d(g^r_\gamma(z), g^i_\omega(x)) < \epsilon, \quad d(g^s_\gamma(z), g^{-j}_\ell(y)) < \epsilon.$$

One can see that the

$$x, g_{\omega_0}(x), \dots, g_{\omega}^{i-1}(x), g_{\gamma}^{r}(z), g_{\gamma}^{r+1}(z), \dots, g_{\gamma}^{s-1}(z), g_{\ell}^{-j}(y), g_{\ell}^{-j+1}(y), y$$

is an (ϵ, ζ)-chain from x to y with

$$\zeta = \omega_0\omega_1 \dots \omega_{i-1}\gamma_r\gamma_{r+1} \dots \gamma_{s-1} \underbrace{\ell \dots \ell}_{j \text{ times}}.$$

□

Let G be a finitely generated semigroup with the finite set of generators $G_1 = \{id, g_1, \dots, g_m\}$. Take $G_1^* = G_1 \setminus id$, and for any $k \in \mathbb{N}$ let G_k^* denote the space of compositions of k elements in G_1^*. That is, $g \in G_k^*$ if $g = g_w^k$ for some finite word $w \in \mathcal{A}^m$ with length k. Denote by G^k the semigroup generated by G_k^*.

In the following, we prove the invariance of $\underline{d}$-shadowing property of the semigroup G under iterations.

Theorem 3.2. *If a semigroup G has the $\underline{d}$-shadowing property, then the semigroup G^k has the$\underline{d}$-shadowing property for any $k \in \mathbb{N}$.*

Proof. Given fixed $\epsilon > 0$ and $k \in \mathbb{N}$, the uniform continuity of mappings g_i, $i = 1, \dots, m$ implies that there exists $\eta < \epsilon/k$ such that $d(x, y) < \eta$ implies that $d(g_w^i(x), g_w^i(y)) < \epsilon/k$, for any $i = 1, \dots, m$ and $w \in \mathcal{A}_k^m$. Thus, for any finite (η, w)-chain $\{r_0, \dots, r_k\}$, we have $d(g_w^i(r_0), r_i) < \epsilon/2, i = 0, 1, \dots, k$. Let $\delta < \eta$ be an η-modulus of ergodic shadowing property for the semigroup G. Suppose that $\xi = \{x_0, x_1, \dots, x_{m_0}; x_{m_0+1}, \dots, x_{m_1}; x_{m_1+1}, \dots\}$ is a δ-ergodic pseudo orbit for the semigroup G^k. It can be verified that there is a sequence $\omega = \omega_0\omega_1 \dots \in \Sigma^m$ such that

$$\{i \in \mathbb{Z}^+; d(g_{\omega_{(i+1)k-1}} \circ \dots \circ g_{\omega_{ik}}(x_i), x_{i+1}) \geq \delta\} \subseteq \{m_i\}_{i \in \mathbb{Z}^+}$$

Take a sequence $\{y_i\}_{i \geq 0}$ with

$$y_i := \begin{cases} x_{i/k}, & i = nk \quad \text{for some} \quad n \in \mathbb{Z}^+ \\ g_{\sigma^{nk}\omega}^{j}(x_n), & i = nk + j \quad \text{for some} \quad n \in \mathbb{Z}^+, 0 < j < k, \end{cases}$$

that is,

$$\begin{aligned} \{y_i\}_{i \geq 0} = \{&x_0, g_{\omega_0}(x_0), g_{\omega_1} \circ f_{\omega_0}(x_0), \dots, g_{\omega_{k-2}} \circ \dots \circ g_{\omega_0}(x_0), \\ &x_1, g_{\omega_k}(x_1), g_{\omega_{k+1}} \circ g_{\omega_k}(x_1), \dots, g_{\omega_{2k-2}} \circ \dots \circ g_{\omega_k}(x_1), x_2, \dots \\ &\vdots \\ &x_n, g_{\omega_{kn}}(x_n), g_{\omega_{kn+1}} \circ g_{\omega_{kn+1}}(x_n), \dots, g_{\omega_{(n+1)k-2}} \circ \dots \circ g_{\omega_{kn}}(x_n), x_{n+1}, \dots\} \end{aligned}$$

is a δ-ergodic pseudo orbit for the semigroup G. Since the semigroup G has the $\underline{d}$-shadowing property, there are a point $z \in X$ and $\gamma \in \Sigma^m$ such that $\{y_i\}_{i \geq 0}$ is $\underline{d}$-η-shadowed by $\{g_\gamma^i(z)\}_{i \geq 0}$. Since $M = \{m_i\}_{i \in \mathbb{Z}^+}$ has zero density, we can assume that $Ns(\{y_i\}_{i \geq 0}, \gamma, z, \eta) \cap M = \emptyset$. If ever, $m \in Ns(\{y_i\}_{i \geq 0}, \gamma, z, \eta) \cap M$, then $d(g_\gamma^m(z), y_m) < \eta$. Let $m = nk + j$ for some $n \in \mathbb{Z}^+$ and $0 < j < k$. Then

$$\{g_\gamma^{m-1}(z), g_{\sigma^{nk}\omega}^{j}(x_n), g_{\sigma^{nk}\omega}^{j+1}(x_n), \dots, g_{\sigma^{nk}\omega}^{k-1}(x_n), x_{n+1}\}$$

is an (η, w)-chain with $w = \gamma_{m-1}\omega_{nk+1} \dots \omega_{(n+1)k-1}\omega_{(n+1)k} \in \mathcal{A}_k^m$. Therefore, we have

$$d(g_\eta^{m+k-j}(z), x_{n+1}) = d(g_w^{k-j}(g_\gamma^{m-1}(z)), x_{n+1}) < \epsilon.$$

Set $B_1 := Ns(\{y_i\}_{i \geq 0}, \gamma, z, \eta)$. With out loss of generality, suppose that $d(z, x_0) < \epsilon$. By an induction argument, we will show that the sequence ξ is (η, ϵ)-traced by the orbit of z under the semigroup G^k along a set $B \subset \mathbb{Z}^+$ with $k\underline{d}(B) \geq \underline{d}(B_1)$. By the definition, x_n is the kn'th element of $\{y_i\}_{i \geq 0}$. Let $\frac{|B \cap \{0,1,2,\dots,n-1\}|}{n} = \frac{q_n}{n}$ and $\frac{|B_1 \cap \{0,1,2,\dots,kn-1\}|}{n} = \frac{p_n}{kn}$ for some

$p_n, q_n \in \mathbb{N}$. Then it is enough to show that $k\frac{q_n}{n} \geq \frac{p_n}{kn}$ for $n \in \mathbb{N}$. Since $d(z, x_0) < \epsilon$, the first step of induction is established. Assume that $k\frac{q_n}{n} \geq \frac{p_n}{kn}$. We will show that

$$k\frac{q_{n+1}}{n+1} \geq \frac{p_{n+1}}{k(n+1)}. \tag{3}$$

We observe that either $q_{n+1} = q_n$ or $q_{n+1} = q_n + 1$. If $q_{n+1} = q_n$, then this means that x_{n+1} in ξ is not ϵ-shadowed by the orbit of z under the semigroup G^k and hence by the above argument x_{n+1} is not ϵ-shadowed by the γ-orbit of z under the semigroup G in sequence $\{y_i\}_{i\geq 0}$ either. So $p_{n+1} = p_n$ and (3) holds. If $q_{n+1} = q_n + 1$, then $p_{n+1} \leq p_n + k$ and again (3) holds. This terminates the proof. □

Corollary 3.1. *Let G be a semigroup generated by a finite family of continuous self-maps on a compact metric space X such that one of the generators is a surjective map. If G has $\underline{d}$-shadowing property, then it is chain mixing.*

Proof. Since G has the $\underline{d}$-shadowing property by Theorem 3.2, for any $k \in \mathbb{N}$, G^k has the $\underline{d}$-shadowing property, so applying Theorem 3.1, we obtain that it is chain transitive and therefore G is chain mixing by [20, Theorem 2.3]. □

Clearly every finitely generated semigroup G with the ergodic shadowing property has the $\underline{d}$-shadowing property. The next example shows that the ergodic shadowing property is strictly stronger than the $\underline{d}$-shadowing of finitely generated semigroups on compact metric spaces.

Example 3.1. *Here, we construct a semigroup with $\underline{d}$-shadowing property that does not have the ergodic shadowing property. Let $g_1, g_2 : [0,1] \to [0,1]$ be two maps defined by*

$$g_1(x) = \sqrt{1-(x-1)^2}, \quad g_2(x) = \sqrt[3]{1-(x-1)^3}.$$

By identifying 0 with 1, g_1 and g_2 are homeomorphisms on S^1 with unique fixed point $x = 0$ and the orbits of any points except zero move by these maps on S^1 counter-clockwise and attract to $x = 0$. Let G be the semigroup generated by $\{id, g_1, g_2\}$. Since g_1 and g_2 do not have the shadowing property, so the semigroup G does not have the shadowing and ergodic shadowing properties (see [2, 17]). Now we show that G has the $\underline{d}$-shadowing property. Let $0 < \epsilon < 1/2$ be given. Since for any $x \in [\epsilon, 1-\epsilon]$ and any $\omega \in \Sigma^m$, after a finite iterate $n \in \mathbb{Z}^+$, $g_\omega^n(x)$ leaves $[\epsilon, 1-\epsilon]$, so we can choose $\delta < \epsilon/2$ sufficiently small such that every (δ, w)-chain with $w \in \mathcal{A}^m$ has at most $N(\epsilon,\delta)$-elements in $[\epsilon, 1-\epsilon]$. Let $\xi = \{x_i\}_{i\geq 0}$ be a (δ, ω)-ergodic pseudo orbit. Then

$$x_0, x_1, \ldots, x_{m_1}; x_{m_1+1}, x_{m_1+1}, \ldots, x_{m_2}; x_{m_2+1}, x_{m_2+2}, \ldots,$$

where $\omega \in \Sigma^m$, $m_0 = -1$, and $\{m_i\}_{i\geq 1}$ is a sequence with zero density. Since every δ-chain of length $N(\epsilon,\delta)+1$ has some element in $S^1 \setminus [\epsilon, 1-\epsilon]$, so every δ-chain $\xi_i = x_{m_i+1}, x_{m_i+1}, \ldots, x_{m_{i+1}}, i \geq 0$ has at least $[(m_{i+1}-m_i)/N(\epsilon,\delta)]$ elements in $S^1 \setminus [\epsilon, 1-\epsilon]$. Set

$$\Lambda_n := \{i \in \{0, 1, \ldots, n-1\} : x_i \in S^1 \setminus [\epsilon, 1-\epsilon]\}.$$

Then

$$|\Lambda_n| \geq \sum_{i=1}^{m_\ell} \frac{m_{i+1}-m_i}{N(\epsilon,\delta)},$$

where $m_\ell = \max_{i\geq 0}\{m_i : m_i \leq n\}$. Put $\Lambda = \cup_{n\in\mathbb{N}} \Lambda_n$. Therefore $d(\Lambda) = 1/N(\epsilon,\delta)$. Let $y = 1-\epsilon/2$; then for any $j \in \Lambda$, we have $d(g_\omega^j(y), x_j) < 2\epsilon$, which implies that G has the $\underline{d}$-shadowing property.

4. **Average shadowing property of semigroup actions**

This section is devoted to show under the assumption of the shadowing property of semigroup G, average shadowing and topologically mixing properties are equivalent. We first prove that every semigroup with shadowing and topologically mixing properties has the average shadowing property.

Theorem 4.1. *If a semigroup G generated by a finite family of continuous self-maps $\{g_i : i = 1, \ldots, m\}$ has the shadowing and topologically mixing properties, then it has the average shadowing property.*

Proof. Without loss of generality, we can assume that $\text{diam}X < 1$. Let $\epsilon > 0$ be given and let $\delta > 0$ be provided by the shadowing property for $\epsilon/2$. Since the mappings g_i are uniformly continuous, there exists $\eta < \delta$ such that $d(x, y) < \eta$ implies that $d(g_i(x), g_i(y)) < \delta$ for any $1 \leq i \leq m$ and $x, y \in X$. Since X is compact, we can find a finite open cover $\mathcal{U} = \{U_1, \ldots, U_M\}$ of X composed of open balls of radius $\eta/2$. For any two open sets $U_i, U_j \in \mathcal{U}$, choose an integer $N_{ij} > 0$ and a sequence $\omega^{ij} \in \Sigma^m$ such that $f^n_{\omega^{ij}}(U_i) \cap U_j \neq \emptyset$ for $n \geq N_{ij}$. Take $N := \max\{N_{ij} : 1 \leq i, j \leq M\}$. Choose $M > 0$ such that $2N/M < \epsilon/2$, and put $\lambda = \delta/M$. Let $\xi = \{x_i\}_{i\geq 0}$ be a (δ, ω)-average pseudo orbit for G. We can construct a sequence

$$0 = a_0 \leq b_0 < a_1 \leq b_1 < \cdots$$

of natural numbers such that the following conditions hold:

(1) $[a_j, b_j] \subset Np^c(\xi, \omega, \lambda)$ for each $j \in \mathbb{Z}^+$,
(2) $a_{j+1} - b_j = N$ for each $j \in \mathbb{Z}^+$.

Then $\xi_j = \{x_{a_j}, x_{a_j+1} \ldots, x_{b_j}\}, j \geq 0$, are (λ, ω^j)-pseudo orbits in X with $\omega^j = \omega_{a_j} \ldots \omega_{b_j-1} \in \mathcal{A}^m$. For any $x \in X$, we denote by $U(x)$ the open set $U \in \mathcal{U}$ containing x. By the choice of N, there exists a sequence $\varphi^j \in \{\omega^{ij} : 1 \leq i, j \leq M\} \subset \Sigma^m$ such that

$$g^N_{\varphi^j}(U(x_{b_j})) \cap U(x_{a_{j+1}}) \neq \emptyset.$$

Therefore there is $y_j \in U(x_{b_j})$ such that $g^N_{\varphi^j}(y_j) \in U(x_{a_{j+1}})$. Now consider the sequence $\gamma = \gamma_0\gamma_1 \ldots \in \Sigma^m$ with

$$\gamma_\ell = \begin{cases} \omega^j_\ell, & a_j \leq \ell \leq b_j - 1, \\ \varphi^j_{\ell-b_j}, & b_j \leq \ell \leq a_{j+1} - 1, \end{cases}$$

and $\zeta_j = \{g_{\varphi^j}(y_j), \ldots, g^N_{\varphi^j}(y_j)\}$. One can see that

$$\{\xi_0, \zeta_0, \xi_1, \zeta_1, \ldots\}$$

is a (δ, γ)-pseudo orbit and can be $\epsilon/2$-shadowed by a point $z \in X$. In particular, for any $j \in [a_j, b_j], j \geq 0$, we have $d(g^j_\gamma(z), x_j) < \epsilon/2$. Now, we show that z, (γ, ϵ)-shadows in the average λ-pseudo orbit $\{x_i\}_{i\geq 0}$. Let $L > 0$ be such that for any $n \geq L$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(g_{\omega_i}(x_i), x_{i+1}) < \delta.$$

Let $L = rM + s$ for some $r \geq 0$ and $0 \leq s < M$. Thus $|Np_n(\{x_i\}_{i\geq 0}, \omega, \lambda)| < s + 1$, since otherwise

$$\frac{1}{n} \sum_{i=0}^{n-1} d(g_{\omega_i}(x_i), x_{i+1}) \geq (r + 1)\delta/L \geq \delta/M = \lambda.$$

Moreover,

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} d(g_\gamma^i(z),x_i) &\leq \frac{\epsilon}{2n}(n-|Ns_n(\{x_i\}_{i\geq 0},\gamma,z,\epsilon)|)+\frac{1}{n}|Ns_n(\{x_i\}_{i\geq 0},\gamma,z,\epsilon)| \\ &\leq \frac{\epsilon}{2}+\frac{N}{n}|Np_n(\{x_i\}_{i\geq 0},\omega,\delta)| \\ &\leq \frac{\epsilon}{2}+\frac{N}{tM}(r+1)\leq \frac{\epsilon}{2}+\frac{2N}{M}<\epsilon,\end{aligned}$$

which completes the proof. □

Corollary 4.1. *Let G be a semigroup generated by the family $\{id, g_1, \dots, g_m\}$ on a compact metric space X, satisfies that g_i is surjective for some $i \in \{1,\dots,m\}$ and let G have the shadowing property. Then G is topologically mixing if and only if it has the average shadowing property.*

Proof. By [20, Corollary 3.3], if G has the average shadowing and ordinary shadowing properties, then it is topologically mixing. This, together with Theorem 4.1, terminates the proof. □

5. **Weak specification property of semigroup action**

In this section, the relation between weak specification and pseudo-orbital specification properties is studied. It is clear from the definition that any semigroup with the pseudo-orbital specification property, has the weak specification property. In the following, we present a finitely generated semigroup action with the weak specification property, that does not have the pseudo-orbital specification property.

Example 5.1. *Let $X = S^1$, let g_1 be any C^1-expanding map on X and $g_2 := R_\alpha : X \to X$ be the rotation of the angle α. Let G be a semigroup with generating set $\{id, g_1, g_2\}$. The semigroup G does not have the shadowing property, as g_2 does not have not the shadowing property. Therefore it does not have the pseudo orbital specification property(since the pseudo-orbital specification property implies shadowing* [17, Theorem 1.1]*). We show that G has the weak specification property. Since g_1 is an expanding map by* [19, Lemma 11.2.7]*, There exists $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$, any $x \in X$, and any $n \in \mathbb{N}$, $g_1^n(B(x,n,\epsilon)) = B(g_1^n(x),\epsilon)$, where,*

$$B(x,n,\epsilon) := \{y \in X, \max_{0\leq i\leq n} d(g_1^i(x), g_1^i(y)) < \epsilon\}.$$

Also for any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that $g_1^N(B(x,\epsilon)) = S^1$ for any $x \in X$. By this observation, for given $\epsilon > 0$, any set $\{x_1,\dots,x_k\}$ of points of X, any sequence of nonnegative integers $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$ with $a_{j+1} - b_j \geq N(\epsilon)$, and any $w^j = w_{a_j}\dots w_{b_j-1} \in \mathcal{A}^m$, $(1 \leq j \leq k)$, we can find a point $z \in X$ such that for $\omega \in \Sigma^m$ with

$$\omega_i := \begin{cases} w_i, & i \in [a_j, b_j - 1], \\ 1, & i \in \mathbb{Z}^+ \setminus [a_j, b_j - 1], \end{cases}$$

we have

$$d(g_\omega^i(z), g_{w^j}^{i-a_j}(x_j)) < \epsilon, \quad \textit{for any} \quad a_j \leq i \leq b_j, \quad 1 \leq j \leq k.$$

Here, we shall show that by assuming the shadowing property for finitely generated semigroup G, weak specification property implies the pseudo-orbital specification property.

Proposition 5.1. *If a semigroup G associated with the family of continuous self maps $\{id, g_1, \dots, g_m\}$ satisfies that g_i is surjective for some $i \in \{1,\dots,m\}$ and has the shadowing and weak specification properties, then it has the pseudo-orbital specification property.*

Proof. Let $\epsilon > 0$ be given and let $\eta < \epsilon/2$. Let $\delta > 0$ be an η-modulus of the shadowing property and let $N(\eta)$ be an η-modulus of the weak shadowing property of the semigroup G. For any sequence of nonnegative integers $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$ with $a_{j+1} - b_j \geq N(\eta)$ and (δ, w^j)-pseudo orbits ξ_j with $\xi_j = \{x_{(j,i)}\}, a_j \leq i \leq b_j$, $1 \leq j \leq k$, and $w^j = w^j_{a_j} \dots w^j_{b_j-1} \in \mathcal{A}^m$, since one of the generators g_i is surjective for some $i \in \{1, \dots, m\}$, we can extend ξ_j to a (δ, ω^j)-pseudo orbit ξ'_j with $\omega^j \in \Sigma^m$ and $\omega^j_i = w^j_i$ for $a_j \leq i \leq b_{j-1}, 1 \leq j \leq k$. By the shadowing property of G, there exits a point $z_j \in X$ that ϵ-shadows ξ'_j. In particular, for any $a_j \leq i \leq b_j, 1 \leq j \leq k$, we have

$$d(g^i_{\omega^j}(z_j), x_{(j,i)}) < \eta. \tag{4}$$

Set $x_j := g^{a_j}_{\omega^j}(z_j), 1 \leq j \leq k$. Since the semigroup G has the weak specification property, there exist a point $z \in X$ and $\omega \in \Sigma^m$ with $\omega_i = w^j_i = \omega^j_i$ for $a_j \leq i \leq b_{j-1}$, such that

$$d(g^i_\omega(z), g^{i-a_j}_{w^j}(x_j)) = d(g^i_\omega(z), g^i_{\omega^j}(z_j)) < \eta. \tag{5}$$

Thus (4) and (5) yield

$$d(g^i_\omega(z), x_{(j,i)}) < 2\eta < \epsilon.$$

This means that G has the pseudo orbital specification property. □

Now, we are ready to state the proof of Theorem 1.1.

Proof of Theorem 1.1. First, we mention that in [17, Theorem 1.1], it was proved that (1) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (7). So, we prove the remaining implications. It is clear from the definition that every semigroup with ergodic shadowing property has the $\underline{d}$-shadowing property, thus (1) implies (2). (2) $\Longrightarrow$ (3) follows from Corollary 3.1. By Corollary 4.1, we obtain (4) $\Longleftrightarrow$ (5). Using the fact that pseudo-orbital specification property implies weak specification property and Proposition 5.1 yield that (6) $\Longleftrightarrow$ (7). This terminates the proof. □

6. **The ergodic shadowing on non-compact metric spaces**

In this section, we define the notion of ergodic shadowing property of semigroup actions on non-compact metric spaces that is a dynamical property.

Definition 6.1. [18] *Let X and Y be two metric spaces. We say that two semigroups F and G with generating sets $\{id, f_1, \dots, f_m\}$ and $\{id, g_1, \dots, g_m\}$ on X and Y, receptively, are (topologically) conjugate if there is a homeomorphism $h : X \to Y$ such that $h \circ f_i = g_i \circ h$ for all $i = 1, \dots, m$. The homeomorphism h is called a conjugacy between F and G.*

A property P is called a *dynamical property* if a semigroup G has the property P, then any other semigroup F which is conjugate to G also has the property P.

Note that shadowing and ergodic shadowing properties of semigroup action on compact metric spaces are independent of metric and they are dynamical properties. However, they depend on the metrics on non-compact metric spaces, as we see in the following example.

Example 6.1. *Let $T : \mathbb{R} \to \mathbb{S}^1 \setminus \{(0,1)\}$ be a map given by*

$$T(t) = \left(\frac{2t}{1+t^2}, \frac{t^2-1}{t^2+1}\right), \quad \textit{for all } t \in \mathbb{R},$$

and let $X = T(\mathbb{Z})$. Let d be the metric on X induced by the Riemannian metric on S^1, and let d' be a discrete metric on X. It is clear that d and d' induce the same topology on X. Let $g_1 : X \to X$ be a homeomorphism defined by $g_1(a_i) = a_{i+1}$. Denote by G the finitely generated semigroup action associated with $\{id, g_1, g_2\}$, where g_2 is any homeomorphism on X. Since the metric d' is discrete, it is easy to see that G has the ergodic shadowing property *with respect to d'. We show that g_1 does not have the shadowing property with respect to d. Therefore the semigroup G does not have the shadowing property. By contradiction, let $g_1 : X \to X$ have the shadowing property. For $\varepsilon = \frac{1}{4}$, let $\delta > 0$ be an ε-modulus of the shadowing property of the mapping g_1. Choose $N_0 \in \mathbb{N}$ such that $d'(a_{N_0}, a_{-N_0}) < \frac{\delta}{2}$. For any $i \geq 0$, put $j := i$ mod $2N_0$. Then, the sequence $\{x_i\}_{i\geq 0}$ given by*

$$x_i = \begin{cases} a_j, & j \in \{0, 1, 2, \ldots, N_0 - 1\}, \\ a_{j-2N_0}, & j \in \{N_0, N_0 + 1, \ldots, 2N_0 - 1\}. \end{cases}$$

is a δ-pseudo orbit for g_1. So, there is a point $z \in X$ such that $d(g_1^i(z), x_i) < \varepsilon$, for any $i \geq 0$. Since for any $z \in X$, $g_1^i(z)$ attract to $(0, 1)$, so we can find an integer $i \in \mathbb{N}$ such that $d(g_1^i(z), x_i) \geq \varepsilon$, which is a contradiction. So, G does not have the shadowing and ergodic shadowing properties with respect to d.

Now, we define the notion of ergodic shadowing property for the finitely generated semigroup actions on non-compact metric spaces, which is independent of metrics.

Let $\mathcal{C}(X)$ be the collection of all continuous functions from X to $(0, \infty)$. Let X be a metrizable space and let G be a finitely generated semigroup action with the set of generators $\{id, g_1, \ldots, g_m\}$. For a sequence $\xi = \{x_i\}_{i\geq 0} \subset X$, $\delta \in \mathcal{C}(X)$, and $\omega = \omega_0\omega_1\omega_2 \ldots \in \Sigma^m$, put

$$Np(\xi, G, \omega, \delta) = \{i \in \mathbb{Z}^+ : d(g_{\omega_i}(x_i), x_{i+1}) \geq \delta(g_{\omega_i}(x_i))\},$$

$$Np^c(\xi, G, \omega, \delta) = \mathbb{Z}^+ \setminus Np(\xi, G, \omega, \delta),$$

and

$$Np_n(\xi, G, \omega, \delta) = Np(\xi, G, \omega, \delta) \cap \{0, \ldots, n - 1\}.$$

Given a sequence $\xi = \{x_i\}_{i\geq 0}$ and a point $z \in X$, consider

$$Ns(\xi, G, \omega, z, \delta) = \{i \in \mathbb{Z}^+ : d(g_\omega^i(z), x_i) \geq \delta(g_\omega^i(z))\},$$

$$Ns^c(\xi, G, \omega, z, \delta) = \mathbb{Z}^+ \setminus Ns(\xi, G, \omega, z, \delta),$$

and

$$Ns_n(\xi, G, \omega, z, \delta) = Ns(\xi, G, \omega, z, \delta) \cap \{0, \ldots, n - 1\}.$$

Definition 6.2. *Let X be a metrizable space, let G be a finitely generated semigroup action with the set of generators $\{id, g_1, \ldots, g_m\}$, and let $\delta \in \mathcal{C}(X)$.*

1. [18] *For $w \in \mathcal{A}^m$ and $x, y \in X$, a (δ, w)-chain of semigroup G from x to y is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $d(g_{w_i}(x_i), x_{i+1}) < \varepsilon(g_{w_i}(x_i))$, for all $i = 1, \ldots, n - 1$.*
2. [18] *We say that $\{x_i\}_{i\geq 0} \subset X$ is a (δ, ω)-pseudo orbit of G for some $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$, if for any $i \in \mathbb{Z}^+$, $d(g_{\omega_i}(x_i), x_{i+1}) < \delta(g_{\omega_i}(x_i))$.*
3. *We say that $\{x_i\}_{i\geq 0} \subset X$ is a (δ, ω)-*ergodic pseudo orbit *of G for some $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$ provided that the set $Np(\xi, G, \omega, \delta)$ has zero density, that is,*

$$\lim_{n\to\infty} \frac{|Np_n(\xi, G, \omega, \delta)|}{n} = 0.$$

Definition 6.3. *Let X be a metrizable space, let G be a finitely generated semigroup action with the set of generators $\{id, g_1, \ldots, g_m\}$. We say that*

1. [18] *G has the shadowing property, if for every $\epsilon \in \mathcal{C}(X)$, there is $\delta \in \mathcal{C}(X)$ such that for every (δ, ω)-pseudo orbit $\{x_i\}_{i\geq 0}$ of G, for some $\omega \in \Sigma^m$, there is a point $z \in X$ satisfying $d(g_\omega^i(x), x_i) < \epsilon(g_\omega^i(x))$ for all $i \geq 0$.*

(2) *G has the ergodic shadowing property if for each $\epsilon \in \mathcal{C}(X)$, there exists $\delta \in \mathcal{C}(X)$ such that every (δ,ω)-ergodic pseudo orbit ξ of G can be ϵ-ergodic shadowed by some point z in X, that is, there exists $\varphi \in \Sigma^m$ with $\varphi_i = \omega_i$ for $i \in Np^c(\{x_i\}_{i\geq 0}, G, \omega, \delta)$, such that*

$$\lim_{n\to\infty} \frac{|Ns_n(\{x_i\}_{i\geq 0}, G, \varphi, z, \epsilon)|}{n} = 0.$$

In the following, we show that Definition 6.3 for the semigroup G on the non-compact metric space X can be preserved by conjugacy. Hence, they do not depend on the choices of metrics on X. For this, we need two lemmas.

Lemma 6.1. [12, Lemmas 2.7 and 2.8] *Let (X,d) and (Y,d') be two metric spaces.*

(1) *A function f from X to Y is continuous if and only if, for any $\varepsilon \in \mathcal{C}(Y)$, there exists $\delta \in \mathcal{C}(X)$ such that if $d(x,y) < \delta(x)$ $(x,y \in X)$, then $d'(f(x),f(y)) < \varepsilon(f(x))$.*
(2) *For every $\alpha \in \mathcal{C}(X)$, there exists $\gamma \in \mathcal{C}(X)$ such that*

$$\gamma(x) \leq \inf\{\alpha(z) : z \in B(x,\gamma(x))\}. \tag{6}$$

The next lemma is cited from [18] and is an immediate result of Lemma 6.1.

Lemma 6.2. [18] *Let (X,d) be a metric space and let $f_i : X \to X$ $(i = 1,\dots,m)$ be continuous maps. Then, for every $\varepsilon \in \mathcal{C}(X)$, there exists $\delta \in \mathcal{C}(X)$ such that if $d(x,y) < \delta(x)$, then $d(f_i(x),f_i(y)) < \varepsilon(f_i(x))$ for all $i = 1,\dots,m$.*

Proposition 6.1. *Let X be a metric space and let G be a semigroup action with generating set $\{id, g_1,\dots,g_m\}$. Then the shadowing and ergodic shadowing properties of G introduced in Definition 6.3, are dynamical properties.*

Proof. Let G and F be two semigroups generated by $G_1 = \{id, g_1,\dots,g_m\}$ and $F_1 = \{id, f_1,\dots,f_m\}$ on the metric space (X,d) and (Y,d'), respectively. Suppose that G and F are topologically conjugate with conjugacy $h : X \to Y$. We show that the ergodic shadowing property preserves by topological conjugacy. Assume that G has the ergodic shadowing property. For every $\varepsilon' \in \mathcal{C}(Y)$, there exists $\varepsilon \in \mathcal{C}(X)$ such that if $d(x,y) < \varepsilon(x)$, then $d'(h(x),h(y)) < \varepsilon'(h(x))$. Take $\delta \in \mathcal{C}(X)$ as an ε-modulus of ergodic shadowing property of G, and let $\delta' \in \mathcal{C}(Y)$ be such that if $d'(x,y) < \delta'(x)$, then $d(h^{-1}(x),h^{-1}(y)) < \delta(h^{-1}(x))$. Let $\{x_i\}_{i\geq 0} \subseteq Y$ be a (δ',ω)-ergodic pseudo orbit of F for $\omega = \omega_0\omega_1\ldots \in \Sigma^m$. We show that $\{h^{-1}(x_i)\}_{i\geq 0}$ is a (δ,ω)-ergodic pseudo orbit of G. Indeed for any $i \in Np^c(\{x_i\}_{i\geq 0}, F, \omega, \delta')$, we have $d'(f_{\omega_i}(x_i), x_{i+1}) < \delta'(f_{\omega_i}(x_i))$ implies that

$$d(g_{\omega_i}(h^{-1}(x_i)), h^{-1}(x_{i+1})) = d(h^{-1}(f_{\omega_i}(x_i)), h^{-1}(x_{i+1})) < \delta(g_{\omega_i}(h^{-1}(x_i))).$$

It yields that $Np^c(\{x_i\}_{i\geq 0}, F, \omega, \delta') \subset Np^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \omega, \delta)$ and so $\{h^{-1}(x_i)\}_{i\geq 0}$ is a δ-ergodic pseudo orbit of G. Since the G has the ergodic shadowing property, there exist $z \in X$ and $\varphi \in \Sigma^m$ with $\varphi_i = \omega_i$ for $i \in Np^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \omega, \delta)$, such that $Ns(\{h^{-1}(x_i)\}_{i\geq 0}, G, \varphi, z, \epsilon)$ has zero density. Since for any $i \in Ns^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \varphi, z, \epsilon)$, $d(g^i_\varphi(z), h^{-1}(x_i)) < \epsilon(g^i_\varphi(z))$, we have

$$d'(h(g^i_\varphi(z)), x_i) = d'(f^i_\varphi(h(z)), x_i) < \epsilon'(f^i_\varphi(h(z))).$$

This means that $Ns^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \varphi, z, \epsilon) \subset Ns^c(\{x_i\}_{i\geq 0}, F, \varphi, h(z), \epsilon')$, which implies that $h(z)$, ϵ'-ergodic shadows $\{x_i\}_{i\geq 0}$. Thus F the has the ergodic shadowing property. □

The proof of the next lemma for a semigroup G on a compact metric space X was appeared in [17]. In the following, we show that it holds for semigroup G on non-compact metric spaces with the new notions for δ-chain and shadowing property introduced in Definitions 6.2 and 6.3.

Lemma 6.3. *Let (X, d) be a metric space and G be a semigroup associated with finite family $\{id, g_1, \dots, g_m\}$ of continuous maps on X. If the semigroup G has the shadowing property, then G is topologically mixing if and only if it is chain mixing.*

Proof. Let $x, y \in X$ and let $\epsilon \in \mathcal{C}(x)$. By Lemmas 6.1 and 6.2, there exists

$$\delta(x) \leq \inf\{\epsilon(z) : z \in B(x, \delta(x))\} \tag{7}$$

such that if $d(x, y) < \delta(x)$, then $d(g_i(x), g_i(y)) < \epsilon(g_i(x)), i = 1, 2, \dots, m$. Since G is topologically mixing, there exists an integer $N > 0$ such that $g^n_\omega(B(x, \delta(x))) \cap B(y, \delta(y)) \neq \emptyset$ for any $n \geq N$, for some $\omega \in \Sigma^m$. Take $z \in B(x, \delta(x))$ such that $g^n_\omega(z) \in B(y, \delta(y))$. We show that

$$\xi = \{x, g_\omega(z), g^2_\omega(z), \dots, g^{n-1}_\omega(z), y\}$$

is an ϵ-chain from x to y. Indeed, $d(x, z) < \delta(x)$ implies that $d(g_{\omega_0}(x), g_{\omega_0}(z)) < \epsilon(g_{\omega_0}(x))$. Since $g^n_\omega(z) \in B(y, \delta(y))$, using equation (7) yields that $\delta(y) < \epsilon(g^n_\omega(z))$. Thus, we have $d(g^n_\omega(z), y) < \delta(y)) < \epsilon(g^n_\omega(z))$. It means that ξ is an ϵ-chain from x to y.

Conversely, fix any nonempty open sets $U, V \subset X$ and choose points $x \in U$ and $y \in V$. Let $\epsilon_0 > 0$ be such that $B(x, \epsilon_0) \subset U$ and $B(y, \epsilon_0) \subset V$. Choose $\epsilon \in \mathcal{C}(X)$ with $\epsilon(x) < \epsilon_0/2$ for any $x \in X$. Let $\delta \in \mathcal{C}(X)$ be provided by the shadowing property for ϵ. For any sufficiently large $n > 0$, choose a (δ, w)-chain $\xi = \{x_0 = x, x_1, \dots, x_n = y\}$ with $w = w_0 w_1 \dots w_{n-1} \in \mathcal{A}^m$, that is, $d(g_{w_i}(x_i), x_{i+1}) < \delta(g_{w_i}(x_i))$, for any $i = 0, 1, \dots, n-1$. Extend ξ to a (δ, ω)-pseudo orbit ξ', where $\omega \in \Sigma^m$ and $\omega_i = w_i$ for $i = 0, 1, \dots, n-1$. By the shadowing property of G, there is a point $z \in X$ such that

$$d(g^i_\omega(z), x_i) < \epsilon(g^i_\omega(z)) < \epsilon_0/2$$

for any $i = 0, 1, \dots, n$. Therefore, $d(z, x) < \epsilon_0$ and $d(g^n_\omega(z), y) < \epsilon_0$. Thus $z \in B(x, \epsilon_0) \subset U$ and $g^n_\omega(z) \in B(y, \epsilon_0) \subset V$. So the semigroup G is topologically mixing. □

7. **Conclusions**

In this paper, we have studied the notions of $\underline{d}$-shadowing, weak specification and average shadowing of semigroup actions on compact metric spaces. Furthermore, the relations between the mentioned notions with the topologically mixing, the ergodic shadowing property and the pseudo-orbital specification property have been discussed. Moreover, we have considered the new notions of shadowing, ergodic shadowing and chain mixing properties of semigroup actions on non-compact metric spaces, that are conjugacy invariant. Studying the relations between various types of shdaowing and specification in non-compact metric spaces via these new definitions is a topic for further research, which is the purpose of our next paper.

Acknowledgements

The authors are grateful to the referee for her or his careful reading of the manuscript and valuable suggestions on this work.

REFERENCES

[1] *N. Aoki and K. Hiraide*, Topological Theory of Dynamical Systems, North-Holland Math. Library, vol. 52, North-Holland, Amsterdam, 1994.

[2] *A.Z. Bahabadi*, Shadowing and average shadowing properties for iterated function systems, Georgian Math. J., **22**(2015), 179-184.

[3] *M.F. Barnsley*, The conley attractor of an iterated function system, Bull. Aust. Math. Soc., **88**(2013), 267-279.

[4] *R. Bowen*, Equilibrium states and ergodic theory of Anosov diffieomorphisms, Trans. Amer. Math. Soc., **154**(1971), 377-397.

[5] *D. A. Dastjerdi and M. Hosseini*, Sub-shadowings, Nonlinear Anal., **72**(2010), 3759-3766.

[6] *J.H. Elton and M. Piccioni* , Iterated function systems a rising from recursive estimation problems, Probab. Theory Relat. Fields., **91**(1992), 103-114.

[7] *A. Fakhari and F. H. Ghane*, On shadowing: ordinary and ergodic, J. Math. Anal. Appl., **364**(2010), 151-155.

[8] *B. Forte and E.R. Vrscay* , Solving the inverse problem for function/image approximation using iterated function systems, Fractals Complex Geom. Patterns Scaling Nat. Soc., **2**(1994), 335-346.

[9] *V. Glavan and V. Gutu*, Shadowing in parameterized IFS, Fixed Point Theory., **7**(2006), 263-274.

[10] *D. Kwietniak and P. Oprocha*, A note on the average shadowing property for expansive maps, Topol. Appl., **159**(2012), 19-27.

[11] *K. Lee*, Continuous inverse shadowing and hyperbolicity, Bull. Austral. Math. Soc., **67**(2003), 15-26.

[12] *K. Lee, N. Nguyen, and Y. Yang*, Topological stability and spectral decomposition for homeomorphisms on noncompact spaces., Discrete Contin. Dyn. Syst., **38**(2018), 2487-2503.

[13] *K. Lee and K. Sakai*, Various shadowing property and their equivalence, Disc. Cont. Dyn. Syst., **13**(2005) 533-539.

[14] *L. Montrucchio and F. Privileggi*, Fractal steady states instochastic optimal control models, Ann. Oper. Res., **88**(1999) 183-197.

[15] *A. V. Osipov and S. B. Tikhomirov*, Shadowing for actions of some finitely generated groups, Dynam. Syst., **29**(2014), 337-351.

[16] *F. B. Rodrigues and P. Varandas*, Specification and thermodynamical properties of semigroup actions, J. Math. Phys., **57**(2016), 052704.

[17] *Z. Shabani*, Ergodic Shadowing of Semigroup Actions, Bull. Iran. Math. Soc., 46 (2020), 303-321.

[18] *Z. Shabani, A. Barzanouni and X. Wu*, Recurrent sets and shadowing for finitely generated semigroup actions on metric spaces, preprint, (2020).

[19] *M. Viana and K. Oliveira*, Foundations of Ergodic Theory, Cambridge University Press, Cambridge, 2016.

[20] *X. Wu, L. Wang and J. Liang*, The chain properties and average shadowing property of iterated function systems J. Qual. Theory Dyn. Syst., **17**(2018), 219-227.

[21] *Y. Zhao, L. Wang and F. Lei*, Shadowing property and topological ergodicity for iterated function systems, Mod. Phys. Lett. B., **33**(2019) 1950272(9 pages).