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CONNES-BIPROJECTIVE DUAL BANACH ALGEBRAS

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In this paper, we introduce a new notion of biprojectivity, called Connes-biprojective, for a dual Banach algebra. We study the relation between this new notion to Connes-amenability and we show that, for a given dual Banach algebra, A it is Connes-amenable if and only if A is Connes-biprojective and has a bounded approximate identity.

*Also, among some useful results, for an Arens regular Banach algebra A, we show that, if A is biprojective, then the dual Banach algebra A** is Connes-biprojective.*

Keywords: Connes-biprojective, dual Banach algebra, σWC-virtual diagonal, Connes-amenable.

1. Introduction

Let A be a Banach algebra and let $A\hat{\otimes}A$ be the projective tensor product of A with itself, which is a Banach A-bimodule for the usual left and right operations with respect to A. Let $\pi : A\hat{\otimes}A \to A$ denote the linearization mapping of the algebra multiplication in A, defined by $\pi(a\otimes b) = ab \quad (a,b\in A)$. Let $\pi^* : A^* \to (A\hat{\otimes}A)^*$ be the adjoint map of π, where A^* is the topological dual of A.

In the *Helemskii's Banach homology* setting, there are two important notions related to that one of *(Johnson's) amenability* for Banach algebras. These are *projectivity* and *flatness*. A Banach algebra A is called biprojective if π is a retraction, which is to say, there is a bounded A-bimodule homomorphism $\rho : A \to A\hat{\otimes}A$ such that $\pi\circ\rho = id_A$. Similarly, a Banach algebra A is called biflat if π^* is a co-retraction; that is, if there is a bounded A-bimodule homomorphism $\gamma : (A\hat{\otimes}A)^* \to A^*$ such that $\gamma\circ\pi^* = id_{A^*}$ [6, section VII]. Then we have that a Banach algebra A is amenable if and only if A is biflat and has a bounded approximate identity; see [10, Proposition 4.3.23 and Exercise 4.3.15] or [6, Theorem VII.2.20]. Clearly, every biprojective Banach algebra is biflat -the converse is generally not true- and, as a consequence, every biprojective Banach algebra with a bounded approximate identity is amenable. There are several analogue notions of biprojectivity and amenability like module biprojectivity and approximate amenability (see [1] and [11] for more details).

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Another important characterization of amenability involving the map $\pi : A\hat{\otimes} A \to A$ is that A is amenable if and only if there is a virtual diagonal for A. The definition of virtual diagonal does not matter here.

Yet in relation with amenability, there is a particular class of Banach algebras which deserves attention. This is the class of dual Banach algebras introduced by V. Runde (2001). Examples of these algebras are the von Neumann algebras, the measure algebras $M(G)$ for locally compact groups G, the algebra of bounded operators $B(H)$, for a Hilbert space or even a reflexive Banach space H, the bidual Banach algebra A^{**} for an Arens regular algebra A, and others. This list shows the interest of the above class.

The original Johnson's amenability is too strong to be handled in connection with dual Banach algebras. Two good samples: S. Wasserman proved (1976) that every amenable von Neumann algebra is subhomogeneous, and H. G. Dales, F. Ghahramani and A. Helemskii proved (2002) that $M(G)$ is amenable if and only if G is amenable -as a group- and *discrete*.

A suitable amenability-type concept to deal with dual Banach algebras seems to be the so-called nowadays *Connes-amenability*, firstly introduced by Johnson, Kadison and Ringrose for von Neumann algebras (1972). In fact, a von Neumann algebra is Connes-amenable if and only if it is injective. Also, $M(G)$ is Connes-amenable if and only if G is amenable (see [2], [3] and [4]).

It is possible to characterize Connes-amenability for dual Banach algebras in terms of virtual diagonals. Let A be a dual Banach algebra. For a given A-bimodule E, let $\sigma WC(E)$ denote the closed submodule of E of all elements x such that the mappings $A \to E; a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are $\sigma(A, A_*) - \sigma(E, E^*)$-continuous. Then $A_* \subseteq \sigma WC((A\hat{\otimes} A)^*)$, from which it follows that π^* maps A_* into $\sigma WC((A\hat{\otimes} A)^*)$. Hence, π^{**} drops to an A-bimodule homomorphism $\pi_{\sigma WC} : \sigma WC((A\hat{\otimes} A)^*)^* \to A$. Any element M in $\sigma WC((A\hat{\otimes} A)^*)^*$ satisfying $a \cdot M = M \cdot a$ and $a \cdot \pi_{\sigma WC} M = a \quad (a \in A)$,
is called a σWC-virtual diagonal for A. Then, a dual Banach algebra A is Connes-amenable if and only if there exists a σWC-virtual diagonal for A (Runde, 2004).

In short, $\pi_{\sigma WC}$ arises as the most convenient mapping, induced by π, to deal with the important class of dual Banach algebras. In this respect, it is now natural to look for a suitable (Helemskii's) homological analogue of biprojectivity, in the setting of dual Banach algebras, that fit well with Connes-amenability.

In this paper, we introduce an analogue of biprojectivity, called Connes-biprojective, studying its homological properties and we show how this concept deals with Connes-amenability. The organization of this paper is as follows. We introduce a suitable notion of Connes-biprojectivity in the setting of dual Banach algebras. We show that a dual Banach algebra is Connes-amenable if and only if it is Connes-biprojective and has a bounded approximate identity (identity, indeed). We prove that if an Arens regular Banach algebra A is biprojective, then the bidual Banach algebra A^{**} is Connes-biprojective.

Given a Banach algebra A and a Banach A-bimodule E, the topological dual space E^* of E becomes a Banach A-bimodule defined by

$$\langle x, a\cdot\varphi\rangle := \langle x\cdot a, \varphi\rangle, \quad \langle x, \varphi\cdot a\rangle := \langle a\cdot x, \varphi\rangle \quad (a\in A, x\in E, \varphi\in E^*).$$

Recall that a Banach space X is a predual for a Banach space Y if we have an isometric isomorphism $X^* \cong Y$.

Definition 1.1. [8, Definition 1.1] A Banach algebra A is called dual, if it is a dual Banach space with predual A_* such that the multiplication in A is separately $\sigma(A, A_*)$-continuous. Equivalently, a Banach algebra A is dual if it has a predual A_* which is a closed submodule of A^* [10, Exercise 4.4.1].

Although a predual may not be unique, we can recognize it from the context. In particular, we may speak of the weak* topology on A without ambiguity.

Definition 1.2. [9, Definition 1.4] Let A be a dual Banach algebra and let E be a dual Banach A-bimodule. An element $x\in E$ is called normal, if the maps $a\mapsto a\cdot x$ and $a\mapsto x\cdot a$ are weak* continuous.

The set of all normal elements in E is denoted by E_σ. We say that E is normal if $E = E_\sigma$. It is easy to see that E_σ is a norm-closed submodule of E. However, there is no need for E_σ to be weak*-closed.

For a given dual Banach algebra A and a Banach A-bimodule E, it is easy to see that $\sigma WC(E)$ is a closed A-submodule of E and so E is canonically mapped into $\sigma WC(E^*)^*$. If F is another Banach A-bimodule and if $\psi : E\to F$ is a bounded A-bimodule homomorphism, then $\psi(\sigma WC(E)) \subseteq \sigma WC(F)$ holds. Runde in [9, Proposition 4.4] showed that $E = \sigma WC(E)$ if and only if E^* is normal Banach A-bimodule, and therefore for any Banach A-bimodule E, the dual module $\sigma WC(E^*)$ is normal.

Recall that for a Banach algebra A and a Banach A-bimodule E, a bounded linear map $D: A\to E$ is called a derivation if $D(ab) = a\cdot D(b) + D(a)\cdot b$ for every $a, b\in A$. A derivation $D: A\to E$ is called inner if there exists an element $x\in E$ such that for every $a\in A$, $D(a) = a\cdot x - x\cdot a$.

Definition 1.3. [9, Definition 1.5] A dual Banach algebra is called Connes-amenable if for every normal dual A-bimodule E, every weak* continuous derivation $D: A\to E$ is inner.

2. Connes-biprojective dual Banach algebras

As mentioned in the introduction, the concept of biprojectivity is important in the Helemskii's Banach homology and is closely related to Johnson's amenability. In this section we define a suitable analogue of Helemskii's homological-type concept to deal with dual Banach algebras, called Connes-biprojectivity and we show that this concept is closely related to Connes-amenability.

Definition 2.1. Let A be a dual Banach algebra. Then A is called *Connes-biprojective* if there exists a bounded A-bimodule homomorphism $\rho : A \to \sigma WC((A \hat{\otimes} A)^*)^*$ such that $\pi_{\sigma WC} \circ \rho = id_A$ (that is, $\pi_{\sigma WC}$ is a retraction).

In the following theorem, we determine the relation between Connes-biprojectivity and Connes-amenability.

Theorem 2.2. The following are equivalent for a dual Banach algebra A:

(i) A is Connes-biprojective and has a bounded approximate identity,

(ii) A is Connes-amenable.

Proof. Suppose that A is Connes-amenable. Then there exists a σWC-virtual diagonal $M \in \sigma WC((A \hat{\otimes} A)^*)^*$ for A. We define $\rho : A \to (\sigma WC(A \hat{\otimes} A)^*)^*$ by $\rho(a) = a \cdot M$, for every $a \in A.$ Then

$$\| \rho(a) \| = \| a \cdot M \| \leq K \| a \| \| M \| .$$

Thus ρ is bounded. Also $a \cdot \rho(b) = a \cdot (b \cdot M) = (ab) \cdot M$. On the other hand, since M is a virtual diagonal, $(ab) \cdot M = M \cdot (ab) = (a \cdot M) \cdot b = \rho(a) \cdot b,$ that is, ρ is an A-bimodule homomorphism. It is easy to see that $\pi_{\sigma WC} \circ \rho = id_A$. Note that since A is Connes-amenable, it has an identity, equivalently, a bounded approximate identity.

Conversely, suppose that A is Connes-biprojective and has a bounded approximate identity $(e_i)_i$. Then there exists a bounded A-bimodule homomorphism $\rho : A \to \sigma WC((A \hat{\otimes} A)^*)^*$ such that $\pi_{\sigma WC} \circ \rho = id_A$. Let $M_i = \rho(e_i) \in \sigma WC((A \hat{\otimes} A)^*)^*$. Then for every $a \in A$, we have

$$a \cdot M_i - M_i \cdot a = \rho(a \cdot e_i) - \rho(e_i \cdot a) \to 0,$$

and

$$\| \pi_{\sigma WC}(M_i)(a) - a \| = \| e_i a - a \| \to 0.$$

Since $(M_i)_i$ is uniformly bounded net in $\sigma WC((A \hat{\otimes} A)^*)^*$, it has a weak* limit point, say M. It is easy to see that M is a σWC-virtual diagonal for A, so A is Connes-amenable. □

Example 2.3. Let A be a biflat dual Banach algebra. Then $\pi^* : A^* \to (A \hat{\otimes} A)^*$ is a co-retraction and so $\pi^* |_{A_*} : A_* \to \sigma WC((A \hat{\otimes} A)^*)$ is a co-retraction again. Thus $\pi_{\sigma WC} : (\sigma WC(A \hat{\otimes} A)^*)^* \to A$ is a retraction, that is, A is Connes-biprojective.

By the previous example, every biprojective dual Banach algebra is Connes-biprojective. In the following example we see that the converse is false in general.

Example 2.4. Let G be a non-discrete amenable locally compact group. Then by [10, Theorem 4.4.13], $M(G)$, the measure algebra of G, is a Connes-amenable dual Banach algebra and thus by Theorem 2.2, $M(G)$ is Connes-biprojective. Since G is not discrete, by [5, Theorem 1.3] $M(G)$ is not amenable, so it is not biflat.

Clearly, every Connes-amenable Banach algebra is Connes-biprojective. Here we give two examples of Connes-biprojective dual Banach algebras, which are not Connes-amenable.

Example 2.5. Let S be a discrete semigroup and let $\ell^1(S)$ be its semigroup algebra. Let $A = \ell^1(S)^*$. If ϕ is a character on $c_0(S)$, then there exists a unique extension of ϕ on $c_0(S)^{**}$, (which is denoted by $\tilde{\phi}$) and defined by

$$\tilde{\phi}(F) = F(\phi) \quad (F \in c_0(S)^{**} \cong A).$$

$\tilde{\phi}$ is a multiplicative map because, for every $F, G \in A$,

$$\tilde{\phi}(FG) = (FG)(\phi) = F(\phi)G(\phi).$$

Now we define a new multiplication on A by

$$ab = \tilde{\phi}(a)b \quad (a, b \in A, \quad \phi \in c_0(S)^*).$$

With this multiplication, A becomes a Banach algebra which is a dual Banach space. We denote this algebra by A_ϕ. We define a map $\rho : A_\phi \to A_\phi \hat{\otimes} A_\phi$ by $\rho(a) = x_0 \otimes a$, where $a \in A_\phi$ and x_0 is an element in A_ϕ such that $\tilde{\phi}(x_0) = 1$. It is easy to see that ρ is a bounded A_ϕ-bimodule homomorphism and $\pi_{A_\phi} \circ \rho(a) = a$ for each $a \in A_\phi$. Hence A_ϕ is a biprojective Banach algebra. If we show that A_ϕ is a dual Banach algebra, then by Example 2.3 A_ϕ is Connes-biprojective. It is enough to show that the multiplication in A_ϕ is separately w^*-continuous. Suppose that $b \in A$ is a fixed element. Let $a \in A$ and $(a_\alpha) \subseteq A$ be such that $a_\alpha \to a$ in the w^*-topology. For each $f \in \ell^1(S) \cong c_0(S)^*$, we have $a_\alpha(f) \to a(f)$. Since the character space of $c_0(S)$ lies in $\ell^1(S)$, so the last statement is also true for $f = \phi$, where ϕ is the corresponding functional to $\tilde{\phi}$.

Hence, $a_\alpha(\phi) \to a(\phi)$, which implies that $\tilde{\phi}(a_\alpha) \to \tilde{\phi}(a)$. Then $\tilde{\phi}(a_\alpha)b \to \tilde{\phi}(a)b$. Now suppose that $(b_\alpha)_\alpha$ is a net in A_ϕ such that $b_\alpha \to b$ in the w^*-topology. It is easy to see that $\tilde{\phi}(a)b_\alpha \to \tilde{\phi}(a)b$ in the w^*-topology, for every $a \in A_\phi$. Hence A_ϕ is a dual Banach algebra.

Now if A_ϕ is Connes-amenable, then A_ϕ has an identity, so $\dim A = 1$, (because for each $a \in A$ we have $a = ae = \tilde{\phi}(a)e$, where e is an identity of A_ϕ), hence a contradiction reveals.

Example 2.6. Consider $A = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ with the usual matrix multiplication and L^1-norm. Since A is finite dimensional, it is a dual Banach algebra. Clearly A has a right identity but it does not have an identity, so it is not Connes-amenable. We define a map $\zeta : A \to A \hat{\otimes} A$ by $\zeta(\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}) = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. We have

$$\| \zeta \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \| = \| \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \| = \| \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \| \| \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \| = 2 \| \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \|$$

so $\| \zeta \| \leq 2$ and an easy calculation shows that ζ is an A-bimodule homomorphism. By composing the canonical map $A \hat{\otimes} A \to \sigma WC((A \hat{\otimes} A)^*)^*$ with ζ, we obtain an A-bimodule homomorphism $\rho : A \to \sigma WC((A \hat{\otimes} A)^*)^*$. Since for every $a \in A, \pi_{\sigma WC} \circ \rho(a) = \pi_{\sigma WC} \circ \zeta(a)$ we have

$(\pi_{\sigma WC} \circ \rho)\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} = \pi_{\sigma WC}(\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$, so ρ is a right inverse of $\pi_{\sigma WC}$. Hence A is Connes-biprojective.

The next theorem shows that how Connes-biprojectivity deals with homomorphisms.

Theorem 2.7. Let A be a Banach algebra, and let B be a dual Banach algebra. Let $\theta : A \to B$ be a continuous homomorphism.

(i) Let $\theta^* : B^* \to A^*$ be such that $\theta^*|_{B_*} : B_* \to A^*$ is surjective. Suppose that the image of the closed unit ball of A by θ is weak* dense in the closed unit ball of B. Then, biprojectivity of A implies Connes-biprojectivity of B.

(ii) If A is dual and Connes-biprojective and θ is weak* continuous, then B is Connes-biprojective.

Proof. Note that since $\theta : A \to B$ is a continuous homomorphism, by [7, Proposition 1.10.10], the map $\theta \otimes \theta : A \otimes A \to B \otimes B$ defined by $(\theta\otimes\theta)(a\otimes b)=\theta(a)\otimes\theta(b)$ can be extended to a bounded linear map $\theta\hat{\otimes}\theta: A\hat{\otimes}A\to B\hat{\otimes}B.$ It is readily seen that, for every $a,b,c\in A,$ $(\theta\hat{\otimes}\theta)([a\otimes b]\cdot c)=(\theta\hat{\otimes}\theta)(a\otimes b)\bullet\theta(c)$ and $(\theta\hat{\otimes}\theta)(c\cdot[a\otimes b])=\theta(c)\bullet(\theta\hat{\otimes}\theta)(a\otimes b)$ where "$\cdot$" is the action of A on $A\hat{\otimes}A$ and "$\bullet$" is the action of B on $B\hat{\otimes}B$ inherited by $\theta.$ Also, consider the canonical map $\imath: B\hat{\otimes}B\to\sigma WC((B\hat{\otimes}B)^*)^*,$ which is norm continuous with weak* dense range.

(i) Suppose that A is biprojective. Then there exists an A-bimodule homomorphism $\rho^A: A\to A\hat{\otimes}A$ such that $\pi^A\circ\rho^A=id_A$. Define a map $\zeta: A\to\sigma WC((B\hat{\otimes}B)^*)^*$ given by $\zeta(a)=(\imath\circ(\theta\hat{\otimes}\theta)\circ\rho^A)(a)$. Then ζ is bounded, and for every $a,a'\in A$ we have

$$\zeta(aa')=\theta(a)\bullet\zeta(a')=\zeta(a)\bullet\theta(a'). \tag{2.1}$$

For every $b\in B$, by assumption, there exists a bounded net $(a_i)\subseteq A$ such that $\theta(a_i)\to b$ in the weak* topology. Since ζ is bounded, $(\zeta(a_i))_i$ has a weak* accumulation point by Banach-Alaoglu theorem. Passing to a subnet (if it is necessary), $w^*-\lim_i\zeta(a_i)$ exists. Thus, we extend ζ to a weak* continuous map $\rho^B: B\to\sigma WC((B\hat{\otimes}B)^*)^*$ defined by $\rho^B(b)=w^*-\lim_i\zeta(a_i)$. We need to verify that ρ^B is well-defined. In order to do this, it is enough to show that $w^*-\lim_i\zeta(a_i)=0$ in $\sigma WC((B\hat{\otimes}B)^*)^*$, whenever $w^*-\lim_i\theta(a_i)=0$ in B. If $\lambda\in A^*$, then by assumption, there is a $\varphi\in B_*$ such that $\theta^*|_{B_*}(\varphi)=\lambda$. Now we have

$$\lim_i\langle\lambda,a_i\rangle=\lim_i\langle\theta^*(\varphi),a_i\rangle=\lim_i\langle\varphi,\theta(a_i)\rangle=0.$$

Hence $a_i\to 0$ in the weak topology of A. Since ζ is weak-weak* continuous, we conclude that $\zeta(a_i)\to 0$ in the weak* topology. Suppose that b and $b'\in B$. Then there exist two nets (a_i) and (a'_β) in A such that $\theta(a_i)\to b$ and $\theta(a'_\beta)\to b'$ in the weak* topology. By the equation (2.1), and by the weak* continuity of the action of B we have,

$$\rho^B(bb')=w^*-\lim_i w^*-\lim_\beta\zeta(a_i a'_\beta)=w^*-\lim_i w^*-\lim_\beta\theta(a_i)\bullet\zeta(a'_\beta)=b\bullet\rho^B(b').$$

Similarly, $\rho^B(bb')=\rho^B(b)\bullet b'$. Thus ρ^B is a B-bimodule homomorphism.

Finally, we prove that $\pi^B_{\sigma WC}\circ\rho^B(b)=b$ for every $b\in B$. Observe that for the elementary tensor element $a\otimes a'\in A\hat{\otimes}A$ we have

$$\pi^B_{\sigma WC}\circ\imath\circ(\theta\hat{\otimes}\theta)(a\otimes a')=\pi^B_{\sigma WC}\circ\imath(\theta(a)\otimes\theta(a'))=\theta(aa')=\theta\circ\pi^A(a\otimes a').$$

Thus for every $a\in A$,

$$\pi_{\sigma WC}^{B} \circ \zeta(a) = \theta(a). \tag{2.2}$$

Now let $b \in B$ and take a net $(a_i) \subseteq A$ such that $\theta(a_i) \to b$ in the weak* topology. Then, by equation (2.2), we have

$$b = w^* - \lim_i \theta(a_i) = w^* - \lim_i \pi_{\sigma WC}^{B} \circ \zeta(a_i) = \pi_{\sigma WC}^{B}(w^* - \lim_i \zeta(a_i)) = \pi_{\sigma WC}^{B} \circ \rho^{B}(b).$$

(ii) Suppose that A is dual and Connes-biprojective. Then there exists an A-bimodule homomorphism $\rho^{A} : A \to \sigma WC((A \hat{\otimes} A)^*)^*$ such that $\pi_{\sigma WC}^{A} \circ \rho^{A} = id_A$. Consider the map $(\theta \hat{\otimes} \theta)^* : (B \hat{\otimes} B)^* \to (A \hat{\otimes} A)^*$ which is an A-bimodule homomorphism. We conclude that $(\theta \hat{\otimes} \theta)^*$ maps $\sigma WC(B \hat{\otimes} B)^*$ into $\sigma WC(A \hat{\otimes} A)^*$. Consequently, we obtain a weak* continuous map

$$\varphi := ((\theta \hat{\otimes} \theta)^* |_{\sigma WC(B \hat{\otimes} B)^*})^* : \sigma WC((A \hat{\otimes} A)^*)^* \to \sigma WC((B \hat{\otimes} B)^*)^*.$$

Now, we define a map $\zeta : A \to \sigma WC((B \hat{\otimes} B)^*)^*$ given by $\zeta(a) = \varphi \circ \rho^{A}(a)$, for every $a \in A$. Since φ and ρ^{A} are weak* continuous, so is ζ and since θ is weak* continuous and weak* dense range, ζ extends to a weak* continuous map $\rho^{B} : B \to \sigma WC((B \hat{\otimes} B)^*)^*$. An argument similar to (i) shows that ρ^{B} is a B-bimodule homomorphism.

Let $a \otimes a' \in A \hat{\otimes} A$. Since the map φ is the double transpose of $\theta \hat{\otimes} \theta$, we have

$$\varphi(a \otimes a') = (\theta \hat{\otimes} \theta)(a \otimes a') = (\theta(a) \otimes \theta(a')).$$

Hence

$$\pi_{\sigma WC}^{B} \circ \varphi(a \otimes a') = \pi_{\sigma WC}^{B}(\theta(a) \otimes \theta(a')) = \theta(aa') = \theta \circ \pi_{\sigma WC}^{A}(a \otimes a').$$

Thus by linearity and continuity and by the hypothesis, for every $a \in A$, we have

$$\pi_{\sigma WC}^{B} \circ \varphi \circ \rho^{A}(a) = \theta \circ \pi_{\sigma WC}^{A} \circ \rho^{A}(a) = \theta(a), \tag{2.3}$$

so that the following diagram is commutative;

$$\begin{array}{ccc} A & \xrightarrow{\rho^A} & \sigma WC((A \hat{\otimes} A)^*)^* \\ \theta \downarrow & & \downarrow \varphi \\ B & \xleftarrow[\pi_{\sigma WC}^{B}]{} & \sigma WC((B \hat{\otimes} B)^*)^* \end{array}$$

Since the range of θ is dense, by (2.3) we have $\pi_{\sigma WC}^{B} \circ \rho^{B}(b) = b$ for every $b \in B$. Hence B is Connes-biprojective. □

Given a Banach algebra A, we may define the bilinear maps $A^{**} \times A^* \to A^*$ and $A^* \times A^{**} \to A^*$ given by $(\Phi, \mu) \mapsto \Phi \cdot \mu$ and $(\mu, \Phi) \mapsto \mu \cdot \Phi$, respectively, where for every $\Phi \in A^{**}$, $\mu \in A^*$ and $a \in A$

$$\langle \Phi \cdot \mu, a \rangle = \langle \Phi, \mu \cdot a \rangle, \quad \langle \mu \cdot \Phi, a \rangle = \langle \Phi, a \cdot \mu \rangle.$$

We define two bilinear maps $\Diamond, \Box : A^{**} \times A^{**} \to A^{**}$ given by $(\Phi, \Psi) \mapsto \Phi \Box \Psi$ and $(\Phi, \Psi) \mapsto \Phi \Diamond \Psi$, where for every $\Phi, \Psi \in A^{**}$ and $\mu \in A^*$

$$\langle \Phi \Box \Psi, \mu \rangle = \langle \Phi, \Psi \cdot \mu \rangle, \quad \langle \Phi \Diamond \Psi, \mu \rangle = \langle \Psi, \mu \cdot \Phi \rangle,$$

One can check that $\Box$ and $\Diamond$ are actually algebra products, called the first and the second Arens products, respectively. If for every $\Phi, \Psi \in A^{**}$ we have $\Phi \Box \Psi = \Phi \Diamond \Psi$, then we say that A is Arens regular.

It is well-known that if A is an Arens regular Banach algebra, then A^{**}, the bidual of A is a dual Banach algebra with predual A^* (see [9] for more details).

Corollary 2.8. Let A be an Arens regular Banach algebra. If A is biprojective, then A^{**} is Connes-biprojective.

Proof. It is clear that the inclusion map $id : A \to A^{**}$ satisfies the conditions of Theorem 2.7(i). □

If A is a dual Banach algebra and I is a weak*-closed ideal of A, then I is a dual Banach algebra with predual $I_* = A_* / I^{\perp}$. To see this, we have

$$(I_*)^* = (A_* / I^{\perp})^* = (I^{\perp})^{\perp} = I.$$

Since the multiplication in A is separately weak* continuous, a simple verification shows that the multiplication on A/I is separately weak* continuous, so A/I is also a dual Banach algebra.

Proposition 2.9. Let A be a Connes-biprojective dual Banach algebra and let I be a weak*-closed ideal of A which is essential as a left Banach A-module. Then A/I is Connes-biprojective.

Proof. Since A is Connes-biprojective, the map $\pi_{\sigma WC} : \sigma WC((A \hat{\otimes} A)^*)^* \to A$ is a retraction, so there exists a bounded A-bimodule homomorphism $\rho : A \to \sigma WC((A \hat{\otimes} A)^*)^*$ as a right inverse of $\pi_{\sigma WC}$. Let $q : A \to A/I$ be the quotient map. Then the map $id \hat{\otimes} q : A \hat{\otimes} A \to A \hat{\otimes} (A/I)$ is a bounded A-bimodule homomorphism, so is $(id \hat{\otimes} q)^* : (A \hat{\otimes} (A/I))^* \to (A \hat{\otimes} A)^*$. Thus $(id \hat{\otimes} q)^*$ maps $\sigma WC((A \hat{\otimes} (A/I))^*)$ into $\sigma WC((A \hat{\otimes} A)^*)$, therefore we obtain an A-bimodule homomorphism

$$((id \hat{\otimes} q)^*|_{\sigma WC(A \hat{\otimes} A)^*})^* : \sigma WC((A \hat{\otimes} A)^*)^* \to \sigma WC((A \hat{\otimes} (A/I))^*)^*.$$

Composing this map with ρ, we obtain the map

$$\varphi := ((id \hat{\otimes} q)^*|_{\sigma WC(A \hat{\otimes} A)^*})^* \circ \rho : A \to \sigma WC((A \hat{\otimes} (A/I))^*)^*,$$

which is a bounded A-bimodule homomorphism since it is a composition of such two maps. If $a \in A$, then

$$\varphi(a) \in \sigma WC((A \hat{\otimes} (A/I))^*)^* = (A \hat{\otimes} (A/I))^{**} / \sigma WC((A \hat{\otimes} (A/I))^*)^{\perp}.$$

Suppose that $\tilde{\varphi}(a)$ is a corresponding element of $\varphi(a)$ in $(A\hat{\otimes}(A/I))^{**}$ (that is, $\tilde{\varphi}(a)$ is the image of $\varphi(a)$ under the quotient map). By the Goldstine's theorem there is a net $(\varphi_\alpha(a))_\alpha \subseteq A\hat{\otimes}(A/I)$ such that $\varphi_\alpha(a) \to \tilde{\varphi}(a)$ in the weak* topology. Since $\sigma WC((A\hat{\otimes}(A/I))^*)^*$ is normal, we have $\varphi_\alpha(a)\cdot i \to \tilde{\varphi}(a)\cdot i$ for every $i \in I$, and since for every α we have $\varphi_\alpha(a)\cdot i = 0$, hence $\tilde{\varphi}(a)\cdot i = 0$. Thus $\varphi(ai) = \varphi(a)\cdot i = 0$. By the fact that I is a left essential ideal, we have $\varphi|_I = 0$, so it induces an A-bimodule homomorphism $\hat{\varphi} : (A/I) \to \sigma WC((A\hat{\otimes}(A/I))^*)^*$. In contrast, we have a bounded A-bimodule homomorphism $q\hat{\otimes} id : A\hat{\otimes}(A/I) \to (A/I)\hat{\otimes}(A/I)$, which gives an A-bimodule homomorphism

$$(q\hat{\otimes} id)^* : ((A/I)\hat{\otimes}(A/I))^* \to (A\hat{\otimes}(A/I))^*,$$

such that it maps $\sigma WC(((A/I)\hat{\otimes}(A/I))^*)$ into $\sigma WC((A\hat{\otimes}(A/I))^*)$. Now consider the adjoint map $((q\hat{\otimes} id)^*|_{\sigma WC(\frac{A}{I}\hat{\otimes}\frac{A}{I})^*})^*$ which is denoted by

$$\psi : \sigma WC(((A\hat{\otimes}(A/I))^*)^* \to \sigma WC(((A/I)\hat{\otimes}(A/I))^*)^*.$$

Take $\zeta := \psi \circ \hat{\varphi} : A/I \to \sigma WC(((A/I)\hat{\otimes}(A/I))^*)^*$. It is easy to see that $\pi_{\sigma WC(A/I)} \circ \zeta = id_{A/I}$, therefore A/I is Connes-biprojective.

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