WEAKLY PRIMARY SUBMODULES AND WEAKLY PRIMARY IDEALS

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Let R be a commutative ring with nonzero identity. Weakly primary submodules was first introduced by S. Ebrahimi Atani and F. Farzalipour in 2005. A proper submodule A of an R-module B is said to be weakly primary if $0 \neq rb \in A$ *implies* $b \in A$ *or* $r \in \sqrt{A:B}$. *In this paper we first provide some results on weakly primary submodules. Various properties of weakly primary submodules are considered. Also we study the relationships among the weakly primary submodules, weakly primary ideals and weakly prime ideals. Finally, we show that how to construct examples of weakly primary ideals using the Method of Idealization in commutative algebra*

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1. Introduction

Weakly prime ideals in a commutative ring with non-zero identity have been introduced by A. G. Aĝargün et al. in [1] and studied in depth by D. D. Anderson and E. Smith in [2]. A proper ideal P of R is said to be weakly prime if for $0 \neq ab \in P$, either $a \in P$ or $b \in P$. It was shown that a proper ideal P of R is weakly prime if and only if $0 \neq IJ \subseteq P$, where I, J are ideals of R, implies $I \subseteq P$ or $J \subseteq P$. Anderson and Smith have shown that a weakly prime ideal P of a ring R that is not a prime ideal satisfies $P^2 = 0$ and $P\sqrt{0} = 0$. They also have proven that every proper ideal of R is a product of weakly prime ideals if and only if R is a finite direct product of Dedekind domains and SPIR's or R is a quasi-local ring with maximal ideal M such that $M^2 = 0$.

The notion of weakly primary ideals in a commutative ring with non-zero identity have been introduced and studied by S. Ebrahimi Atani and F. Farzalipour in [4]. They extended some results in [2] to the weakly primary ideals. Following [4], a proper ideal I of R is said to be weakly primary ideal if whenever $0 \neq rs \in I$ implies that $r \in I$ or $s \in \sqrt{I}$. In the last section of the paper they defined the notion of a weakly primary submodule and obtained two results respect to it. A proper submodule N of a module M over a commutative ring R is said to be a weakly primary submodule if whenever $0 \neq rm \in N$, for some $r \in R$, $m \in M$ then

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$m \in N$ or $r^n M \subseteq N$ for some n. Clearly, every primary submodule of a module is a weakly primary submodule. However, since 0 is always weakly primary by definition, a weakly primary submodule need not be primary. (See also [2], Theorem 2.14 and 2.15.) After that, in another paper [5], Atani and Ghaleh studied this notion further on multiplication modules and have proved that if M is a free multiplication R-module and N is a non-zero strongly pure submodule of M, then N is a primary submodule of M iff it is a weakly primary submodule of M. Here an R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R.

Here we study weakly primary submodule. The corresponding results are obtained by modification and here we give a number of results concerning weakly primary submodules.

2. Preliminaries and Notations

Throughout this paper all rings will be commutative with non-zero identity. If R is a ring and A is a submodule of an R-module B, the ideal $\{r \in R : rB \subseteq A\}$ will be denoted by $(A:B)$. Then $(0:B)$ is the annihilator of B. For an ideal I in R we set, $\sqrt{I} = \{r \in R \mid r^n \in I$ for some positive integer $n\}$, the radical ideal of I.

A proper submodule A of an R-module B is primary if $rb \in A$, then either $b \in A$ or $r \in \sqrt{A:B}$. Also A is a weakly primary submodule of B if $0 \neq rb \in A$ then $b \in A$ or $r \in \sqrt{A:B}$. For unexplained notations and terminology we refer the reader to [3], [6] and [7].

We start this section with the following lemma:

Lemma 2.1 Let A be a proper submodule of an R-module B. Then the following are equivalent:

(i) A is a primary submodule of B.

(ii) For every $b \in B - A$, $(A:Rb) \subseteq \sqrt{A:B}$.

Proof. The proof is clear.□

We shall use the following equivalent form of the definition of weakly primary submodules. See also [4], Proposition 2.15.

Proposition 2.2 Let A be a proper submodule of B. Then the following assertions are equivalent:

(i) A is a weakly primary submodule of B.

(ii) For every $b \in B - A$, $(A:Rb) \subseteq \sqrt{A:B} \cup (0:Rb)$.

Proof. (i) $\Rightarrow$ (ii) Let $b \in B - A$ and $r \in (A:Rb)$. Then $rb \in A$. If $rb = 0$, then $r \in (0:Rb)$. If $rb \neq 0$ then $r \in \sqrt{A:B}$ by (i).

(ii) $\Rightarrow$ (i) Assume that $0 \neq rb \in A$ and $b \notin A$. Then, $r \in (A : Rb)$ and so, $r \in \sqrt{A : B}$ or $r \in (0 : Rb)$ by (ii). As $rb \neq 0$, hence $r \in \sqrt{A : B}$.□

3. Main Results

In this section we present some new results related to weakly primary submodules, but first we start with a definition.

Definition 3.1 If A is a weakly primary submodule of an R-module B and $\sqrt{A : B} = P$ then, A is called P-weakly primary submodule.

Theorem 3.2 If $A_i (1 \leq i \leq n)$ are P-weakly primary submodules of B, then $A = \bigcap_{i=1}^{n} A_i$ is P-weakly primary.

Proof. First of all, note that

$$\sqrt{A : B} = \sqrt{(\bigcap_{i=1}^{n} A_i : B)} = \bigcap_{i=1}^{n} \sqrt{A_i : B} = \bigcap_{i=1}^{n} P = P$$

Let $0 \neq rb \in A$ and $b \notin A$, so there exists $1 \leq j \leq n$ such that $b \notin A_j$ and $0 \neq rb \in A_j$. Since A_j is P-weakly primary so, $r \in P$.□

Suppose that R is an integral domain. Let B be an R-module, it will be a torsion-free module if $rb = 0$ then either $r = 0$ or $b = 0$. Note that every free module is torsion free, but not vice versa. (See [6])

Following [4], a proper ideal I of R is said to be weakly primary ideal if whenever $0 \neq rs \in I$ implies that $r \in I$ or $s \in \sqrt{I}$. Now we have

Proposition 3.3 Let R be an integral domain and A a weakly primary submodule of a torsion free R-module B. Then the ideal $(A : B)$ is a weakly primary ideal.

Proof. Suppose that $0 \neq rs \in (A : B)$ with $s \notin (A : B)$, so there is an element $b \in B - A$ such that $sb \notin A$. We know that $rsb \in A$. If $rsb = 0$, then $b = 0$ which is a contradiction. Otherwise, $rsb \neq 0$ and so $r \in \sqrt{A : B}$, as needed.□

As we mentioned in the introduction, a weakly primary submodule need not be primary submodule. However, in the following situation we have the converse.

Proposition 3.4 Let A be a weakly primary submodule of B such that $\sqrt{0} = \sqrt{0 : Rb}$ for every $b \in B - A$. Then A is a primary submodule.

Proof. Let $rb \in A$ with $b \notin A$. If $rb = 0$, then $r \in \sqrt{0 : Rb}$ and so $r^n = 0$ for some positive integer n and clearly $r \in \sqrt{A : B}$. If $rb \neq 0$, since A is a weakly primary submodule of B, we get $r \in \sqrt{A : B}$.□

Proposition 3.5 Consider the ring R as an R-module. Let Q be a proper ideal of R. Then Q is a weakly primary submodule of R iff Q is a weakly primary ideal of R.

Proof. This follows immediately from the equality $\sqrt{Q:R} = \sqrt{Q}$.□

Next we are studying the behavior of weakly primary submodules with respect to a homomorphism.

Proposition 3.6 Let $f: B \to C$ be an R-module homomorphism and A a submodule of C. Then the following hold:

(i) If f is an epimorphism and $f^{-1}(A)$ is a weakly primary submodule of B then, A is a weakly primary submodule of C.

(ii) If f is a monomorphism and A is a weakly primary submodule of C, then $f^{-1}(A)$ is a weakly primary submodule of B .

Proof. (i) First of all, since $f^{-1}(A)$ is a proper submodule of B there exists $b \in B - f^{-1}(A)$, so $f(b) \notin A$. Therefore $A \neq C$. Next, suppose that $0 \neq rc \in A$ and $c \notin A$. Then there exists $b \in B$ such that $f(b) = c$. So $0 \neq rc = rf(b) = f(rb) \in A$. This means that $0 \neq rb \in f^{-1}(A)$ with $b \notin f^{-1}(A)$. Hence $r \in \sqrt{f^{-1}(A):B}$. We get that $r^n B \subseteq f^{-1}(A)$ for some positive integer n. It is enough to show $r^n C \subseteq A$. Let $c_1 \in C$. Since f is an epimorphism, then there exists $b_1 \in B$ such that $f(b_1) = c_1$. Therefore $r^n b_1 \in f^{-1}(A)$. It follows that $f(r^n b_1) = r^n f(b_1) = r^n c_1 \in A$, as desired.

(ii) Let $0 \neq rb \in f^{-1}(A)$ where $b \notin f^{-1}(A)$. Then $0 \neq f(rb) = rf(b) \in A$ and $f(b) \notin A$, so $r \in \sqrt{A:C}$. We shall write $r^n C \subseteq A$ for some positive integer n. We will show that $r^n f^{-1}(C) \subseteq f^{-1}(A)$. If $b \in f^{-1}(C)$ then $f(b) \in C$. Hence $r^n f(b) = f(r^n b) \in A$. It follows that $r^n(b) \in f^{-1}(A)$, as desired. Therefore, $r^n f^{-1}(C) \subseteq f^{-1}(A)$, as needed.□

It is frequently necessary to apply these results to the case where *f* is the natural mapping of *B* onto a factor module. We shall therefore restate the theorem for this case and get the following result.

Corollary 3.7 Suppose that *B* is an *R*- module, *A* a proper submodule of *B* and *C* is a submodule of *A*. If *A* is a weakly primary submodule of *B*, then A/C is a weakly primary submodule of B/C.

For the converse we have

Theorem 3.8 Let *B* be an *R*- module, *A* a proper submodule of *B* and *C* is a submodule of *A*. If A/C is a weakly primary submodule of B/C and *C* is a weakly primary submodule of *B* then, *A* is a weakly primary submodule of *B*.

Proof. Let $0 \neq rb \in A$ and $b \notin A$. Then, $r\bar{b} \in A/C$ and $\bar{b} \notin A/C$. If $r\bar{b} \neq 0$, then $r \in \sqrt{A/C : B/C}$ and so $r \in \sqrt{A:B}$. If $r\bar{b} = 0$, then $0 \neq rb \in C$ and $b \notin C$. Therefore, $r \in \sqrt{A:B}$. □

The definition of weakly primary submodule in fact only depends on the quotient module. It can be restated as the following. We will use it to prove theorem 3.10.

Theorem 3.9 Let *B* be an *R*- module. A submodule *A* of *B* is weakly primary iff $B/A \neq 0$ and for every zero devisor *r* of *B/A* there exists $b \in B - A$ such that $r \in (0:Rb) \cup \sqrt{0:B/A}$.

Proof. $(\Rightarrow)$ Let *r* be a zero devisor of *B/A*. So there exists $\bar{0} \neq \bar{b} = b + A \in B/A$ such that $r\bar{b} = \bar{0}$. Then $rb \in A$ and $b \notin A$. If $rb = 0$ then $r \in (0:Rb)$. If $rb \neq 0$, then $r \in \sqrt{0:B/A}$.

$(\Leftarrow)$ Since, $B/A \neq 0$ therefore A is a proper submodule of *B*. If $0 \neq rb \in A$ and $b \notin A$ then $r\bar{b} = \bar{0}$ also $\bar{b} \neq \bar{0}$ where $\bar{b} = b + A \in B/A$. It means *r* is a zero devisor of B/A so $r \in \sqrt{0:B/A}$ and therefore, $r \in \sqrt{A:B}$. □

Now we consider the behavior of polynomial modules related to this notion.

Theorem 3.10 Let A be a proper submodule of an R-module B. If *A* is a weakly primary submodule of *B*, then $A[x]$ is a weakly primary submodule of $B[x]$.

Proof. We know $\psi : B[x] \to (B/A)[x]$ given by $\psi(\sum_{i=1}^{n} b_i x^i) = \sum_{i=1}^{n} \bar{b}_i x^i$ is an *R*-epimorphism. Here by $\bar{b}_i$ we mean $b_i + A$. The kernel of the homomorphism is obtained by reducing coefficients modulo *A*. Thus $B[x]/A[x] \cong (B/A)[x]$. As $B/A \neq 0$ implies $B[x]/A[x] \neq 0$. Let *r* be a zero devisor of $(B/A)[x]$ so there exists $0 \neq \bar{f} = \sum_{i=1}^{n} \bar{b}_i x_i \in (B/A)[x]$ such that $r\bar{f} = 0$. Hence there exists $1 \leq j \leq n$ such that $r\bar{b}_j = 0$ and $\bar{b}_j \neq 0$. So $rb_j \in A$ and $b_j \notin A$. If $rb_j = 0$ then $r \in (0:Rb_j)$. If $rb_j \neq 0$ then $r \in \sqrt{0:B/A}$. So $r \in (0:Rb_j) \cup \sqrt{0:B/A}$. Therefore $r \in (0:Rb_j) \cup \sqrt{0:B/A[x]}$. By previous theorem $A[x]$ is a weakly primary submodule of $B[x]$.□

Theorem 3.11 Let $B = B_1 \times B_2$ be a decomposable module, A_1 a proper submodule of B_1 and A_2 a proper submodule of B_2. Assume $rb_1 = 0$ iff $rb_2 = 0$ for $(b_1, b_2) \in B$. Then the following holds:

(i) If A_1 is a weakly primary submodule of B_1 then, $A_1 \times B_2$ is a weakly primary submodule of *B*.

(ii) If A_2 is a weakly primary submodule of B_2 then, $B_1 \times A_2$ is a weakly primary submodule of *B*.

Proof. (i) If $(b_1, b_2) \in B - (A_1 \times B_2)$ then, $b_1 \in B_1 - A_1$ and

$$\left[(A_1 \times B_2) : R(b_1, b_2)\right] = (A_1 : Rb_1).$$

Also $\left[0 : R(b_1, b_2)\right] = (0 : Rb_1)$. Hence by proposition 2.2,

$$\left[(A_1 \times B_2) : R(b_1, b_2)\right] \subseteq \sqrt{\left[(A_1 \times B_2) : (B_1 \times B_2)\right]} \cup \left[0 : R(b_1, b_2)\right].$$

(ii) The proof is similar to (i). □

We next show that how to construct examples of weakly primary ideals using the Method of Idealization. Let *R* be a commutative ring and *M* an *R*-module. Put $R(M) = R \oplus M$. Then *R(M)* with multiplication $(a, m)(b, n) = (ab, an + bm)$ is a commutative ring with identity and $0 \oplus M$ is an ideal of *R(M)* with $(0 \oplus M)^2 = 0$. (See also [2].)

Theorem 3.12 Let *R* be a commutative ring and *M* an *R*-module. Let *I* a proper ideal of *R*. Then,

(i) If $I \oplus M$ is a weakly primary ideal of the ring *R(M)* then, *I* is a weakly primary ideal of *R*.

(ii) If *I* is a primary ideal of *R* then, $I \oplus M$ is a primary ideal of *R(M)*.

(iii) If *I* is a weakly primary ideal and for $a, b \in R$ with $ab=0$ but $a \notin I$ and $b \notin \sqrt{I}$, $a \in 0 : M$ and $b \in 0 : M$, then $I \oplus M$ is weakly primary.

Proof. (i) Assume that $0 \neq r_1 r_2 \in I$ then $0 \neq (r_1, 0)(r_2, 0) = (r_1 r_2, 0) \in I \oplus M$, so $(r_1, 0) \in I \oplus M$ or $(r_2, 0) \in \sqrt{I \oplus M}$. In the end $r_1 \in I$ or $r_2 \in \sqrt{I}$.

(ii) If $(r_1, a)(r_2, b) = (r_1 r_2, r_1 b + r_2 a) \in I \oplus M$, then $r_1 r_2 \in I$ so $r_1 \in I$ or $r_2 \in \sqrt{I}$. As a result, $(r_1, a) \in I \oplus M$ or $(r_2, b) \in \sqrt{I \oplus M}$.

(iii) Suppose that $0 \neq (r_1, a)(r_2, b) \in I \oplus M$. If $r_1 r_2 \neq 0$, then we are done since *I* is a weakly primary ideal of *R*. Thus assume that $r_1 r_2 = 0$ and $r_1 \notin I$ also $r_2 \notin \sqrt{I}$. Then $r_1, r_2 \in 0 : M$. So $r_1 b + r_2 a = 0$. It means $(r_1, a)(r_2, b) = 0$ which is a contradiction. □

Recall [6] that an R-module N is simple if $N \neq 0$ and it has no submodules other than 0 and N itself. N is semi-simple if it is a sum of simple submodules. Now we have the following. (Compare with [2] Theorem 17.)

Corollary 3.13 Let (R, M) be a quasi-local ring, I an ideal of R and N a semi-simple R-module. Then $I \oplus N$ is a weakly primary ideal of $R(M)$ if and only if I is a weakly primary ideal of R.

Proof. ($\Rightarrow$) By theorem 3.2 is obvious.

($\Leftarrow$) It is enough to show that if $r_1 r_2 = 0$ and $r_1 \notin I$ also $r_2 \notin \sqrt{I}$ then $r_1, r_2 \in 0 : N$. Since N is semi-simple so there is a family of simple submodules of N like $\{T_\alpha \mid \alpha \in \Lambda\}$ with $N = \oplus_{\alpha \in \Lambda} T_\alpha$ also $T_\alpha \cong R/\mathrm{M}$. Then

$$(0 : N) = (0 : \oplus_{\alpha \in \Lambda} T_\alpha) = \bigcap_{\alpha \in \Lambda} (0 : T_\alpha) = \bigcap_{\alpha \in \Lambda} (0 : R/\mathrm{M}) = \mathrm{M}.$$

If r_1 or r_2 is unit then $r_2 = 0$ or $r_1 = 0$. So $r_1 \in I$ or $r_2 \in \sqrt{I}$, a contradiction. Therefore r_1, r_2 are not units and are elements of $\mathrm{M} = (0 : N)$. □

Corollary 3.14 Let R be a ring with $\bigcap_{\alpha \in \Lambda} Q_\alpha = 0$ where Q_α is P_α-primary of R. Suppose that I is a weakly primary ideal of R and M is an R-module in which for any $\alpha \in \Lambda$, $P_\alpha \subseteq 0 : M$. Then $I \oplus M$ is a weakly primary ideal.

Proof. It is enough to show that if $r_1 r_2 = 0$ and $r_1 \notin I$, $r_2 \notin \sqrt{I}$ then $r_1 \in 0 : M$, $r_2 \in 0 : M$. Let $r_1 \notin 0 : M$ so $r_1 \notin P_\alpha$ for every $\alpha \in \Lambda$. On the other hand, $r_1 r_2 \in Q_\alpha$ therefore $r_2 \in Q_\alpha$ for every $\alpha \in \Lambda$. Hence $r_2 \in \bigcap_{\alpha \in \Lambda} Q_\alpha = 0$ and so $r_2 \in \sqrt{I}$. A contradiction! □

Theorem 3.15 Let M be a Noetherian R-module. Then every submodule of M is an intersection of a finite family of weakly primary submodules of M.

Proof. Recall that every proper submodule of M is an intersection of a finite family of primary submodules of M and every primary submodules is weakly primary, as requested. □

Theorem 3.16 Suppose that $(R, \mathcal{M})$ is a zero-dimensional quasi-local ring. If A is a P-weakly primary submodule of a torsion-free R-module B, then $P = \mathcal{M}$ or $\mathcal{M}P = 0$.

Proof. P is weakly prime by Proposition 3.3 then either $P = \mathcal{M}$ or $\mathcal{M}P = 0$ (See [2]). □

Theorem 3.17 If $(R, \mathcal{M})$ is a quasi-local ring with $\mathcal{M}^2 = 0$ or $R = F_1 \times F_2$ where F_1 and F_2 are fields, then every proper submodule of an R-module B with multiplication $(r_1, r_2).b = r_1 b$ is weakly primary.

Proof. Let $(R,\mathcal{M})$ be a quasi-local ring with $\mathcal{M}^2=0$. Let A be a proper submodule of B and $0\neq rb\in A$. If r is unit then $b\in A$ and when r is not unit then $r\in \mathrm{M}$. So $r^2=0$, finally $r\in\sqrt{A:B}$. Next, let $R=F_1\times F_2$ where F_1 and F_2 are fields. If $0\neq(r_1,r_2)b=r_1b\in A$ then $b\in A$ because $0\neq r_1\in F_1$ so it is unit. □

Definition 3.18 Let B be an R-module. An element $a\in B$ is prime if $a\,|\,rb$ then either $a\,|\,b$ or $r\in Ra:B$. (As usual, here $a\,|\,b$ means there exists $r\in R$ such that $b=ra$.) Also a is weakly prime if $a\,|\,rb$ and $rb\neq 0$ then $a\,|\,b$ or $r\in Ra:B$. The element $a\in B$ is called irreducible if $a=rb$ then $Ra=Rb\cup rB$.

We next investigate the relationship between weakly prime elements and irreducible elements.

Proposition 3.19 Let B be an R-module and $a\in B$. Then a prime $\Rightarrow a$ is weakly prime $\Rightarrow a$ is irreducible.

Proof. It is clear. (See [1, Theorem 9])□

Proposition 3.20 Let A be a proper submodule of B. Suppose that every nonzero element of A is irreducible then A is a weakly primary.

Proof. Let $0\neq rb\in A$. Then rb is irreducible. So $Rrb=Rb\cup rB$. Hence $b\in Rrb$ and so $b\in A$.□

Proposition 3.21 Suppose that $(R,\mathcal{M})$ is a zero-dimensional quasi-local ring. If A is a P-weakly primary submodule of a torsion-free module B then $P=\mathcal{M}$ or every non zero element of P is irreducible. Also if $P\subseteq\mathcal{M}$ then every ideal of R is contained in P is weakly prime.

Proof. By Proposition 2.5, P is weakly prime ideal so by [1, Theorem 9] the result follows.□

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