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ALMOST H-CONFORMAL SEMI-SLANT RIEMANNIAN MAPS TO QUATERNIONIC HERMITIAN MANIFOLDS

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We introduce the concept of almost h-conformal semi-slant Riemannian maps from Riemannian manifolds to almost quaternionic Hermitian manifolds, with the aim of exploring new geometric properties and map behaviors within these structures. Specifically, we define several key types of maps, including invariant, pluriharmonic, and geodesic maps, and investigate conditions for these maps to exhibit harmonicity and total geodesicity. Through these conditions, we show that h-conformal slant Riemannian maps can, in certain cases, act as pseudo-horizontally weakly conformal maps and pseudo-harmonic morphisms.

Keywords: conformal Riemannian map, quaternionic manifold, pseudo-horizontally weakly conformal.

MSC2020: 53C15, 53C26, 53C43.

1. Introduction

The study of Riemannian manifolds and their structures is fundamental in differential geometry, with applications spanning mathematics, physics, and computer science. The incorporation of geometrical structures such as almost contact, Sasakian, Kahler, and quaternionic Hermitian structures enhances the understanding of both intrinsic and extrinsic properties of these manifolds. Among them, quaternionic Hermitian manifolds provide a generalized framework for complex and Kahler geometry, playing a significant role in theoretical physics, including supergravity, Yang-Mills theory, and Kaluza-Klein theory, as well as in applied fields like computer vision and medical imaging ([21], [18], [2], [12], [15], [5], [13], [23], [24]).

In addition to manifold structures, the study of maps between Riemannian manifolds is essential for understanding how geometric properties transfer between spaces. Riemannian submersions and slant Riemannian maps offer insight into the behavior of distributions under different geometrical constraints. Almost h-slant Riemannian maps, in particular, exhibit pseudo-horizontally weakly conformal properties and can act as pseudo-harmonic morphisms [17], making them valuable tools for analyzing target manifolds and their applications in mathematical and physical theories ([16], [9], [6], [10], [7], [20], [1]).

This paper extends these ideas by introducing almost h-conformal semi-slant Riemannian maps from Riemannian manifolds to almost quaternionic Hermitian manifolds. These maps integrate conformal and slant conditions within quaternionic structures, allowing for new geometric insights. We investigate their harmonicity, contributing to the study of rigidity and stability in quaternionic Hermitian geometry. The paper is structured as follows: Section 2 reviews preliminary concepts, Section 3 introduces h-conformal and almost h-conformal semi-slant Riemannian maps, Section 4 examines the associated distributions and their harmonicity conditions, and Section 5 explores pseudo-horizontally weakly conformal maps and pseudo-harmonic morphisms.

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2. **Preliminaries**

Let (P, g_P) and (Q, g_Q) be Riemannian manifolds, where g_P and g_Q are Riemannian metrics on the C^∞-manifolds P and Q, respectively.

Let $\varphi : (P, g_P) \mapsto (Q, g_Q)$ be a C^∞-map.

Then we define the *adjoint map* ${}^*(\varphi_*)_x$ of the differential $(\varphi_*)_x$, $x \in P$, as follows:

$$g_Q((\varphi_*)_x Y,\ W) = g_P(Y,\ {}^*(\varphi_*)_x W) \tag{2.1}$$

for $Y \in T_x P$ and $W \in T_{\varphi(x)} Q$.

The *second fundamental form* of φ is defined by

$$(\nabla \varphi_*)(Y, Z) := \nabla^\varphi_Y \varphi_* Z - \varphi_*(\nabla_Y Z) \quad \text{for } Y, Z \in \Gamma(TP),$$

where ∇^φ is the pullback connection [3].

We call the map φ *harmonic* and *totally geodesic* if the tension field $\tau(\varphi) = \text{trace}\,(\nabla \varphi_*) = 0$ and $(\nabla \varphi_*)(Y, Z) = 0$ for $Y, Z \in \Gamma(TP)$, respectively [3].

Lemma 2.1. ([22]) Let (P, g_P) and (Q, g_Q) be Riemannian manifolds and $\varphi : (P, g_P) \mapsto (Q, g_Q)$ a C^∞-map. Then we have

$$\nabla^\varphi_Y \varphi_* Z - \nabla^\varphi_Z \varphi_* Y - \varphi_*([Y, Z]) = 0 \tag{2.2}$$

for $Y, Z \in \Gamma(TP)$.

Remark 2.1. *From (2.2), $\nabla \varphi_*$ is symmetric.*

Let $\ker(\varphi_*)_x = \{Y \in T_x P \mid (\varphi_*)_x Y = 0\}$ for $x \in P$ and denote by $(\ker(\varphi_*)_x)^\perp$ the orthogonal complement of $\ker(\varphi_*)_x$ in $T_x P$. Let $\text{range}\,(\varphi_*)_x = \{(\varphi_*)_x Y \mid Y \in T_x P\}$ for $x \in P$ and denote by $(\text{range}\,(\varphi_*)_x)^\perp$ the orthogonal complement of $\text{range}\,(\varphi_*)_x$ in $\varphi^{-1} T_{\varphi(x)} Q$.

The map φ is said to be *horizontally weakly conformal* (HWC) at $x \in P$ if either (i) $(\varphi_*)_x = 0$ or (ii) the differential $(\varphi_*)_x$ maps $(\ker(\varphi_*)_x)^\perp$ conformally into $T_{\varphi(x)} Q$. i.e.,

$$g_Q((\varphi_*)_x Y,\ (\varphi_*)_x Z) = \lambda^2(x) \cdot g_P(Y,\ Z) \tag{2.3}$$

for $Y, Z \in (\ker(\varphi_*)_x)^\perp$. We call the positive number $\lambda(x)$ the *dilation* of φ at x. If it satisfies the case (i), then we call the point x a *critical point*. If it satisfies the case (ii), then we call the point x a *regular point*. The map φ is said to be a *horizontally weakly conformal* (HWC) map if it is horizontally weakly conformal at any point of P [3].

A HWC map φ is called a *conformal Riemannian map* if every point of P is a regular point and $0 < \text{rank}\,(\varphi_*)_x = \text{rank}\,(\varphi_*)_y \le \min(\dim P, \dim Q)$ for $x, y \in P$. A HWC map φ is said to be *horizontally homothetic* if $Y(\lambda) = 0$ for $Y \in \Gamma((\ker \varphi_*)^\perp)$.

Let $\varphi : (P, g_P) \mapsto (Q, g_Q)$ be a conformal Riemannian map with dilation λ.

Then we have

$$(\varphi_*)_x\ {}^*(\varphi_*)_x W = \lambda^2 W \quad \text{for } W \in \text{range}\,(\varphi_*)_x \tag{2.4}$$

and

$${}^*(\varphi_*)_x\ (\varphi_*)_x Y = \lambda^2 Y \quad \text{for } Y \in (\ker(\varphi_*)_x)^\perp. \tag{2.5}$$

Given $U \in \Gamma(\varphi^{-1} TQ)$, we write

$$U = \overline{P} U + \overline{Q} U, \tag{2.6}$$

where $\overline{P} U \in \Gamma(\text{range}\,\varphi_*)$ and $\overline{Q} U \in \Gamma((\text{range}\,\varphi_*)^\perp)$.

Given $Y, Z \in \Gamma((\ker \varphi_*)^\perp)$ and $U \in \Gamma((\text{range}\,\varphi_*)^\perp)$, we define

$$\widehat{\nabla}^\varphi_Y \varphi_* Z := \overline{P} \nabla^\varphi_Y \varphi_* Z \tag{2.7}$$

and

$$\nabla^\varphi_Y U = -\mathcal{S}_U \varphi_* Y + \nabla^{\varphi\perp}_Y U, \tag{2.8}$$

where $-\mathcal{S}_U \varphi_* Y = \overline{P} \nabla^\varphi_Y U \in \Gamma(\text{range}\,\varphi_*)$ and $\nabla^{\varphi\perp}_Y U = \overline{Q} \nabla^\varphi_Y U \in \Gamma((\text{range}\,\varphi_*)^\perp)$.

Theorem 2.1. ([20]) Let $\varphi : (P, g_P) \mapsto (Q, g_Q)$ be a conformal Riemannian map with dilation λ. Then given $Y, Z \in \Gamma((\ker \varphi_*)^\perp)$, we obtain

$$\begin{aligned}
&(\nabla\varphi_*)(Y, Z) \\
&= \overline{P}(\nabla\varphi_*)(Y, Z) + \overline{Q}(\nabla\varphi_*)(Y, Z) \\
&= Y(\ln\lambda)\varphi_* Z + Z(\ln\lambda)\varphi_* Y - g_P(Y, Z)\varphi_*(\nabla \ln\lambda) + (\nabla\varphi_*)^\perp(Y, Z),
\end{aligned} \tag{2.9}$$

where $(\nabla\varphi_*)^\perp(Y, Z) = \overline{Q}(\nabla\varphi_*)(Y, Z)$.

Conveniently, we also define $(\nabla\varphi_*)^r(Y, Z) := \overline{P}(\nabla\varphi_*)(Y, Z)$, $\widehat{\nabla}^\varphi_Y W := \overline{P}\nabla^\varphi_Y W$, and $\nabla^{\varphi\perp}_Y \varphi_* Z := \overline{Q}\nabla^\varphi_Y \varphi_* Z$ for $Y, Z \in \Gamma((\ker\varphi_*)^\perp)$ and $W \in \Gamma(\mathrm{range}\,\varphi_*)$.

Throughout this paper, we will use these notations.

3. Almost h-conformal semi-slant Riemannian maps

In this paper, we denote by (Q, E, g_Q) an almost quaternionic Hermitian manifold, where E is an almost quaternionic structure on Q (See [19]). We also denote by (Q, I, J, K, g_Q) a hyperkähler manifold, where (I, J, K, g_Q) is a hyperkähler structure on Q (See [19]). In this section, we introduce the notions of h-conformal semi-slant Riemannian maps, almost h-conformal semi-slant Riemannian maps, h-conformal slant Riemannian maps, h-conformal semi-invariant Riemannian maps, almost h-conformal semi-invariant Riemannian maps and give some examples of such maps.

Definition 3.1. *Let $\varphi : (P, g_P) \mapsto (Q, E, g_Q)$ be a conformal Riemannian map. We call the map φ an* almost h-conformal semi-slant Riemannian map *if given $p \in P$ with a neighborhood U, there is an open set $V \subset Q$ with $\varphi(U) \subset V$ and a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on V such that for $R \in \{I, J, K\}$, there exist two orthogonal complementary distributions $\mathcal{D}_1^R, \mathcal{D}_2^R \subset (\ker\varphi_*)^\perp$ on U such that*

$$(\ker\varphi_*)^\perp = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \;\; R(\varphi_*\mathcal{D}_1^R) = \varphi_*\mathcal{D}_1^R, \tag{3.1}$$

and the angle $\theta_R = \theta_R(Y)$ between $R\ (\varphi_*)_q Y$ and the space $(\varphi_*)_q(\mathcal{D}_2^R)_q$ is constant for nonzero $Y \in (\mathcal{D}_2^R)_q$ and $q \in U$.

We call such a basis $\{I, J, K\}$ an *almost h-conformal semi-slant Riemannian basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-conformal semi-slant Riemannian angles.*

Remark 3.1. *Let $\varphi : (P, g_P) \mapsto (Q, E, g_Q)$ be an almost h-conformal semi-slant Riemannian map. Furthermore,*

(a) If $\mathcal{D}_1 = \mathcal{D}_1^I = \mathcal{D}_1^J = \mathcal{D}_1^K$ and $\mathcal{D}_2 = \mathcal{D}_2^I = \mathcal{D}_2^J = \mathcal{D}_2^K$, then we call the map φ an h-conformal semi-slant Riemannian map*, the basis $\{I, J, K\}$ an* h-conformal semi-slant Riemannian basis *and the angles $\{\theta_I, \theta_J, \theta_K\}$* h-conformal semi-slant Riemannian angles.

(b) If $\theta_I = \theta_J = \theta_K = \frac{\pi}{2}$, then we call the map φ an almost h-conformal semi-invariant Riemannian map *and the basis $\{I, J, K\}$ an* almost h-conformal semi-invariant Riemannian basis.

(c) If $\mathcal{D}_1 = \mathcal{D}_1^I = \mathcal{D}_1^J = \mathcal{D}_1^K$, $\mathcal{D}_2 = \mathcal{D}_2^I = \mathcal{D}_2^J = \mathcal{D}_2^K$, and $\theta_I = \theta_J = \theta_K = \frac{\pi}{2}$, then we call the map φ an h-conformal semi-invariant Riemannian map *and the basis $\{I, J, K\}$ an* h-conformal semi-invariant Riemannian basis.

(d) If $\mathcal{D}_2^I = \mathcal{D}_2^J = \mathcal{D}_2^K = (\ker\varphi_*)^\perp$, then we call the map φ an h-conformal slant Riemannian map*, the basis $\{I, J, K\}$ an* h-conformal slant Riemannian basis *and the angles* $\{\theta_I, \theta_J, \theta_K\}$ h-conformal slant Riemannian angles.

(e) If $\mathcal{D}_2^I = \mathcal{D}_2^J = \mathcal{D}_2^K = (\ker\varphi_*)^\perp$ and $\theta_I = \theta_J = \theta_K = \frac{\pi}{2}$, then we call the map φ an h-conformal anti-invariant Riemannian map *and the basis $\{I, J, K\}$ an* h-conformal anti-invariant Riemannian basis.

In convenience, we denote an almost h-conformal semi-slant Riemannian map, an h-conformal semi-slant Riemannian map, an almost h-conformal semi-invariant Riemannian map, an h-conformal semi-invariant Riemannian map, an h-conformal slant Riemannian map, an h-conformal anti-invariant Riemannian map, an almost h-conformal semi-slant Riemannian basis, an h-conformal semi-slant Riemannian basis, an almost h-conformal semi-invariant Riemannian basis, an h-conformal semi-invariant Riemannian basis, an h-conformal slant Riemannian basis, an h-conformal anti-invariant Riemannian basis, almost h-conformal semi-slant Riemannian angles, h-conformal semi-slant Riemannian angles, h-conformal slant Riemannian angles by an ahssR map, an hssR map, an ahsiR map, an hsiR map, an hsR map, an haiR map, an ahssR basis, an hssR basis, an ahsiR basis, an hsiR basis, an hsR basis, an haiR basis, ahssR angles, hssR angles, hsR angles, respectively.

We obtain some examples of such maps. Consider a hyperkähler manifold $(\mathbb{R}^{4m}, I, J, K, \langle\ ,\ \rangle)$ such that

$$
\begin{aligned}
&I(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+2}}, I(\tfrac{\partial}{\partial x_{4k+2}}) = -\tfrac{\partial}{\partial x_{4k+1}}, I(\tfrac{\partial}{\partial x_{4k+3}}) = \tfrac{\partial}{\partial x_{4k+4}}, I(\tfrac{\partial}{\partial x_{4k+4}}) = -\tfrac{\partial}{\partial x_{4k+3}},\\
&J(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+3}}, J(\tfrac{\partial}{\partial x_{4k+2}}) = -\tfrac{\partial}{\partial x_{4k+4}}, J(\tfrac{\partial}{\partial x_{4k+3}}) = -\tfrac{\partial}{\partial x_{4k+1}}, J(\tfrac{\partial}{\partial x_{4k+4}}) = \tfrac{\partial}{\partial x_{4k+2}},\\
&K(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+4}}, K(\tfrac{\partial}{\partial x_{4k+2}}) = \tfrac{\partial}{\partial x_{4k+3}}, K(\tfrac{\partial}{\partial x_{4k+3}}) = -\tfrac{\partial}{\partial x_{4k+2}}, K(\tfrac{\partial}{\partial x_{4k+4}}) = -\tfrac{\partial}{\partial x_{4k+1}}
\end{aligned}
$$

for $k \in \{0, 1, \cdots, m-1\}$, where $\langle\ ,\ \rangle$ is the Euclidean metric on $\mathbb{R}^{4m}$.

Example 3.1. *Let (Q_1, E_1, g_1) and (Q_2, E_2, g_2) be almost quaternionic Hermitian manifolds. Let $\varphi : Q_1 \mapsto Q_2$ be a quaternionic submersion. Then the map φ is an h-conformal slant Riemannian map (hsR map) with the hsR angles $\theta_I = \theta_J = \theta_K = 0$ and dilation $\lambda = 1$* [11].

Example 3.2. *Let (P, g_P) be an n-dimensional Riemannian manifold and (Q, E, g_Q) a 4m-dimensional almost quaternionic Hermitian manifold. Let $\varphi : (P, g_P) \mapsto (Q, E, g_Q)$ be a conformal Riemannian map such that rank $\varphi = 4m - 1$ and dilation a smooth function λ. Then the map φ is an almost h-conformal semi-invariant Riemannian map (ahsiR map) such that with a local quaternionic Hermitian basis $\{I, J, K\}$ of E, $\varphi_* \mathcal{D}_2^R = R((\varphi_*[(\ker \varphi_*)^\perp])^\perp)$ for $R \in \{I, J, K\}$ and dilation λ.*

Example 3.3. *Let $\varphi : \mathbb{R}^n \mapsto \mathbb{R}^{4m}$ be a conformal Riemannian map such that rank $\varphi = 4m-1$ and dilation a smooth function λ. Then the map φ is an almost h-conformal semi-invariant Riemannian map (ahsiR map) such that $\varphi_* \mathcal{D}_2^R = R((\varphi_*[(\ker \varphi_*)^\perp])^\perp)$ for $R \in \{I, J, K\}$ and dilation λ.*

Example 3.4. *Define a map $\varphi : \mathbb{R}^7 \mapsto \mathbb{R}^8$ by*

$$\varphi(s_1, \cdots, s_7) = (t_1, \cdots, t_8) = \pi(s_3, s_5, s_2 \cos\alpha, s_7, 0, 78, s_2 \sin\alpha, 56),$$

where $\alpha \in (0, \frac{\pi}{2})$. *Then the map* φ *is an almost h-conformal semi-slant Riemannian map (ahssR map) such that*

$$\begin{aligned}
&\ker \varphi_* =< \frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_4}, \frac{\partial}{\partial s_6} >,\\
&(\ker \varphi_*)^\perp =< \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3}, \frac{\partial}{\partial s_5}, \frac{\partial}{\partial s_7} >,\\
&\mathcal{D}_1^I =< \frac{\partial}{\partial s_3}, \frac{\partial}{\partial s_5} >, \mathcal{D}_2^I =< \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_7} >,\\
&\varphi_* \mathcal{D}_1^I =< \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2} >, \varphi_* \mathcal{D}_2^I =< \frac{\partial}{\partial t_4}, \sin\alpha \frac{\partial}{\partial t_7} + \cos\alpha \frac{\partial}{\partial t_3} >,\\
&\mathcal{D}_1^J =< \frac{\partial}{\partial s_5}, \frac{\partial}{\partial s_7} >, \mathcal{D}_2^J =< \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3} >,\\
&\varphi_* \mathcal{D}_1^J =< \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_4} >, \varphi_* \mathcal{D}_2^J =< \frac{\partial}{\partial t_1}, \sin\alpha \frac{\partial}{\partial t_7} + \cos\alpha \frac{\partial}{\partial t_3} >,\\
&\mathcal{D}_1^K =< \frac{\partial}{\partial s_3}, \frac{\partial}{\partial s_7} >, \mathcal{D}_2^K =< \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_5} >,\\
&\varphi_* \mathcal{D}_1^K =< \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_4} >, \varphi_* \mathcal{D}_2^K =< \frac{\partial}{\partial t_2}, \sin\alpha \frac{\partial}{\partial t_7} + \cos\alpha \frac{\partial}{\partial t_3} >,
\end{aligned}$$

ahssR angles $\{\theta_I = \alpha, \theta_J = \alpha, \theta_K = \alpha\}$, *and dilation* π.

Example 3.5. *Define a map* $\varphi : \mathbb{R}^5 \mapsto \mathbb{R}^4$ *by*

$$\varphi(s_1, \cdots, s_5) = (t_1, \cdots, t_4) = e(s_2, s_1, 0, 68).$$

Then the map φ *is an almost h-conformal semi-slant Riemannian map (ahssR map) such that*

$$\begin{aligned}
&\ker \varphi_* =< \frac{\partial}{\partial s_3}, \frac{\partial}{\partial s_4}, \frac{\partial}{\partial s_5} >,\\
&(\ker \varphi_*)^\perp =< \frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2} >,\\
&\mathcal{D}_1^I = (\ker \varphi_*)^\perp, \varphi_* \mathcal{D}_1^I =< \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2} >,\\
&\mathcal{D}_2^J = (\ker \varphi_*)^\perp, \varphi_* \mathcal{D}_2^J =< \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2} >,\\
&\mathcal{D}_2^K = (\ker \varphi_*)^\perp, \varphi_* \mathcal{D}_2^K =< \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2} >,
\end{aligned}$$

ahssR angles $\{\theta_I = 0, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$, *and dilation* e.

4. Geometry of distributions

In this section we consider the geometry of distributions and introduce invariant maps, pluriharmonic maps, and geodesic maps.

Let $\varphi : (P, g_P) \mapsto (Q, E, g_Q)$ be an ahssR map with an ahssR basis $\{I, J, K\}$.

Given $Y \in \Gamma((\ker \varphi_*)^\perp)$ and $R \in \{I, J, K\}$, we write

$$Y = P_R Y + Q_R Y, \tag{4.1}$$

where $P_R Y \in \Gamma(\mathcal{D}_1^R)$ and $Q_R Y \in \Gamma(\mathcal{D}_2^R)$.

Given $U \in \Gamma(\text{range}\,\varphi_*)$ and $R \in \{I, J, K\}$, we have

$$RU = \phi_R U + \omega_R U, \tag{4.2}$$

where $\phi_R U \in \Gamma(\text{range}\,\varphi_*)$ and $\omega_R U \in \Gamma((\text{range}\,\varphi_*)^\perp)$.

Given $V \in \Gamma((\text{range}\,\varphi_*)^\perp)$ and $R \in \{I, J, K\}$, we get

$$RV = B_R V + C_R V, \tag{4.3}$$

where $B_R V \in \Gamma(\text{range}\,\varphi_*)$ and $C_R V \in \Gamma((\text{range}\,\varphi_*)^\perp)$.

Define the tensor

$$\varphi_R := {}^*(\varphi_*)\phi_R\varphi_* \quad \text{for } R \in \{I, J, K\}. \tag{4.4}$$

Remark 4.1. *In Example 3.4, we have* $\varphi_I(\frac{\partial}{\partial s_3}) = \pi^2\frac{\partial}{\partial s_5}$, $\varphi_I(\frac{\partial}{\partial s_5}) = -\pi^2\frac{\partial}{\partial s_3}$, $\varphi_I(\frac{\partial}{\partial s_2}) = \pi^2\cos\alpha\frac{\partial}{\partial s_7}$, $\varphi_I(\frac{\partial}{\partial s_7}) = -\pi^2\cos\alpha\frac{\partial}{\partial s_2}$, $\varphi_J(\frac{\partial}{\partial s_5}) = -\pi^2\frac{\partial}{\partial s_7}$, $\varphi_J(\frac{\partial}{\partial s_7}) = \pi^2\frac{\partial}{\partial s_5}$, $\varphi_J(\frac{\partial}{\partial s_2}) = -\pi^2\cos\alpha\frac{\partial}{\partial s_3}$, $\varphi_J(\frac{\partial}{\partial s_3}) = \pi^2\cos\alpha\frac{\partial}{\partial s_2}$, $\varphi_K(\frac{\partial}{\partial s_3}) = \pi^2\frac{\partial}{\partial s_7}$, $\varphi_K(\frac{\partial}{\partial s_7}) = -\pi^2\frac{\partial}{\partial s_3}$, $\varphi_K(\frac{\partial}{\partial s_2}) = -\pi^2\cos\alpha\frac{\partial}{\partial s_5}$, $\varphi_K(\frac{\partial}{\partial s_5}) = \pi^2\cos\alpha\frac{\partial}{\partial s_2}$.

Lemma 4.1. *Let $\varphi : (P, g_P) \mapsto (Q, E, g_Q)$ be an ahssR map with an ahssR basis $\{I, J, K\}$ and ahssR angles $\{\theta_I, \theta_J, \theta_K\}$. Then we get*

$$\phi_R^2\varphi_* Y = -\cos^2\theta_R \cdot \varphi_* Y \tag{4.5}$$

for $Y \in \Gamma(\mathcal{D}_2^R)$ and $R \in \{I, J, K\}$.

Proof. Given nonzero $Y \in \Gamma(\mathcal{D}_2^R)$ and $R \in \{I, J, K\}$, we have

$$\cos\theta_R = \frac{g_Q(R\varphi_* Y, \phi_R\varphi_* Y)}{|R\varphi_* Y|\,|\phi_R\varphi_* Y|} = \frac{|\phi_R\varphi_* Y|}{|\varphi_* Y|}.$$

It deduces

$$\cos^2\theta_R\ g_Q(\varphi_* Y, \varphi_* Y) = g_Q(\phi_R\varphi_* Y, \phi_R\varphi_* Y) = -g_Q(\phi_R^2\varphi_* Y, \varphi_* Y).$$

Hence, by polarization,

$$\cos^2\theta_R\ g_Q(\varphi_* Y_1, \varphi_* Y_2) = -g_Q(\phi_R^2\varphi_* Y_1, \varphi_* Y_2)$$

for $Y_1, Y_2 \in \Gamma(\mathcal{D}_2^R)$. Therefore, the result follows. □

Corollary 4.1. *Let $\varphi : (P, g_P) \mapsto (Q, E, g_Q)$ be an ahssR map with an ahssR basis $\{I, J, K\}$, ahssR angles $\{\theta_I, \theta_J, \theta_K\}$, and dilation λ. Then given $R \in \{I, J, K\}$, we have*

$$\varphi_R^2 Y = -\lambda^4\cos^2\theta_R \cdot Y \quad \textit{for } Y \in \Gamma(\mathcal{D}_2^R) \tag{4.6}$$

Definition 4.1. *Let $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ be an ahssR map with dilation λ such that $\{I, J, K\}$ is an ahssR basis with the ahssR angles $\{\theta_I, \theta_J, \theta_K\}$. Given $R \in \{I, J, K\}$, we call the map φ* R-pluriharmonic, $(\ker\varphi_*)^\perp$-R-pluriharmonic, $\ker\varphi_*$-R-pluriharmonic, $\mathcal{D}_1^R$-R-pluriharmonic, $\mathcal{D}_2^R$-R-pluriharmonic, $(\ker\varphi_*)^\perp$-$\ker\varphi_*$-R-pluriharmonic *if*

$$(\nabla\varphi_*)(\varphi_R Y, Z) - (\nabla\varphi_*)(Y, \varphi_R Z) = 0. \tag{4.7}$$

for $Y, Z \in \Gamma(TP)$, for $Y, Z \in \Gamma((\ker\varphi_*)^\perp)$, for $Y, Z \in \Gamma(\ker\varphi_*)$, for $Y, Z \in \Gamma(\mathcal{D}_1^R)$, for $Y, Z \in \Gamma(\mathcal{D}_2^R)$, for $Y \in \Gamma((\ker\varphi_*)^\perp)$ and $Z \in \Gamma(\ker\varphi_*)$, respectively.

Definition 4.2. *Let $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ be an ahssR map with dilation λ such that $\{I, J, K\}$ is an ahssR basis with the ahssR angles $\{\theta_I, \theta_J, \theta_K\}$. Given $R \in \{I, J, K\}$, we call the map φ* R-invariant, $(\ker\varphi_*)^\perp$-R-invariant, $\ker\varphi_*$-R-invariant, $\mathcal{D}_1^R$-R-invariant, $\mathcal{D}_2^R$-R-invariant, $(\ker\varphi_*)^\perp$-$\ker\varphi_*$-R-invariant *if*

$$(\nabla\varphi_*)(\varphi_R Y, Z) + (\nabla\varphi_*)(Y, \varphi_R Z) = 0. \tag{4.8}$$

for $Y, Z \in \Gamma(TP)$, for $Y, Z \in \Gamma((\ker\varphi_*)^\perp)$, for $Y, Z \in \Gamma(\ker\varphi_*)$, for $Y, Z \in \Gamma(\mathcal{D}_1^R)$, for $Y, Z \in \Gamma(\mathcal{D}_2^R)$, for $Y \in \Gamma((\ker\varphi_*)^\perp)$ and $Z \in \Gamma(\ker\varphi_*)$, respectively.

Definition 4.3. *Let* $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ *be an ahssR map with dilation* λ *such that* $\{I, J, K\}$ *is an ahssR basis with the ahssR angles* $\{\theta_I, \theta_J, \theta_K\}$. *Given* $R \in \{I, J, K\}$, *we call the map* φ totally geodesic, $(\ker \varphi_*)^\perp$-geodesic, $\ker \varphi_*$-geodesic, $\mathcal{D}_1^R$-geodesic, $\mathcal{D}_2^R$-geodesic, $(\ker \varphi_*)^\perp$-$\ker \varphi_*$-geodesic *if*

$$(\nabla \varphi_*)(Y, Z) = 0. \tag{4.9}$$

for $Y, Z \in \Gamma(TP)$, *for* $Y, Z \in \Gamma((\ker \varphi_*)^\perp)$, *for* $Y, Z \in \Gamma(\ker \varphi_*)$, *for* $Y, Z \in \Gamma(\mathcal{D}_1^R)$, *for* $Y, Z \in \Gamma(\mathcal{D}_2^R)$, *for* $Y \in \Gamma((\ker \varphi_*)^\perp)$ *and* $Z \in \Gamma(\ker \varphi_*)$, *respectively.*

Remark 4.2. *(1) Given* $R \in \{I, J, K\}$, *we have* $(\nabla \varphi_*)(\varphi_R Y, \varphi_R Z) = 0$ *for* $Y \in \Gamma(\ker \varphi_*)$ *and* $Z \in \Gamma(TP)$.

(2) By (2.4), (2.5), (4.4), and (4.6),

$$\varphi_R^2 Y = \begin{cases} 0, & Y \in \Gamma(\ker \varphi_*) \\ -\lambda^4 Y, & Y \in \Gamma(\mathcal{D}_1^R) \\ -\lambda^4 \cos^2 \theta_R Y, & Y \in \Gamma(\mathcal{D}_2^R) \end{cases} \tag{4.10}$$

for $R \in \{I, J, K\}$.

Lemma 4.2. *Let* $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ *be an ahssR map with dilation* λ *such that* $\{I, J, K\}$ *is an ahssR basis with the ahssR angles* $\{\theta_I, \theta_J, \theta_K\}$. *If the map* φ *is* $\mathcal{D}_1^R$-*R-pluriharmonic for some* $R \in \{I, J, K\}$, *then we obtain*

$$(\nabla \varphi_*)(\varphi_R Y, \varphi_R Z) = -\lambda^4 (\nabla \varphi_*)(Y, Z) \quad \text{for } Y, Z \in \Gamma(\mathcal{D}_1^R). \tag{4.11}$$

Proof. Given $Y, Z \in \Gamma(\mathcal{D}_1^R)$, by (4.7) and (4.10), we have

$$\begin{aligned}(\nabla \varphi_*)(\varphi_R Y, \varphi_R Z) &= (\nabla \varphi_*)(Y, \varphi_R^2 Z) \\ &= (\nabla \varphi_*)(Y, -\lambda^4 Z) \\ &= -\lambda^4 (\nabla \varphi_*)(Y, Z).\end{aligned}$$

□

Similarly,

Lemma 4.3. *Let* $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ *be an ahssR map with dilation* λ *such that* $\{I, J, K\}$ *is an ahssR basis with the ahssR angles* $\{\theta_I, \theta_J, \theta_K\}$. *If the map* φ *is* $\mathcal{D}_1^R$-*R-invariant for some* $R \in \{I, J, K\}$, *then we get*

$$(\nabla \varphi_*)(\varphi_R Y, \varphi_R Z) = \lambda^4 (\nabla \varphi_*)(Y, Z) \quad \text{for } Y, Z \in \Gamma(\mathcal{D}_1^R). \tag{4.12}$$

Lemma 4.4. *Let* $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ *be an ahssR map with dilation* λ *such that* $\{I, J, K\}$ *is an ahssR basis with the ahssR angles* $\{\theta_I, \theta_J, \theta_K\}$. *If the map* φ *is* $\mathcal{D}_2^R$-*R-invariant for some* $R \in \{I, J, K\}$, *then we obtain*

$$(\nabla \varphi_*)(\varphi_R Y, \varphi_R Z) = \lambda^4 \cos^2 \theta_R (\nabla \varphi_*)(Y, Z) \quad \text{for } Y, Z \in \Gamma(\mathcal{D}_2^R). \tag{4.13}$$

Lemma 4.5. *Let* $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ *be an ahssR map with dilation* λ *such that* $\{I, J, K\}$ *is an ahssR basis with the ahssR angles* $\{\theta_I, \theta_J, \theta_K\}$. *If the map* φ *is* $\mathcal{D}_2^R$-*R-pluriharmonic for some* $R \in \{I, J, K\}$, *then we obtain*

$$(\nabla \varphi_*)(\varphi_R Y, \varphi_R Z) = -\lambda^4 \cos^2 \theta_R (\nabla \varphi_*)(Y, Z) \quad \text{for } Y, Z \in \Gamma(\mathcal{D}_2^R). \tag{4.14}$$

Theorem 4.1. *Let* $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ *be an ahssR map with dilation* λ. *If the map* φ *is* $(\ker \varphi_*)^\perp$-*geodesic, then the map* φ *is horizontally homothetic.*

Proof. Since the map φ is a conformal Riemannian map, given $X, Y, Z \in \Gamma((\ker \varphi_*)^\perp)$, we have

$$g_Q(\varphi_* Y, \varphi_* Z) = \lambda^2 g_P(Y, Z) \tag{4.15}$$

so that

$$Xg_Q(\varphi_*Y, \varphi_*Z) = X(\lambda^2 g_P(Y,Z)). \tag{4.16}$$

We obtain

$$\begin{aligned} Xg_Q(\varphi_*Y, \varphi_*Z) &= g_Q(\nabla^{\varphi}_X \varphi_*Y, \varphi_*Z) + g_Q(\varphi_*Y, \nabla^{\varphi}_X \varphi_*Z) \\ &= g_Q(\varphi_*\nabla_X Y, \varphi_*Z) + g_Q(\varphi_*Y, \varphi_*\nabla_X Z) \\ &= \lambda^2 g_P(\nabla_X Y, Z) + \lambda^2 g_P(Y, \nabla_X Z) \\ &= \lambda^2 X g_P(Y,Z) \end{aligned}$$

and

$$\begin{aligned} X(\lambda^2 g_P(Y,Z)) &= X(\lambda^2) g_P(Y,Z) + \lambda^2 X g_P(Y,Z) \\ &= 2\lambda X(\lambda) g_P(Y,Z) + \lambda^2 X g_P(Y,Z). \end{aligned}$$

Hence,

$$X(\lambda) = 0 \quad \text{for } X \in \Gamma((\ker \varphi_*)^{\perp}).$$

Therefore, the result follows. □

Now, we deal with a rigidity theorem on ahssR maps.

Theorem 4.2. *Let $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ be an ahssR map with dilation λ such that $\{I, J, K\}$ is an ahssR basis with the ahssR angles $\{\theta_I, \theta_J, \theta_K\} \subset [0, \frac{\pi}{2})$. Assume that the tensor ω_R is parallel for some $R \in \{I, J, K\}$ and $(\nabla\varphi_*)^{\perp}(Y, Z) \neq 0$ for some $Y, Z \in \Gamma(\mathcal{D}_2^R)$. Then there does not exist a map $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ such that the map φ is $\mathcal{D}_2^R$-R-invariant.*

Proof. Suppose that there exists a map $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ such that the map φ is $\mathcal{D}_2^R$-R-invariant. Given $Y, Z \in \Gamma(\mathcal{D}_2^R)$, from (**??**) and (4.13), we have

$$(\nabla\varphi_*)^{\perp}(\varphi_R Y, \varphi_R Z) = -\lambda^4 \cos^2\theta_R \cdot (\nabla\varphi_*)^{\perp}(Y, Z)$$

and

$$(\nabla\varphi_*)^{\perp}(\varphi_R Y, \varphi_R Z) = \lambda^4 \cos^2\theta_R \cdot (\nabla\varphi_*)^{\perp}(Y, Z).$$

It is not possible.

Therefore, we get the result. □

Let $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ be an ahssR map with dilation λ such that $\{I, J, K\}$ is an ahssR basis with the ahssR angles $\{\theta_I, \theta_J, \theta_K\}$.

Since $R(\varphi_*\mathcal{D}_1^R) = \varphi_*\mathcal{D}_1^R$ for $R \in \{I, J, K\}$, we can choose a local orthonormal frame $\{\varphi_*v_1, R\varphi_*v_1, \cdots, \varphi_*v_{l_R}, R\varphi_*v_{l_R}\}$ of $\varphi_*\mathcal{D}_1^R$ so that $\{\lambda v_1, \frac{1}{\lambda}\varphi_R v_1, \cdots, \lambda v_{l_R}, \frac{1}{\lambda}\varphi_R v_{l_R}\}$ is a local orthonormal frame of $\mathcal{D}_1^R$. We can also choose a local orthonormal frame $\{\varphi_*e_1, \sec\theta_R \phi_R \varphi_* e_1, \cdots, \varphi_* e_{k_R}, \sec\theta_R \phi$ of $\varphi_*\mathcal{D}_2^R$ for $\theta_R \in [0, \frac{\pi}{2})$ so that $\{\lambda e_1, \frac{1}{\lambda}\sec\theta_R \varphi_R e_1, \cdots, \lambda e_{k_R}, \frac{1}{\lambda}\sec\theta_R \varphi_R e_{k_R}\}$ is a local orthonormal frame of $\mathcal{D}_2^R$.

Theorem 4.3. *Let $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ be an ahssR map with dilation λ such that $\{I, J, K\}$ is an ahssR basis with the ahssR angles $\{\theta_I, \theta_J, \theta_K\} \subset [0, \frac{\pi}{2})$. Assume that all the fibers are minimal submanifolds of P. Then each of the following assertions implies that φ is harmonic.*

(a) The map φ is $\mathcal{D}_1^I$-I-pluriharmonic and $\mathcal{D}_2^I$-I-pluriharmonic.
(b) The map φ is $\mathcal{D}_1^J$-J-pluriharmonic and $\mathcal{D}_2^J$-J-pluriharmonic.
(c) The map φ is $\mathcal{D}_1^K$-K-pluriharmonic and $\mathcal{D}_2^K$-K-pluriharmonic.

Proof. Since $TP = \ker \varphi_* \oplus (\ker \varphi_*)^\perp = \ker \varphi_* \oplus \mathcal{D}_1^R \oplus \mathcal{D}_2^R$ for $R \in \{I, J, K\}$,

$$\begin{aligned}\tau(\varphi) &= \operatorname{trace}(\nabla \varphi_*) \\ &= \operatorname{trace}(\nabla \varphi_*)|_{\ker \varphi_*} + \operatorname{trace}(\nabla \varphi_*)|_{\mathcal{D}_1^R} + \operatorname{trace}(\nabla \varphi_*)|_{\mathcal{D}_2^R}.\end{aligned}$$

Furthermore, all the fibers are minimal submanifolds of P if and only if trace $(\nabla \varphi_*)|_{\ker \varphi_*} = 0$.

By Corollary 4.2, we have

$$\begin{aligned}\operatorname{trace}(\nabla \varphi_*)|_{\mathcal{D}_1^R} &= \sum_{i=1}^{l_R} \{(\nabla \varphi_*)(\lambda v_i, \lambda v_i) + (\nabla \varphi_*)(\frac{1}{\lambda}\varphi_R v_i, \frac{1}{\lambda}\varphi_R v_i)\} \\ &= \sum_{i=1}^{l_R} \{\lambda^2 (\nabla \varphi_*)(v_i, v_i) + \frac{1}{\lambda^2}(\nabla \varphi_*)(\varphi_R v_i, \varphi_R v_i)\} \\ &= \sum_{i=1}^{l_R} \{\lambda^2 (\nabla \varphi_*)(v_i, v_i) + \frac{1}{\lambda^2}(-\lambda^4 (\nabla \varphi_*)(v_i, v_i))\} \\ &= 0.\end{aligned}$$

By Corollary 4.5, we obtain

$$\begin{aligned}\operatorname{trace}(\nabla \varphi_*)|_{\mathcal{D}_2^R} &= \sum_{j=1}^{k_R} \{(\nabla \varphi_*)(\lambda e_j, \lambda e_j) + (\nabla \varphi_*)(\frac{1}{\lambda}\sec\theta_R \varphi_R e_j, \frac{1}{\lambda}\sec\theta_R \varphi_R e_j)\} \\ &= \sum_{j=1}^{k_R} \{\lambda^2 (\nabla \varphi_*)(e_j, e_j) + \frac{1}{\lambda^2}\sec^2\theta_R (\nabla \varphi_*)(\varphi_R e_j, \varphi_R e_j)\} \\ &= \sum_{j=1}^{k_R} \{\lambda^2 (\nabla \varphi_*)(e_j, e_j) + \frac{1}{\lambda^2}\sec^2\theta_R (-\lambda^4 \cos^2\theta_R (\nabla \varphi_*)(e_j, e_j))\} \\ &= 0.\end{aligned}$$

Therefore, the result follows. □

5. **Pseudo-horizontally weakly conformal maps**

In this section we deal with some notions: pseudo-horizontally weakly conformal (PHWC) maps, pseudo-harmonic morphisms (PHM). From [3], we get

Lemma 5.1. *Let $\varphi : (P, g_P) \mapsto (Q, g_Q)$ be a C^∞-map and let $x \in P$. Then the following assertions are equivalent:*

(a) The map φ is HWC at x with dilation $\lambda(x)$.

(b) The adjoint ${}^*(\varphi_*)_x$ of $(\varphi_*)_x$ satisfies

$$(\varphi_*)_x \circ {}^*(\varphi_*)_x = \lambda(x)^2 \cdot id \quad on \quad range\,(\varphi_*)_x. \tag{5.1}$$

Let φ be a C^∞-map from a Riemannian manifold (P, g_P) to an almost Hermitian manifold (Q, J, g_Q). The map φ is called a *pseudo-horizontally weakly conformal* (PHWC) at $x \in P$ if ${}^*(\varphi_*)_x(T^{1,0}_{\varphi(x)}Q)$ is isotropic. i.e.,

$$g_P({}^*(\varphi_*)_x U,\ {}^*(\varphi_*)_x V) = 0 \quad \text{for } U, V \in T^{1,0}_{\varphi(x)}Q. \tag{5.2}$$

We call the map φ a *pseudo-horizontally weakly conformal* (PHWC) map if it is pseudo-horizontally weakly conformal at each point of P [3]. From [3], we have

Lemma 5.2. *Let φ be a C^∞-map from a Riemannian manifold (P, g_P) to an almost Hermitian manifold (Q, J, g_Q) and $x \in P$. Then the following assertions are equivalent:*

(a) The map φ is PHWC at x.

(b) The composition of ${}^*(\varphi_*)_x$ and $(\varphi_*)_x$ commutes with J. i.e.,

$$[(\varphi_*)_x \circ {}^*(\varphi_*)_x,\ J] = 0. \tag{5.3}$$

Corollary 5.1. *Let $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ be an hsR map with dilation λ such that $\{I, J, K\}$ is an hsR basis with the hsR angles $\{\theta_I, \theta_J, \theta_K\}$. Assume that $\theta_R = 0$ for some $R \in \{I, J, K\}$. Then the map $\varphi : (P, g_P) \mapsto (Q, R, g_Q)$ is PHWC.*

Proof. Given $x \in P$, since $\theta_R = 0$, we deduce $R(\text{range}\,(\varphi_*)_x) \subset \text{range}\,(\varphi_*)_x$ so that by (2.4),

$$\begin{aligned}(\varphi_*)_x\ {}^*(\varphi_*)_x RZ &= \lambda^2 RZ \\ &= R(\lambda^2 Z) \\ &= R(\varphi_*)_x\ {}^*(\varphi_*)_x Z\end{aligned}$$

for $Z \in \text{range}\,(\varphi_*)_x$.

By Lemma 5.2, the result follows. □

Let φ be a PHWC map from a Riemannian manifold (P, g_P) to a Kähler manifold (Q, J, g_Q). We call the map φ a *pseudo-harmonic morphism* (PHM) if it is harmonic.

Remark 5.1. *Let φ be a C^∞-map from a Riemannian manifold (P, g_P) to a Kähler manifold (Q, J, g_Q). Loubeau [14] proved that the map φ pulls back local $\pm$ holomorphic functions to local harmonic functions if and only if the map φ is a PHM.*

Using Theorem 4.3, Corollary 5.1, and Remark 5.1, we obtain

Theorem 5.1. *Let $\varphi : (P, g_P) \mapsto (Q, I, J, K, g_Q)$ be an hsR map with dilation λ such that $\{I, J, K\}$ is an hsR basis with the hsR angles $\{\theta_I, \theta_J, \theta_K\}$. Assume that all the fibers are minimal submanifolds of P and the map φ is $\mathcal{D}_1^R$-R-pluriharmonic and $\mathcal{D}_2^R$-R-pluriharmonic for some $R \in \{I, J, K\}$.*

If $\theta_R = 0$, then the map $\varphi : (P, g_P) \mapsto (Q, R, g_Q)$ is a PHM so that φ pulls back local $\pm$ holomorphic functions to local harmonic functions.

6. **Applications**

The study of almost h-conformal semi-slant Riemannian maps has significant implications in both theoretical and applied fields, particularly in areas involving quaternionic structures and harmonic maps. In theoretical physics, these maps play a role in Kaluza-Klein theory, Yang-Mills theory, and supergravity, offering geometric tools for unifying physical forces and modeling smooth transitions in gravitational and particle physics contexts [[2], [12], [15]]. Their harmonic properties provide insights into continuous mappings essential for stability in these frameworks.

In applied fields, quaternionic and Riemannian structures are widely used in computer vision and medical imaging, particularly in shape analysis, object recognition, and 3D image registration [[13], [23], [24]]. Almost h-conformal semi-slant Riemannian maps contribute to these applications by providing tools for precise spatial transformations and geometric modeling. Additionally, the criteria developed for pseudo-horizontally weakly conformal and pseudo-harmonic morphisms lay the groundwork for applying these maps in conformal and harmonic transformations, advancing computational algorithms in various disciplines.

7. **Conclusion**

In this paper, we introduced almost h-conformal semi-slant Riemannian maps from Riemannian manifolds to almost quaternionic Hermitian manifolds, extending classical Riemannian maps by incorporating conformal and slant conditions within quaternionic structures. We defined various types of maps, such as invariant, pluriharmonic, and geodesic maps, and established conditions for harmonicity. These results showed that, under certain conditions, h-conformal slant Riemannian maps can act as pseudo-horizontally weakly conformal maps and pseudo-harmonic morphisms, expanding their theoretical and applied significance.

Our findings deepen the understanding of quaternionic Hermitian manifolds and provide a foundation for future research. Potential directions include applications in mathematical physics and computer science, as well as investigations into the stability and rigidity of these maps in dynamic settings. Further exploration of their properties in real-world modeling, from theoretical physics simulations to computational imaging, could enhance their practical relevance.

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