ON ERROR ESTIMATION OF AUTOMATIC QUADRATURE SCHEME FOR THE EVALUATION OF HADAMARD INTEGRAL OF SECOND ORDER SINGULARITY

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This paper presents an automatic quadrature scheme (AQS) for the evaluation of hypersingular integrals (HSI)

$$Q_i(f,x) = \int_{-1}^{1} \frac{w_i(t)f(t)}{(t-x)^2}dt, \quad x \in [-1,1], \; i = 0,1,2, \tag{1}$$

where $w_0(t) = 1,\; w_1(t) = \sqrt{1-t^2},\; w_2(t) = \frac{1}{\sqrt{1-t^2}}$ *are the weights, and the given function* f *imperative to have certain smoothness or continuity properties. Particular attention is paid to error estimate of the developed AQS, where it shows the acquired AQS scheme is obtained in the class of functions* $C^{N+2,\alpha}[-1,1]$ *which converges to the exact very fast by increasing the knot points. The first and second kind of Chebyshev polynomials are used in the conjecture. Several numerical examples clearly demonstrate the developed AQS rendering efficient, accurate and reliable results. This research gives comparative performances of the present method with others.*

Keywords: Hypersingular integrals, Chebyshev series, Interpolation, Automatic quadrature scheme, Error estimate.

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1. Introduction

One of the most valuable mathematical tools is the subject of singular integrals in both pure and applied mathematics. There are many papers in the literature on singular and hypersingular integral. A precise evaluation of HSI is only possible in rare cases, therefore there is a need to enrich the approximate methods for evaluating it. By the implementation of various techniques, the transformation of HSIs into singular or weakly singular integrals by using different techniques, provides basis to one group of methods [1, 2, 3, 4, 5, 9] whereas another group is based on the numerical computation of finite part integrals by various quadrature or cubature formulas [6, 7, 10]. In 2012, Tadeu and Antonio [1] present an analytical evaluation of the singular and hypersingular integrals for three-dimensional boundary acoustic problems. A review of dual boundary element methods on hypersingular integrals and divergent series can be found in the work Chen [8]. Yang [10] proposed a general class of methods for the evaluation of hypesingular and supersingular integrals with a periodic integrand of singularity higher than or equal to 2. Furthermore, HSI

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turns out in many engineering issues [11, 12] with a common usage in fracture mechanics for the prediction of stress intensity and crack propagation [8]. The automatic quadrature problem has been considered by Clenshaw and Curtis, for a simple integral [9]. One of the nodal methods is the automatic quadrature formula [3, 4, 9] which is combined with the collocation method to solve Cauchy principal-value and hypersingular integrals. The Gaussian quadrature rule have been used in [7] and the formula is then specialized into the Legendre and Chebyshev series expansion, which use classical orthogonal polynomials and give high order accuracy for hypersingular integrals with second-order singularities of the form

$$Q(f,x) = \neq\!\!\!\!\!\int_a^b \frac{f(t)dt}{(t-x)^2}. \tag{2}$$

The widely used definition of HSI in the engineering society is

$$\neq\!\!\!\!\!\int_{-1}^1 \frac{f(t)dt}{(t-x)^2} = \lim_{\alpha\to 0}\left[\left(\int_{-1}^{x-\varepsilon} + \int_{x+\varepsilon}^1\right)\frac{f(t)dt}{(t-x)^2} - \frac{2f(x)}{\varepsilon}\right], \tag{3}$$

where this limits of integration are exists and bounded. Providing that the regular part of the integrand in Eq. (3) is a function $f(t), a \le t \le b$, which satisfies a Hölder continuous first-derivative condition

$$|f(t) - f(x) - (t-x)f'(x)| \le A|t-x|^{\alpha+1} \tag{4}$$

where A is a positive constant and $\alpha \in (0,1]$. Nevertheless, the differentiation of Cauchy principle-value integral (CPVI)

$$I(f,x) = \int_{-1}^1 \frac{f(t)}{t-x}dt; \qquad x \in (a,b), \tag{5}$$

with respect to the singular point x, gives another definition of HSI [13]

$$\neq\!\!\!\!\!\int_{-1}^1 \frac{f(t)}{(t-x)^2}dx = \frac{d}{dx}\int_{-1}^1 \frac{f(t)}{(t-x)}dt, \quad x \in (-1,1) \tag{6}$$

which is very serviceable in evaluating HSI. The relation in (6) implies that the HSI represent a natural extension of singular integrals in the Cauchy principal-value. Other properties of finite-part integrals and many references to the related literature can be found in [13, 14].

In this paper, we develop AQS for the evaluation of hypersingular integrals, which has a global error rate calculated by Chebyshev norm for functions of $C^{N+2,\alpha}[-1,1]$. We extend Hasegawa's work [3] for CPV integral to evaluate the HSI (2). To construct the AQS for the HSI (2), we use the orthogonal Chebyshev series of the first and second kinds. We have selected the same singular point and weight function as in Hui and Shia [7] at different nodes. In our approach the parameter x is not equal to the roots of the Chebyshev orthogonal polynomials of the first and second kinds, and hence give more convenient in the practical computations.

The paper is organized as follows, the basic concepts pertaining to Chebyshev polynomials are considered in Section 2. In Section 3, the details of construction of AQS by Chebyshev series expansion of the first kind is given along with the derivation of the recurrence relation for the sequence of the interpolating polynomials $\{P_N(t)\}$. Section 4 is related to the direct derivation of AQS for special weight function. Section 5 covers the proof of the efficiency of the proposed algorithm where the error estimates have been studied in details. Finally, the numerical results for the constructed AQS are presented, along with the comparison performance of Kutt [21], Hui and Shia's [7] results.

2. **Basic concepts**

The Chebyshev polynomials $T_n(t)$, $U_n(t)$ of the first and second kinds are elucidated on the interval [-1,1] according to the following trigonometric formulae [15]

$$T_n(t) = \cos n\theta,$$
$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad n = 0, ..., N,$$

where $t = \cos\theta, 0 \leq \theta \leq \pi$. These polynomials are orthogonal on [-1,1] with respect to

$$W_1(t) = \sqrt{1-t^2} \qquad \text{and} \qquad W_2(t) = \frac{1}{\sqrt{1-t^2}}$$

respectively, and they both share the same recurrence relation

$$P_n(t) = 2tP_{n-1}(t) - P_{n-2}(t), \tag{7}$$

with a common starting value $P_0(t) = 1$, and

$$P_1(t) = T_1(t) = t, \qquad P_1(t) = U_1(t) = 2t. \tag{8}$$

The indefinite integral of $T_n(t)$ and $U_n(t)$ can be expressed in terms of Chebyshev polynomials as follows

$$\int T_n(t)dt = \frac{1}{2}\left(\frac{T_{n+1}(t)}{n+1} - \frac{T_{n-1}(t)}{n-1}\right), \quad n \geq 2, \tag{9}$$

$$\int U_n(t)dt = \frac{T_{n+1}(t)}{n+1}, \quad n \geq 2, \tag{10}$$

and after setting $t = \cos\theta$, the differentiation formulae are

$$\frac{d}{dt}T_n(t) = \frac{n\sin n\theta}{\sin\theta} = nU_{n-1}(t). \tag{11}$$

and

$$\frac{d}{dt}U_n(t) = \frac{(n+1)T_{n+1}(t) - tU_n(t)}{t^2-1}. \tag{12}$$

It is well revealed that the Chebyshev polynomials of first and second kinds are integral transforms of each other with respect to weighted Hilbert kernels,

$$\frac{d}{dt}\left(\sqrt{1-t^2}U_{n-1}(t)\right) = \frac{-nT_n(t)}{\sqrt{1-t^2}}\ , \tag{13}$$

and

$$⨍_{-1}^{1} \frac{\sqrt{1-t^2}U_{n-1}(t)}{t-x}dx = -\pi T_n(x), \tag{14}$$

which may readily be proved by induction [16]. Notice that, for the Chebyshev polynomials of first kind, we have [15]

$$⨍_{-1}^{1} \frac{T_{n+1}(t)}{\sqrt{1-t^2}(t-x)}dt = \pi U_n(x). \tag{15}$$

From Eq. (14), it is easy to show

$$\fint_{-1}^{1} \frac{\sqrt{1-t^2}U_k(t)}{(t-x)^2}dt = -\pi(k+1)U_k(x). \tag{16}$$

3. Construction of the automatic quadrature scheme

For simplicity, we take $a=-1, b=1$ in (2).

Let

$$P_N(t) = \sum_{k=0}^{N}{}'' a_k^N T_k(t), \quad -1 \le t \le 1 \tag{17}$$

where the double prime means the first and last terms are halved. The sample nodes $t_j = \cos(\pi j/N),\ 0 \le j \le N$, are the zeros of the polynomial $\omega_{N+1}(t)$ defined by

$$\omega_{N+1}(t) = T_{N+1}(t) - T_{N-1}(t) = 2(t^2-1)U_{N-1}(t). \tag{18}$$

It is known that this approximation leads directly to the Clenshaw-Curtis method (CC method). To be specific, the approximation (17) yields the integration rule $I_N(f,x)$ to $I(f,x)$ (see [3]). Subtracting out the singularity, $I(f,x)$ in (5) gives

$$I(f,x) = \fint_{-1}^{1} \frac{f(t)-f(x)}{t-x}dt + f(x)\log\left(\frac{1-x}{1+x}\right), \tag{19}$$

and by applying the approximate polynomial $P_N(t)$ in (17) to interpolate $f(t)$, the above integral becomes

$$I(f,x) \approx I_N(f,x) = \fint_{-1}^{1} \frac{P_N(t)-P_N(x)}{t-x}dt + f(x)\log\left(\frac{1-x}{1+x}\right). \tag{20}$$

The integrant in (20), can be written as

$$\frac{P_N(t)-P_N(x)}{t-x} = \sum_{k=0}^{N-1}{}' d_k T_k(t). \tag{21}$$

Taking into account (9)-(10) and by integrating term by term, Eq. (21) becomes

$$\fint_{-1}^{1} \frac{P_N(t)-P_N(x)}{t-x}dt = 2d_0 + \sum_{k=2}^{N-1}{}' d_k \frac{1}{2}\left[\frac{T_{k+1}(t)}{k+1} - \frac{T_{k-1}(t)}{k-1}\right]_{-1}^{1}. \tag{22}$$

Substitution (22) into (19) yields an AQS for Cauchy value integral [3]

$$I_N(f,x) = \fint_{-1}^{1} \frac{f(t)}{t-x}dt = 2\sum_{k=0}^{[\frac{N}{2}-1]}{}' \frac{d_{2k}}{1-4k^2} + f(x)\log\left(\frac{1-x}{1+x}\right), \tag{23}$$

where the prime means the first term is halved and assume that N is even. The polynomial coefficients d_k in (21) can be stably calculated by employing the recurrence relation

$$d_{k+1} - 2xd_k + d_{k-1} = 2a_k^N, \qquad k = N, N-1, \ldots, 1, \tag{24}$$

in the backward direction with the starting condition $d_N = d_{N+1} = 0$.
From the interpolation condition

$$P_N(\cos \pi j/N) = f(\cos \pi j/N), \qquad 0 \le j \le N \tag{25}$$

the coefficients a_k^N in (17) are determined as follows

$$a_k^N = \frac{2}{N} \sum_{j=0}^{N}{}'' f(\cos \pi j/N) \cos(\pi k j/N), \ 0 \le k \le N. \tag{26}$$

It is known that the Fast Fourier Transform (FFT) is valuable for efficiently evaluating equation (26) [17].
Differentiate (23), and assuming that $d_{2k} = d_{2k}(x)$, and taking into account (6), yields

$$Q_0(f,x) = \int_{-1}^{1} \frac{f(t)}{(t-x)^2} dt = 2 \sum_{k=0}^{[\frac{N}{2}-1]}{}' \frac{d'_{2k}(x)}{1-4k^2} + f'(x) \log\left(\frac{1-x}{1+x}\right) - \frac{2f(x)}{1-x^2} + R_0(f,x), \tag{27}$$

which is the AQS formula for Eq. (2).
In Eq. (27) the coefficients $d'_{2k}(x)$ are computed in a similar way as in (24). The derivative of (24) with respect to x leading to the following backward recurrence relation.

$$d'_{k+1}(x) - 2x d'_k(x) + d'_{k-1}(x) = 2d_k^N, \qquad k = N, N-1, \ldots, 1, \tag{28}$$

with the initial values $d'_N(x) = d'_{N+1}(x) = 0$, while the values of d_k^N in (28) are calculated using Eq. (24) along with the starting condition $d_N = d_{N+1} = 0$ and N takes

$$N = 3, 4, 5, \ldots, 3 \times 2^n, 4 \times 2^n, 5 \times 2^n, \ldots \ (n = 1, 2, \ldots). \tag{29}$$

4. **Automatic quadrature scheme for special weight functions**

Consider HSIs of the form

$$Q_1(f,x) = \int_{-1}^{1} \frac{\sqrt{1-t^2} f(t)}{(t-x)^2} dt. \tag{30}$$

The AQS for Eq. (30) is acquired by approximating $f(x)$ by the Chebyshev polynomial of the second kind,

$$f(t) \approx \sum_{k=1}^{N} b_k U_{k-1}(t), \quad -1 \le t \le 1. \tag{31}$$

Substituting (31) into (30) and applying (16), gives

$$Q_1(f,x) \cong \sum_{k=1}^{N} b_k \int_{-1}^{1} \frac{\sqrt{1-t^2} U_{k-1}(t)}{(t-x)^2} dt = -\pi \sum_{k=1}^{N} k b_k U_{k-1}(x). \tag{32}$$

Coefficients b_k in (32) are determined by involving Eqs. (31) and (11) along with the orthogonality relation [19]

$$\sum_{k=0}^{N}{}'' \sin\left(\frac{ki\pi}{N}\right)\sin\left(\frac{kj\pi}{N}\right) = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{N}{2} & \text{if } i = j \neq 0 \text{ or } N, \\ N & \text{if } i = j = 0 \text{ or } N. \end{cases} \tag{33}$$

The collocation points are chosen as the roots of extrema of $T_N(x)$ on [-1,1], i.e.

$$t = t_j = \cos\frac{j\pi}{N}, \quad j = 0, \cdots, N, \tag{34}$$

implies,

$$\sum_{k=1}^{N} b_k \sin\frac{kj\pi}{N} \approx g(\cos\frac{j\pi}{N})\sin\frac{j\pi}{N}. \tag{35}$$

From Eqs. (34) and (35), it follows that

$$b_k = \frac{2}{N}\sum_{j=1}^{N} f(\cos\pi j/N)\sin(\pi j/N)\sin(\pi kj/N), \ 1 \leq j \leq N. \tag{36}$$

For the case of $i = 2$ in Eq. (1), we have

$$Q_2(f,x) = \fint_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}(t-x)^2}dt \tag{37}$$

The AQS for Eq (37) is obtained by using the following Chebyshev series [18],

$$f(t) \approx \sum_{k=0}^{N}{}'' a_k T_k(t), \quad -1 \leq t \leq 1. \tag{38}$$

With the help of Eq. (6) along with the concepts (12) and (15), yields

$$\begin{aligned} \fint_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}(t-x)^2}dt &\cong \sum_{k=0}^{N} a_k \fint_{-1}^{1} \frac{T_k(t)}{\sqrt{1-t^2}(t-x)^2}dt \\ &= \pi\sum_{k=0}^{N} a_k \left(\frac{kU_k(x) - (k+1)xU_{k-1}(x)}{x^2-1}\right). \end{aligned} \tag{39}$$

From Eq. (38) and choosing the collocation points as in Eq. (34) along with the orthogonality relation [19]

$$\sum_{k=0}^{N}{}'' \cos\left(\frac{ki\pi}{N}\right)\cos\left(\frac{kj\pi}{N}\right) = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{N}{2} & \text{if } i = j \neq 0 \text{ or } N, \\ N & \text{if } i = j = 0 \text{ or } N. \end{cases} \tag{40}$$

We obtain

$$a_k = \frac{2}{N}\sum_{j=0}^{N}{}'' f(\cos\frac{j\pi}{N})\cos\frac{jk\pi}{N}, \qquad 0 \leq j \leq N. \tag{41}$$

5. **Error estimate**

Let $e_N(t) = f(t) - P_N(t)$ and the Chebyshev norm of the form

$$\| e_N \|_c = \max_{-1\le a\le t\le b\le 1} |f(t) - P_N(t)|, \tag{42}$$

and

$$R_{0N}(f,x) = \Big| Q_0(f,x) - Q_{0N}(f,x) \Big|. \tag{43}$$

Theorem 5.1. Let $f(t) \in C^{N+2,\alpha}[-1,1]$ and $x \in (-1,1)$, then the error term of AQS (27) is estimated as

$$||R_{0N}(f,x)|| \le \frac{9.23L_1}{2^{N-1}(N-1)!}\frac{\ln(N+1)}{N} + \frac{L_3}{2^{N-1}(N+1)!(N+1)^\alpha}\left(21.58 + \frac{11.16C}{\alpha}\right), \tag{44}$$

where

$L_1 = \max\{M_1, M_2\}; \qquad M_1 = \max_{-1\le\eta_t\le 1}\left|f^{(N+1)}(\eta_t)\right|, \quad M_2 = \max_{-1\le\eta_t\le 1}\left|\frac{d}{dx}f^{(N+1)}(\eta_t)\right|$

$L_3 = \max\{A_2, M_2B_2, A_1, M_1B_{1,3}\},\ B_{1,3} = \max\{B_1, B_3\},$

and A_2 is a Holder constant of $\frac{d}{dx}f^{(N+1)}(\eta_x)$, A_1 is a Lipschitz constant of $f^{(N+1)}(\eta_x)$, B_1 and B_3 are Lipschitz constant of Chebyshev polynomial of first kind $T_N(\xi_x)$ and second kind $U_N(\xi_x)$ times polynomial ξ_x respectively, and
$|\eta_t - x| \le c|t-x|,\ \eta_t, t \in (x-\epsilon, x+\epsilon)$.

Proof:

Without losing the generality, we prove for $x \in [0,1)$. The case $x \in (-1,0]$ is proved in a similar way to $x \in [0,1)$. The error term of AQS (27) can be written as

$$R_N(f,x) = \frac{d}{dx}\left(\int_{-1}^{1}\frac{e_N(t)-e_N(x)}{t-x}dt\right). \tag{45}$$

By dividing the interval [-1,1] into two parts, we get

$$\begin{aligned} R_N(f,x) &= \frac{d}{dx}\left(\left(\int_{-1}^{x-\epsilon} + \int_{x+\epsilon}^{1}\right)\frac{e_N(t)-e_N(x)}{t-x}dt + \int_{x-\epsilon}^{x+\epsilon}\frac{e_N(t)-e_N(x)}{t-x}dt\right) \\ &= R_1(x) + R_2(x), \end{aligned} \tag{46}$$

where

$$R_1(x) = \frac{d}{dx}\left(\int_{-1}^{x-\epsilon} + \int_{x+\epsilon}^{1}\right)\frac{e_N(t)-e_N(x)}{t-x}dt,$$

$$R_2(x) = \frac{d}{dx}\int_{x-\epsilon}^{x+\epsilon}\frac{e_N(t)-e_N(x)}{t-x}dt.$$

It is known that [20]

$$\frac{d}{dx}\int_{A(x)}^{B(x)} K(x,t)dt = \int_{A(x)}^{B(x)}\frac{\partial K(x,t)}{\partial x}dt + K(x,B(x))B'(x) - K(x,A(x))A'(x). \tag{47}$$

The kernel $K(x,t)$ in Eq. (46) is given by

$$K(x,t)=\frac{e_N(t)-e_N(x)}{t-x},$$

and its derivative is

$$\frac{\partial K}{\partial x}=\frac{e_N(t)-e_N(x)}{(t-x)^2}-\frac{e'_N(x)}{t-x}.$$

Applying (47) for $R_1(x)$ in Eq. (46), yields

$$\begin{aligned}R_1(x)&=\left(\int_{-1}^{x-\epsilon}+\int_{x+\epsilon}^{1}\right)\frac{e_N(t)-e_N(x)}{(t-x)^2}dt-e'_N(x)\left(\ln\frac{1-x}{1+x}\right)+\\&+\frac{1}{\epsilon}\left(e_N(x)-e_N(x-\epsilon)-(e_N(x+\epsilon)-e_N(x))\right)\\&=R_{11}(x)+R_{12}(x)+R_{13}(x),\end{aligned}\tag{48}$$

where

$$\begin{aligned}R_{11}(x)&=\left(\int_{-1}^{x-\epsilon}+\int_{x+\epsilon}^{1}\right)\frac{e_N(t)-e_N(x)}{(t-x)^2}dt,\\R_{12}(x)&=-e'_N(x)\ln\frac{1-x}{1+x},\\R_{13}(x)&=\frac{1}{\epsilon}\left[e_N(x)-e_N(x-\epsilon)-(e_N(x+\epsilon)-e_N(x))\right].\end{aligned}$$

In the sequel, we implement (42), and by applying the Mean Value Theorem, we acquire

$$\begin{aligned}|R_{11}(x)|&\le\left|\int_{-1}^{x-\epsilon}\frac{e'_N(\xi_{1t})}{t-x}dt\right|+\left|\int_{x+\epsilon}^{1}\frac{e'_N(\xi_{2t})}{t-x}dt\right|\\&\le\| e'_N(\xi_1)\|_1\, C_1(\epsilon)+\ \| e'_N(\xi_2)\|_2\, C_2(\epsilon)\\&\le(C_1(\epsilon)+C_2(\epsilon))\ \| e'_N\|_c,\end{aligned}\tag{49}$$

where

$$\begin{aligned}\| e'_N\|_c&=\max_{-1\le t\le 1}|\, e'_N(t)\,|,\\C_1(\epsilon)&=\left\|\int_{-1}^{x-\epsilon}\frac{dt}{t-x}\right\|=\left\|\ln\frac{\epsilon}{1+x}\right\|=\max_{0<x<1}\left|\ln\frac{\epsilon}{1+x}\right|=\left|\ln\frac{\epsilon}{2}\right|,\\C_2(\epsilon)&=\left\|\int_{x+\epsilon}^{1}\frac{dt}{t-x}dt\right\|=\left\|\ln\frac{1-x}{\epsilon}\right\|=\max_{0\le x<1}\left|\ln\frac{1-x}{\epsilon}\right|=\left|\ln\frac{1}{\epsilon}\right|.\end{aligned}$$

Applying (13), and from [4]

$$e_N(t)=\frac{f^{(N+1)}(\eta_t)}{2^{N-1}(N+1)!}(t^2-1)U_{N-1}(t),\qquad \xi_t\in(-1,1)\tag{50}$$

we have

$$e'_N(t)=\frac{-1}{2^{N-1}(N+1)!}\left[\frac{d}{dt}f^{(N+1)}(\eta_t)(1-t^2)U_{N-1}(t)+f^{(N+1)}(\eta_t)\Big(NT_N(t)+tU_{N-1}(t)\Big)\right].\tag{51}$$

Since $\xi_{1t}, \eta_t \in [-1, x-\epsilon]$, we obtain

$$\left|e'_N(t)\right| \leq \frac{1}{2^{N-1}(N+1)!}\left[M_2 + M_1\left(N + \frac{1}{\sqrt{\epsilon(2-\epsilon)}}\right)\right] \leq \frac{L_1}{2^{N-1}(N+1)!}\left[N+1+\frac{1}{\sqrt{\epsilon(2-\epsilon)}}\right], \tag{52}$$

where

$$L_1 = \max\{M_1, M_2\}, \quad M_1 = \max_{-1\leq\eta_t\leq 1}\left|f^{(N+1)}(\eta_t)\right|, \quad M_2 = \max_{-1\leq\eta_t\leq 1}\left|\frac{d}{dx}f^{(N+1)}(\eta_t)\right|.$$

From Eq. (49), and let $\epsilon = \frac{2}{(N+1)^2}$; $N \geq 5$, the error for R_{11} is as follows:

$$\parallel R_{11} \parallel \leq \frac{6.11 L_1}{2^{N-1}(N-1)!}\frac{\ln(N+1)}{N} \tag{53}$$

For the estimation of R_{12}, we have

$$\begin{aligned}|R_{12}| = |e'_N(x)|\left|\ln\frac{1-x}{1+x}\right| &\leq \parallel e'_N \parallel_c \left\|\ln\frac{1-x}{1+x}\right\| \\ &\leq \frac{L_1}{2^{N-1}(N+1)!}\left[N+1+\frac{1}{\sqrt{\epsilon(2-\epsilon)}}\right]\left|\ln\frac{\epsilon}{2-\epsilon}\right|,\end{aligned}$$

and substitute the chosen $\epsilon = \frac{2}{(N+1)^2}$; $N \geq 5$, and following the same procedures as in (52) for the norm of $e'_N(x)$, results

$$|R_{12}| \leq \frac{3.12 L_1}{2^{N-1}(N-1)!}\frac{\ln(N+1)}{N}. \tag{54}$$

For $R_{13}(x)$, we apply the Mean Value Theorem, yields

$$R_{13}(x) = e'_N(\xi_{1x}) - e'_N(\xi_{2x}),$$

where $\xi_{1x} \in (x-\epsilon, x)$, $\xi_{2x} \in (x, x+\epsilon)$ and $\| \xi_{2x} - \xi_{1x} \| = 2\epsilon$, then by following Mushkelishvili [22] p.13, for the inequality of product function, we obtain

$$
\begin{aligned}
|R_{13}(x)| \le \frac{1}{2^{N-1}(N+1)!} \Bigg[& \left| \frac{d}{dx} f^{(N+1)}(\eta_{1x}) - \frac{d}{dx} f^{(N+1)}(\eta_{2x}) \right| \left| (1-\xi_{1x}^2) U_{N-1}(\xi_{1x}) \right| \\
& + \left| \frac{d}{dx} f^{(N+1)}(\eta_{2x}) \right| \left| (1-\xi_{1x}^2) U_{N-1}(\xi_{1x}) - (1-\xi_{2x}^2) U_{N-1}(\xi_{2x}) \right| \\
& + \left| f^{(N+1)}(\eta_{1x}) - f^{(N+1)}(\eta_{2x}) \right| \left| N T_N(\xi_{1x}) + \xi_{1x} U_{N-1}(\xi_{1x}) \right| \\
& + \left| f^{(N+1)}(\eta_{2x}) \right| \left| N(T_N(\xi_{1x}) - T_N(\xi_{2x})) + \xi_{1x} U_{N-1}(\xi_{1x}) - \xi_{2x} U_{N-1}(\xi_{2x}) \right| \Bigg]
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
&\le \frac{1}{2^{N-1}(N+1)!} \Bigg[A_2 |\eta_{1x} - \eta_{2x}|^\alpha + M_2 B_2 |\xi_{1x} - \xi_{2x}| + A_1 |\eta_{1x} - \eta_{2x}| \left(N + \frac{1}{\sqrt{1-\xi_{1x}^2}} \right) \\
&\qquad + M_1 \Big(N B_1 \Big| \xi_{1x} - \xi_{2x} \Big| + B_3 \Big| \xi_{1x} - \xi_{2x} \Big| \Big) \Bigg] \\
&\le \frac{L_3}{2^{N-1}(N+1)!(N+1)^\alpha} \left((2/3)^\alpha + \frac{10}{(N+1)^{1-\alpha}} \right) \\
&\le \frac{10.79 L_3}{2^{N-1}(N+1)!(N+1)^\alpha}.
\end{aligned} \tag{56}
$$

From (53), (54) and (56), Eq. (48) gives

$$
R_1(x) \le \frac{9.23 L_1}{2^{N-1}(N-1)!} \frac{\ln(N+1)}{N} + \frac{10.79 L_3}{2^{N-1}(N+1)!(N+1)^\alpha}. \tag{57}
$$

Finally, we estimate R_2 in Eq (46). Using (47), we obtain

$$
R_2(x) = \fint_{x-\epsilon}^{x+\epsilon} \frac{e_N'(\eta_t) - e_N'(x)}{t-x} dt + e_N'(\xi_{1x}) - e_N'(\xi_{2x}) = R_{21}(x) + R_{22}(x),
$$

where $\xi_{1x} \in (x, x+\epsilon)$, $\xi_{2x} \in (x-\epsilon, x)$ and

$$
R_{21}(x) = \fint_{x-\epsilon}^{x+\epsilon} \frac{e_N'(\xi_t) - e_N'(x)}{t-x} dt, \qquad R_{22}(x) = e_N'(\xi_{1x}) - e_N'(\xi_{2x}), \tag{58}
$$

Since $|\eta_t - x| \le c|t-x|$ and by applying Eqs. (51) and (55), we get

$$
\begin{aligned}
\left| e_N'(\eta_t) - e_N'(x) \right| \le \frac{1}{2^{N-1}(N+1)!} \Big[& A_2 |\eta_{1t} - \eta_x|^\alpha + M_2 B_2 |\eta_t - x| + \\
& + A_1 |\eta_{1t} - \eta_x| \left(N + \frac{1}{\sqrt{1-\eta_t^2}} \right) + M_1 (N B_1 |\eta_t - x| + B_3 |\eta_t - x|) \Big] \\
\le \frac{L_3}{2^{N-1}(N+1)!} \Big[& c_1 |t-x|^\alpha + c_2 |t-x| + \\
& + c_1 |t-x| \left(N + \frac{1}{\sqrt{\epsilon(2-\epsilon)}} \right) + (N+1) c_2 |t-x| \Big] \\
\le \frac{C L_3}{2^{N-1}(N+1)!} & \left(1 + (2 + 3/\sqrt{35})(N+1)\epsilon^{1-\alpha} \right) |t-x|^\alpha,
\end{aligned}
$$

so that

$$|R_{21}| \leq \frac{CL_3}{2^{N-1}(N+1)!}\left(1+2.62(N+1)\epsilon^{1-\alpha}\right)\int_{x-\epsilon}^{x+\epsilon}\frac{dt}{|t-x|^{1-\alpha}}$$
$$\leq \frac{11.16CL_3}{\alpha 2^{N-1}(N+1)!(N+1)^{\alpha}}, \tag{59}$$

where $C=\max\{c_1,c_2\}$, and L_3 is the constant stated in R_{13}.
For the last part of $R_N(f,x)$, we estimate R_{22} in the sense of Eq. (56), and since $\xi_{1x}\in(x,x+\epsilon),\xi_{2x}\in(x-\epsilon,x)$,

$$|R_{22}(x)| = |e'_N(\xi_{1x})-e'_N(\xi_{2x})| \leq \frac{10.79L_3}{2^{N-1}(N+1)!(N+1)^{\alpha}}.$$

Thus

$$|R_2(x)| = \frac{L_3}{2^{N-1}(N+1)!(N+1)^{\alpha}}\left(10.79+\frac{11.16C}{\alpha}\right). \tag{60}$$

Substitution Eqs. (46), (57) and (60), we obtaine the desired estimation •

6. Numerical examples

Example 1. Consider the following HSIs

$$Q_i(f,x) = \int_{-1}^{1}\frac{f_i(t)}{(t-x)^2}dt, \qquad i=0,1,2,3$$

where

$$f_1(t) = e^t\cos t, \tag{61}$$

$$f_2(t) = \sqrt{1-t^2}\cos t, \tag{62}$$

and

$$f_3(t) = e^t\sqrt{1-t^2} \tag{63}$$

For each function, there is no analytical solution, therefore we compared our results with Hui and Shia's method [7] and Kutt's result [21] at singular point $x=0$.

Table I: Numerical results of AQS (27), compared with Hui and Shia [7], and Kutt [21] for the function (61)

n	Eq. (27)	Hui and Shia [7]	Kutt [21]
4	-2.01335	-2.11100	-2.11100
5	-2.11159	NA	-2.11100
6	-2.11054	-2.11100	-2.11102
7	-2.11087	NA	-2.11100
8	-2.11107	-2.11100	-2.11187

Table II: Numerical results of AQS (32), compared with Hui and Shia [7], and Kutt [21] for the function (62)

n	Eq. (32)	Hui and Shia [7]	Kutt [21]
4	-3.89481	-3.90699	-3.90719
5	-3.87100	NA	-3.90916
6	-3.91076	-3.90945	-3.90997
7	-3.91043	NA	-3.91033
8	-3.91090	-3.91022	-3.91140

Table III: Numerical results of AQS (32), compared with Hui and Shia [7], and Kutt [21] for the function (63)

n	Eq. (32)	Legendre Polynomial, [7]	Chebyshev Polynomial, [7]	Kutt's method, [21]
4	-2.32292	-2.32613	-2.33956	-2.32683
5	-2.29762	NA	NA	-2.33406
6	-2.33970	-2.33503	-2.33956	-2.33668
7	-2.34004	NA	NA	-2.33784
8	-2.33956	-2.33751	-2.33956	-2.33933

Example 2. Consider the quadratic density function and N takes the value 2^n, $(n = 2, 5)$.

$$\varphi(f,x) = \int_{-1}^{1} \frac{t^2+1}{(t-x)^2} dt, \qquad N = 4, 32. \tag{64}$$

The exact solution is

$$\varphi(f,x) = 2 + 2x \ln\left(\frac{1-x}{1+x}\right) - (1+x^2)\frac{2}{1-x^2}. \tag{65}$$

From the results showed in Table IV, it can be seen that the present scheme is exact for the polynomial of degree 2, for a set of x-values in $(-1,1)$, where we have defined the error term by

$$E_N(f) = \|\varphi(f,x) - \varphi_N(f,x)\|_c \tag{66}$$

Table IV. The results of problem (64)

N	x	$E_N(f)$	N	x	$E_N(f)$
4	-0.998	$1.13687e^{-013}$	32	-0.998	$2.38742e^{-012}$
	-0.5	$8.88178e^{-016}$		-0.5	$9.94760e^{-014}$
	0	$4.44089e^{-016}$		0	$2.10054e^{-013}$
	0.5	$8.88178e^{-016}$		0.5	$1.09246e^{-013}$
	0.998	$1.13687e^{-013}$		0.998	$1.53477e^{-011}$

Example 3. Consider the rational function $f(t) = 1/(1+t^2)$ and

$$\varphi(f,x) = \int_{-1}^{1} \frac{1}{(1+t^2)(t-x)^2} dt, \qquad N = 8, 16, 32, 64. \tag{67}$$

Table V. The results of problem (67)

N	x	$E_N(f)$	N	x	$E_N(f)$
8	-0.998	0.32583	32	-0.998	$2.84297e^{-009}$
	-0.5	0.03381		-0.5	$9.44458e^{-011}$
	0	0.02454		0	$7.59712e^{-011}$
	0.5	0.033814		0.5	$9.44964e^{-011}$
	0.998	0.32583		0.998	$2.83890e^{-009}$
16	-0.998	0.00129	64	-0.998	$3.18323e^{-012}$
	-0.5	$1.07772e^{-005}$		-0.5	$3.21076e^{-013}$
	0	$4.79270e^{-005}$		0	$3.04201e^{-013}$
	0.5	$1.07772e^{-005}$		0.5	$2.93543e^{-013}$
	0.998	0.00129		0.998	$8.86757e^{-012}$

From the figures in Table V, it shows that the error term $E_N(f)$ in (66) goes to zero very fast as N increases. The method also provide a good convergence numerical results even when x is close to the boundary points.

Example 4. Consider the special case when $f(x)$ is a rational function and N takes the value $2^n, (n = 2, 6)$.

$$\varphi(f, x) = \int_{-1}^{1} \frac{t^2 + 1}{\sqrt{1 - t^2}(t - x)^2} dt, \qquad N = 4, 64. \tag{68}$$

The exact solution is

$$\varphi(f, x) = \pi. \tag{69}$$

From the results showed in Table VI, it can be seen that the present scheme is exact, for a set of x-values in $(-1, 1)$, where the error term is defined by Eq (66).

Table VI. The results of problem (68)

N	x	$E_N(f)$	N	x	$E_N(f)$
4	-0.998	$8.79297e^{-014}$	64	-0.998	$9.47602e^{-011}$
	-0.5	$1.33227e^{-015}$		-0.5	$4.70735e^{-013}$
	0	$8.88178e^{-016}$		0	$6.83453e^{-013}$
	0.5	$1.33227e^{-015}$		0.5	$-2.62013e^{-013}$
	0.998	$8.79297e^{-014}$		0.998	$2.28972e^{-010}$

7. Conclusions

In this paper, an AQS for hypersingular integrals is developed. This method is an extension of the CC method for Cauchy principal value integrals, where the AQS exerts a classical Chebyshev polynomials of the first and second kinds. It is found that our approach has the potency to assess (27) for all values of N, whereas Hui and Shia's [7] approach works only for even N. It is also found that the Chebyshev polynomials give better approximation to Legendre polynomial [7]. Moreover, the numerical results show that by increasing the value of N and by choosing the appropriate weight function, the error decreases very quickly and the convergence is very fast, even when x is close to the end points.

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