CLASSICAL MOTION OF A TEST PARTICLE IN THE BERTOTTI-ROBINSON SPACETIME

Gholam Reza Soleimani[1], Morteza Yavari[2]

In this paper, the motion of a test particle in the Bertotti-Robinson spacetime by using the Hamilton-Jacobi method is studied. Then, in threading formalism, the gravitoelectromagnetism[1] force acting on this particle in this spacetime is calculated.

Keywords: Bertotti-Robinson spacetime, gravitoelectromagnetism force, test particle trajectory.

1. **Introduction**

A stationary spacetime[3] $(\mathrm{M}, \mathrm{g}_{\alpha\beta})$ is a 4-dimensional Lorentzian manifold with a global timelike Killing vector field η. We now assume that there exists a global time function $t : \mathrm{M} \rightarrow \mathrm{R}$ such that $\eta = \frac{\partial}{\partial t}$. Thus, the quotient space of M by the integral curves of η is a 3-dimensional orbit manifold Σ with projected metric tensor γ_{ij} to which we refer as space manifold. Hence, this decomposition attempts to decompose spacetime quantities into pieces orthogonal to the given congruence of curves and pieces tangent to the congruence, [3,4]. The threading decomposition leads to the following line element, [4,5]:

$$ds^2 = \mathrm{g}_{\alpha\beta} dx^\alpha dx^\beta = h(dt - \mathrm{g}_i dx^i)^2 - \gamma_{ij} dx^i dx^j, \tag{1}$$

where the components of metric are

$$\mathrm{g}_{00} = h,\ \mathrm{g}_{0i} = -h\mathrm{g}_i,\ \mathrm{g}_{ij} = -\gamma_{ij} + h\mathrm{g}_i\mathrm{g}_j, \tag{2}$$

and their inverse are

$$\mathrm{g}^{00} = -\mathrm{g}^2 + \frac{1}{h},\ \mathrm{g}^{0i} = -\mathrm{g}^i,\ \mathrm{g}^{ij} = -\gamma^{ij}, \tag{3}$$

in which $\mathrm{g}^2 = \mathrm{g}^i\mathrm{g}_i = \gamma^{ij}\mathrm{g}_i\mathrm{g}_j$. We now consider a moving test particle of mass m in a spacetime with the time dependent metric tensor (1). The gravitoelectromagnetism force acting on this particle, as measured by the threading observers, is described

[1] Corresponding author, Islamic Azad University, Kashan Branch, Kashan, Iran

[2]Islamic Azad University, Kashan Branch, Kashan, Iran

[3]The Greek indices run from 0 to 3 while the Latin indices take the values 1 to 3.

by the following equation[4,5], [6]:

$$^*\mathbf{F} = \frac{^*d^*\mathbf{p}}{dt} - \frac{m}{\sqrt{1 - {}^*v^2}}\{^*\mathbf{E} + {}^*v \times {}^*\mathbf{B} + \mathbf{f}\}, \tag{4}$$

here $^*\mathrm{p}^i = \frac{m^*v^i}{\sqrt{1 - {}^*v^2}}$ such that $^*v^2 = \gamma_{ij}{}^*v^i\,{}^*v^j$ in which $^*v^i = \frac{v^i}{\sqrt{h}\,(1 - \mathrm{g}_k v^k)}$ and $v^i = \frac{dx^i}{dt}$. Also, the starry total derivative with respect to time is defined as $\frac{^*d}{dt} = \frac{^*\partial}{\partial t} + {}^*v^i\,{}^*\partial_i$ where $\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{h}}\frac{\partial}{\partial t}$ and $_{*i} = {}^*\partial_i = \partial_i + \mathrm{g}_i\frac{\partial}{\partial t}$. In equation (4), the last term is defined as

$$f^i = -({}^*\lambda^i_{jk}{}^*v^j + 2\mathrm{D}^i_k)^*v^k, \tag{5}$$

where 3-dimensional starry Christoffel symbols are defined as $^*\lambda^i_{jk} = \frac{1}{2}\gamma^{il}(\gamma_{jl*k} + \gamma_{kl*j} - \gamma_{jk*l})$ and deformation rates of the reference frame with respect to the observer are represented by the following tensors

$$\mathrm{D}_{ij} = \frac{1}{2}\frac{^*\partial\gamma_{ij}}{\partial t},\ \mathrm{D}^{ij} = -\frac{1}{2}\frac{^*\partial\gamma^{ij}}{\partial t}. \tag{6}$$

Finally, time dependent gravitoelectromagnetism fields are defined in terms of gravoelectric potential $\Phi = \ln\sqrt{h}$ and gravomagnetic vector potential $\mathrm{g} = (\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3)$ as follows[6]

$$^*\mathbf{E} = -{}^*\nabla\Phi - \frac{\partial \mathrm{g}}{\partial t}\ ;\ {}^*\mathrm{E}_i = -\Phi_{*i} - \frac{\partial \mathrm{g}_i}{\partial t}, \tag{7}$$

$$\frac{^*\mathbf{B}}{\sqrt{h}} = {}^*\nabla \times \mathrm{g}\ ;\ \frac{^*\mathrm{B}^i}{\sqrt{h}} = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\,\mathrm{g}_{[k*j]}. \tag{8}$$

For more details about applications of gravitoelectromagnetism fields, see references [7,8].

2. Motion of a test particle in the Bertotti-Robinson spacetime

2.1. Calculation of trajectory

At first, we assume that the metric of spacetime is given by the following line element

$$ds^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - d\Omega^2, \tag{9}$$

[4]The vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ has components as $\mathrm{C}^i = \frac{\varepsilon^{ijk}}{\sqrt{\gamma}}\mathrm{A}_j\mathrm{B}_k$ in which $\gamma = \det(\gamma_{ij})$ and 3-dimensional Levi-Civita tensor ε_{ijk} is skew-symmetric in any exchange of indices while $\varepsilon^{123} = \varepsilon_{123} = 1$, [5].

[5]We use the gravitational units with c=1.

[6]Here, curl of an arbitrary vector in a 3-space with metric γ_{ij} is defined by $({}^*\nabla\times\mathbf{A})^i = \frac{\varepsilon^{ijk}}{2\sqrt{\gamma}}\mathrm{A}_{[k*j]}$ while the symbol [ ] represents the anticommutation over indices.

which is usually assumed to cover all static spherical spacetimes. As is well known, the only static conformally flat spacetime of the form (9) is the Bertotti-Robinson solution which may be written in the following form, [9]:

$$ds^2 = \frac{1}{r^2}(dt^2 - dr^2 - r^2 d\Omega^2). \tag{10}$$

Below, we will calculate the trajectory of a test particle in the Bertotti-Robinson spacetime by using the Hamilton-Jacobi equation, [10-12]. Therefore, this equation is of the form

$$\left(r\frac{\partial \mathrm{S}}{\partial t}\right)^2 - \left(r\frac{\partial \mathrm{S}}{\partial r}\right)^2 - \left(\frac{\partial \mathrm{S}}{\partial \theta}\right)^2 - \left(\frac{1}{\sin\theta}\frac{\partial \mathrm{S}}{\partial \phi}\right)^2 - m^2 = 0, \tag{11}$$

where m is the mass of particle. For solving this partial differential equation, we use the method of separation of variables for the Hamilton-Jacobi function as follows

$$\mathrm{S}(t, r, \theta, \phi) = -Et + \mathrm{A}(r) + \mathrm{B}(\theta) + L\phi, \tag{12}$$

where E and L are arbitrary constants and can be identified respectively as energy and angular momentum of test particle along ϕ-direction. With substituting the last relation in the Hamilton-Jacobi equation, the integral expressions for the unknown functions A and B are given by

$$\mathrm{A} = \epsilon E \int \frac{\sqrt{r^2 - n^2}}{r}\, dr, \tag{13}$$

$$\mathrm{B} = \epsilon \int \sqrt{c^2 - L^2 \csc^2\theta}\, d\theta, \tag{14}$$

where $n = \epsilon\frac{\sqrt{m^2 + c^2}}{E}$ while c is the constant of separation and $\epsilon = \pm 1$ stands for the sign changing whenever $r(\text{or}\,\theta)$ passes through a zero of the integrand, [13]. Now the trajectory of the test particle can be obtained by using the Hamilton-Jacobi method as follows, [10-12]:

$$\frac{\partial \mathrm{S}}{\partial E} = \text{constant}, \quad \frac{\partial \mathrm{S}}{\partial L} = \text{constant}, \quad \frac{\partial \mathrm{S}}{\partial c} = \text{constant}. \tag{15}$$

Consequently, the set of equations (15) change to the following relations (respectively)

$$r = \sqrt{t^2 + n^2}, \tag{16}$$

$$\phi = \epsilon L \int \frac{d\theta}{\sin^2\theta\sqrt{c^2 - L^2\csc^2\theta}}, \tag{17}$$

$$\int \frac{dr}{rE\sqrt{r^2 - n^2}} = \int \frac{d\theta}{\sqrt{c^2 - L^2\csc^2\theta}}, \tag{18}$$

we have taken the constants in equations (15) to be zero, without any loss of generality. In continuation, after solving the equation (18), we obtain

$$\sin^2\theta = a + \epsilon b \sin(\zeta \arcsin(\frac{n}{r})), \tag{19}$$

where $a = \frac{c^2+L^2}{2c^2}$, $b = \frac{c^2-L^2}{2c^2}$ and $\zeta = \frac{2c}{nE}$. Also, from equations (17) and (18), we conclude

$$\phi = \epsilon L \int \frac{dr}{rE\sin^2\theta\sqrt{r^2-n^2}}. \tag{20}$$

However, calculations show that the above integral can be solved exactly only for the values $\zeta = 1, 2$. Here, we discuss these two cases as follows:

Case (1) : $\zeta = 1$.

It can be shown that the exact solution of the equation (20) is of the form

$$r = \frac{\epsilon n b e^{-4i\epsilon\phi} + 4n^2 a e^{-2i\epsilon\phi} + 4\epsilon n^3 b}{a e^{-4i\epsilon\phi} + 4\epsilon n b e^{-2i\epsilon\phi} + 4n^2 a}, \tag{21}$$

where $i = \sqrt{-1}$. Let us now restrict our analysis to the following cases:

subcase (1) : $a = 0$.

In this case, the equation (21) is transformed to

$$r = n^2 e^{2i\epsilon\phi} + \frac{1}{4}\, e^{-2i\epsilon\phi}. \tag{22}$$

If we choose $n = \pm\frac{1}{2}$, then we have the following results

$$r = \frac{1}{2}\cos(2\phi), \tag{23}$$

$$\theta = \arcsin\sqrt{\sec(2\phi)}\,. \tag{24}$$

Therefore, the trajectories that describe by these equations will be bounded, i.e. the particle can be trapped by the extended object with the Bertotti-Robinson geometry.

subcase (2) : $b = 0$.

In this case, the equation (21) is transformed to

$$\frac{1}{r} = e^{2i\epsilon\phi} + \frac{1}{4n^2}\, e^{-2i\epsilon\phi}. \tag{25}$$

Like in previous case, if we choose $n = \pm\frac{1}{2}$, then we have

$$\frac{1}{r} = 2\cos(2\phi). \tag{26}$$

From this equation, we can see that the trajectory of particle is bounded and motion occurs on surface $\theta = \dfrac{\pi}{2}$ with time period $\mathrm{T} = \pi$. In these two subcases, the energy of particle is[7] $E = \dfrac{4m}{\sqrt{3}}$ which depends on the particle mass.

subcase (3) : $a = b$.

In this case, particle is at rest at point $(\epsilon n, \dfrac{\pi}{2}, 0)$ at time $t = 0$.

[7] In the next section, we will show that the energy of particle is equal to E.

Case (2) : $\zeta = 2$.

As before, it is easy to show that the following identity is valid, [14]:

$$2\arcsin x = \begin{cases} \arcsin(2x\sqrt{1-x^2}) & |x| \leq \frac{1}{\sqrt{2}}, \\ \pi - \arcsin(2x\sqrt{1-x^2}) & \frac{1}{\sqrt{2}} < x \leq 1, \\ -\pi - \arcsin(2x\sqrt{1-x^2}) & -1 \leq x < -\frac{1}{\sqrt{2}}, \end{cases} \tag{27}$$

where x is an arbitrary variable. By considering this identity and applying the equation (16), the solution of equation (20) in terms of t will be

$$\tan\phi = \frac{(at + \epsilon nb)c}{nL}. \tag{28}$$

But for value $\zeta = 2$, we have $m = 0$. Hence, we ignore the study of this case.

2.2. Calculation of the gravitoelectromagnetism force

At first, from the equations (16-18), we can deduce

$$^*v^i = \epsilon \begin{cases} \sqrt{r^2 - n^2} & i = 1, \\ \frac{\sqrt{c^2 - L^2\csc^2\theta}}{rE} & i = 2, \\ \frac{\epsilon L}{rE\sin^2\theta} & i = 3. \end{cases} \tag{29}$$

With applying this equation and after simplifying, we infer

$$\frac{m}{\sqrt{1 - {^*v^2}}} = rE. \tag{30}$$

Before continuing, we know that the energy of particle, as measured by threading observers located at the trajectory, is given by $\mathcal{E} = \frac{m\sqrt{h}}{\sqrt{1-{^*v^2}}}$, which is a conserved quantity during the motion of particle, [3]. As a result, from the equation (30), we conclude that $\mathcal{E} = E$. In the next step, all nonzero components of starry Christoffel symbols are calculated as[8]

$$\begin{aligned} {^*\lambda^1_{11}} &= -\frac{1}{r}, \\ {^*\lambda^2_{33}} &= -\sin\theta\cos\theta, \\ {^*\lambda^3_{23}} &= \cot\theta. \end{aligned} \tag{31}$$

Also, the nonzero components of deformation tensors are

$$\mathrm{D}_{11} = -\frac{\epsilon\sqrt{r^2 - n^2}}{r^3}, \tag{32}$$

$$\mathrm{D}_{33} = \frac{\epsilon\sin 2\theta\sqrt{c^2 - L^2\csc^2\theta}}{2rE}. \tag{33}$$

[8]In our notation $(r, \theta, \phi) \equiv (1, 2, 3)$.

At this stage, from the equations (29-33), we can derive the following expressions

$$f^i = \begin{cases} \frac{3(r^2-n^2)}{r} & i=1, \\ \frac{L^2\cot\theta}{r^2E^2\sin^2\theta} & i=2, \\ -\frac{4\epsilon L\cot\theta\sqrt{c^2-L^2\csc^2\theta}}{r^2E^2\sin^2\theta} & i=3. \end{cases} \tag{34}$$

and

$$\frac{{}^*d\,{}^*\mathrm{p}^i}{dt} = \begin{cases} 4(2r^2-n^2)E & i=1, \\ \frac{4L^2\cot\theta}{rE\sin^2\theta} & i=2, \\ -\frac{8\epsilon L\cot\theta\sqrt{c^2-L^2\csc^2\theta}}{rE\sin^2\theta} & i=3. \end{cases} \tag{35}$$

By considering equations (30), (34), (35) and some calculations, we finally obtain

$${}^*\mathbf{F} = \left((4r^2-n^2)E\,,\ \frac{3\sqrt{b}\,L^2\sqrt{r-\epsilon n}}{(ar+\epsilon nb)^{\frac{3}{2}}E}\,,\ -\frac{2\epsilon nbL\sqrt{r^2-n^2}}{(ar+\epsilon nb)^2}\right). \tag{36}$$

To continue our analysis, we are going to determine the potential function corresponding to the gravitoelectromagnetism force. For doing this, we can prove that in a 3-space with time dependent metric γ_{ij} the following identity is valid, [7]:

$$[{}^*\partial_i\,,{}^*\partial_j] = \sqrt{h}\,\mathrm{g}_{[j*i]}\frac{{}^*\partial}{\partial t}. \tag{37}$$

With the help of this identity and equation (8), we obtain the following relation[9]

$${}^*\nabla\times{}^*\nabla\Psi = \frac{{}^*\partial\Psi}{\partial t}\,{}^*\mathbf{B}, \tag{38}$$

where Ψ is an arbitrary scalar function. But the gravitomagnetic fields for the metric (10) are zero, so the above relation changes to

$${}^*\nabla\times{}^*\nabla\Psi = 0. \tag{39}$$

On the other hand, by taking the curl of the force, we get

$$({}^*\nabla\times{}^*\mathbf{F})^i = r^{\frac{3}{2}}\begin{cases} \frac{4\epsilon L\sqrt{b}\sqrt{r-\epsilon n}}{ar+\epsilon nb}\left(\frac{a\sqrt{r^2-n^2}}{ar+\epsilon nb}-\frac{n^2}{r\sqrt{r^2-n^2}}\right)+\frac{3\epsilon L\sqrt{b}\sqrt{r+\epsilon n}\,(2ar-n\epsilon(3a+b))}{2(ar+\epsilon nb)^2} & i=1, \\ \frac{2\epsilon n^2E^2\sqrt{ar+\epsilon nb}\sqrt{r^2-n^2}}{Lr^3}-\frac{2\epsilon nbL}{r(ar+\epsilon nb)^{\frac{3}{2}}}\left(\frac{a\sqrt{r^2-n^2}}{ar+\epsilon nb}-\frac{n^2}{r\sqrt{r^2-n^2}}\right) & i=2, \\ -\frac{3L^2\sqrt{b}\,(2ar-n\epsilon(3a+b))}{2E\sqrt{r-\epsilon n}\,(ar+\epsilon nb)^3}-\frac{4nE\sqrt{r-\epsilon n}}{r^2} & i=3, \end{cases}$$
$$\neq 0. \tag{40}$$

Consequently, by comparing the equations (39) and (40), we cannot define the potential function (V) corresponding to the gravitoelectromagnetism force with the following familiar form

$${}^*\mathbf{F} = -{}^*\nabla\mathrm{V}. \tag{41}$$

[9]Proof of this identity is simple and have been omitted.

3. **Conclusions**

In this work, we studied the classical motion of a test particle in the Bertotti-Robinson spacetime. We proved that the particle can be trapped by this gravitational field. By determining the gravitoelectromagnetism force, it was shown that the existence of the potential function with the classical definition is impossible.

Acknowledgment

This research has been supported in part by the Islamic Azad University-Kashan Branch.

REFERENCES

[1] *R.T. Jantzen, P. Carini and D. Bini*, The many faces of gravitoelectromagnetism, Ann. Phys. **215** (1992) 1-50.

[2] *B. Mashhoon*, Gravitoelectromagnetism: A brief review, 2008, arXiv:gr-qc/0311030v2.

[3] *R. Jantzen and P. Carini*, Understanding spacetime splittings and their relationships in classical mechanics and relativity: Relationship and consistency, ed. by G. Ferrarese, Bibliopolis, Naples (1991) 185.

[4] *S. Boersma and T. Dray*, Slicing, threading and parametric manifolds, Gen. Relativ. Gravit. **27** (1995) 319-339; *Z. Perje's*, The parametric manifold picture of space-time, Nucl. Phys. B **403** (1993) 809-837.

[5] *L.D. Landau and E.M. Lifishitz*, The classical theory of fields, Pergamon Press, New York, 1975.

[6] *A. Zel'manov*, Chronometric invariants and co-moving coordinates in the general relativity theory, Soviet. Phys. Doklady. **1** (1956) 227-230; Chronometric invariants, American Research Press, Rehoboth, New Mexico, 2006.

[7] *M. Yavari*, Derivation of gravitational continuity equation from time-dependent quasi-Maxwell equations, Nuovo Cimento B **124** (2009) 197-204.

[8] *M. Yavari*, Exact solution of Petrov type $\{3, 1\}$ metric via time dependent quasi-Maxwell equations, Int. J. Theor. Phys. **48** (2009) 3169-3172.

[9] *B. Bertotti*, Uniform electromagnetic field in the theory of general relativity, Phys. Rev. **116** (1959) 1331-1333; *I. Robinson*, A solution of the Einstein-Maxwell equations, Bull. Acad. Pol. Sci., Ser. Mat. Fis. Astr. **7** (1959) 351-372.

[10] *S. Chakraborty*, Motion of test particles around monopoles, Gen. Relativ. Gravit. **28** (1996) 1115-1119.

[11] *S. Chakraborty and L. Biswas*, Motion of test particles in the gravitational field of cosmic strings in different situations, Class. Quantum Gravit. **13** (1996) 2153-2161.

[12] *J. Gamboa and A.J. Segui-Santonja*, Motion of test particles around cosmic strings, Class. Quantum Gravit. **9** (1992) L111-L114.

[13] *J.L. Synge and B.A. Griffith*, Principles of mechanics, McGraw-Hill, New York, 1970.

[14] *I.S. Gradshteyn and I.M. Ryzhik*, Table of integrals, series, and products, Seventh edition, ed. by A. Jeffrey and D. Zwillinger, Academic Press, New York, 2007, p. 59.