THE ASYMPTOTIC FORMULAS FOR THE SUM OF SQUARES OF NEGATIVE EIGENVALUES OF A SINGULAR STURM-LIOUVILLE OPERATOR

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In this work, I find the asymptotic formulas for the sum of squares of negative eigenvalues of the operator L which is formed by differential expression

$$\ell(y) = -(p(x)y'(x))' - q(x)y(x)$$

in the space $L_2[0,\infty)$, with the boundary condition $y(0) = 0$.

Keywords: :Hilbert Space, self-adjoint operator, semi bounded operator, eigenvalue, discrete spectrum

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1. Introduction

Let us consider the operator L, which is formed by the differential expression

$$\ell(y) = -(p(x)y'(x))' - q(x)y(x) \tag{1.1}$$

in the space $L_2[0,\infty)$, with the boundary condition $y(0) = 0$. Suppose that the functions $p(x)$ and $q(x)$ which placed in the expression $\ell(y)$ satisfy the following conditions:

1.There are constants c_1 and c_2 such that $0 < c_1 \le p(x) \le c_2$.

2.$p(x)$ is continuous, nondecreasing function and it has bounded derivative on $[0,\infty)$.

3. $q(x)$ is continuous, decreasing and positive valued function on $[0,\infty)$.

4. $\lim\limits_{x\to\infty} q(x) = 0$.

5. $\lim\limits_{x\to\infty} q(x)x^{k-\eta} = \lim\limits_{x\to\infty} (q(x)x^{k+\eta})^{-1} = 0$ for every $\eta > 0$, where k is a constant which belongs to the interval $(0,\ \frac{2}{5})$.

6.Let us denote the functions of the form $\ln_0 x = x$, $\ln_j x = \ln(\ln_{j-1} x)$ by $\ln_j x \quad (j = 0,\ 1,\ 2\cdots)$.

There are a positive number $\xi > 0$ and a natural number $n \ge 1$ such that the function $q(x) - (\ln_n x)^{-\xi}$ is neither negative valued and nor monotonous increasing in an interval $[a,\ \infty)$ $(a > 0)$.

$D(L)$ denotes the set of all functions $y(x)$ satisfying the following conditions in $L_2[0,\infty)$:

a.$y(x)$ and $y'(x)$ are absolutely continuous in every finite interval $[0,a) \quad (a \in (0,\infty))$.

b. $y(0) = 0$.

c. $-(p(x)y'(x))' - q(x)y(x) \in L_2[0,\infty)$.

Let the operator L be defined by $Ly = \ell(y)$ from $D(L)$ to $L_2[0,\infty)$. It is known that the operator L is self-adjoint, semi-bounded below and negative part of its spectrum is discrete [1].

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Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ be negative eigenvalues of the operator L.
In this work, I find the following asymptotic formulas for the sum of squares of negative eigenvalues of the singular Sturm-Liouville Operator L as $\epsilon \to +0$, $(\epsilon > 0)$

$$\sum_{\lambda_j<-\epsilon} \lambda_j^2 = \frac{1}{15\pi}\left(1+O(\epsilon^{t_0})\right) \int\limits_{q(x)\geq\epsilon} \sqrt{\frac{q(x)-\epsilon}{p(x)}}\left[8q^2(x)+4q(x)\epsilon+3\epsilon^2\right]dx,$$

under the conditions 1,2,3,5; and

$$\sum_{\lambda_j<-\epsilon} \lambda_j^2 = \frac{1}{15\pi}\left[1+O(e^{-\epsilon^{-\beta}})\right] \int\limits_{q(x)\geq\epsilon} \sqrt{\frac{q(x)-\epsilon}{p(x)}}\left[8q^2(x)+4q(x)\epsilon+3\epsilon^2\right]dx,$$

under the conditions 1,2,3,4,6.
In the work [2], some asymptotic formulas are found for the number of negative eigenvalues of the operator L. In the work [4], the asymptotic behavior of the negative part of the spectrum of a differential operator with the operator coefficient is investigated. Later, the asymptotic formula for the number of eigenvalues of Sturm-Liouville operator with the operator coefficient which has singularity is studied in [3]. The works [5] and [8] analyzes the asymptotic behavior of the negative eigenvalues of the operator in the space $L_2[0,\infty)$ which is formed by the differential expression $-y''(x) - q(x)y(x)$, with the boundary condition $y'(0) = 0$. The papers [10] and [9] are related to asymptotics of the number of negative eigenvalues of a differential operator with operator coefficient. In many other works such as [11], [12], [13], [14] negative spectrum of different type of differential operators is prospected by using other methods. My work concerns the asymptotics of the summation of the squares of eigenvalues. To do this I use Courant's variational principle.

2. **Some Inequalities About The Eigenvalues**

Since the function $q(x)$ is monotone decreasing, it has inverse. Let $g(x)$ be inverse function of $q(x)$. Let ϵ is a number belonging the interval $(0, q(0))$. Let us consider the following operators:
Let L' be operator in the space $L_2[0, g(\epsilon)]$, which is formed by the expression (1.1), with the boundary condition $y(0) = y(g(\epsilon)) = 0$.
Let L'' be operator in the space $L_2[0, g(\epsilon)]$, which is formed by the expression (1.1), with the boundary condition $y'(0) = y'(g(\epsilon)) = 0$.
Let the partition points of the interval $[0, g(\epsilon)]$ be $0 = x_0 < x_1 < \cdots < x_m = g(\epsilon)$.
Let L_i' be operator in the space $L_2[x_{i-1}, x_i]$, which is formed by expression (1.1), with the boundary condition $y(x_{i-1}) = y(x_i) = 0$.
Let L_i'' be operator in the space $L_2[x_{i-1}, x_i]$, which is formed by expression (1.1), with the boundary condition $y'(x_{i-1}) = y'(x_i) = 0$.
Let $L_i^{(1)}$ be an operator in the space $L_2[x_{i-1}, x_i]$, which is formed by expression $-p(x_i)y''(x)-q(x_i)y(x)$ with the boundary condition $y(x_{i-1}) = y(x_i) = 0$.
Let $L_i^{(2)}$ be an operator in the space $L_2[x_{i-1}, x_i]$ which is formed by expression $-p(x_{i-1})y''(x)-q(x_{i-1})y(x)$ with boundary condition $y'(x_{i-1}) = y'(x_i)) = 0$.
Let us divide the interval $[0, g(\epsilon)]$ into the intervals the with length

$$\delta = \frac{g(\epsilon)}{[|\ g^{\alpha}(\epsilon)\ |]+1}. \tag{2.1}$$

Here ϵ is a positive constant satisfying the conditions $g^{\alpha}(\epsilon) \geq 2$ and $\alpha \in (0,1)$.
Let $N(\lambda), N'(\lambda), N''(\lambda),\ n_i'(\lambda), n_i^{(1)}(\lambda)$ be numbers of eigenvalues smaller than $-\lambda$ $(\lambda \in (0,\infty))$ of the operators $L, L', L'', L_i', L_i^{(1)}$.
Instead of $n_i'(\epsilon)$ and $n_i^{(1)}(\epsilon)$ we will simply write n_i' and $n_i^{(1)}$.

In the work [2], the following inequalities

$$N'(\epsilon) \leq N(\epsilon) \leq N''(\epsilon) \tag{2.2}$$

and

$$L_i' \leq L_i^{(1)}, \quad L_i'' > L_i^{(2)}$$

are proved. By the similar way the inequalities (2.2), the inequalities

$$N'(\lambda) \leq N(\lambda) \leq N''(\lambda), \quad (\lambda \geq \epsilon) \tag{2.3}$$

can be proved.
Since the inequality $L_i' \leq L_i^{(1)}$ and from the work [6], we obtain that

$$n_i'(\lambda) \geq \ n_i^{(1)}(\lambda). \tag{2.4}$$

On the other hand, from the variation principles of R. Courant [7] we have

$$N'(\lambda) \geq \sum_{i=1}^{M} n_i'(\lambda), \tag{2.5}$$

and from (2.3), (2.4) and (2.5) we find

$$N(\lambda) \geq \sum_{i=1}^{M} n_i^{(1)}(\lambda), \tag{2.6}$$

where M any natural number. Let $\mu_{i1} \leq \mu_{i2} \leq \mu_{i3} \leq \cdots$ be the eigenvalues of the operator $L_i^{(1)}$.
Moreover let us take the following equality $f(x,\epsilon) = \sqrt{\frac{q(x)-\epsilon}{p(x)}}[8q^2(x) + 4q(x)\epsilon + 3\epsilon^2]$

Theorem 2.1. *For the eigenvalues smaller than* $-\epsilon$ *of the operator* $L_i^{(1)}$ *, the inequality* $\sum\limits_{m=1}^{n_i^{(1)}} \mu_{im}^2 > \frac{\delta}{15\pi} f(x_i,\epsilon) - 2q^2(x_i)$ *is satisfied.*

Proof: Since the eigenvalues of the operator $L_i^{(1)}$ are of the form
$\mu_{im} = p(x_i)\left(\frac{m\pi}{x_i - x_{i-1}}\right)^2 - q(x_i) \qquad (m = 1, 2, \cdots)$ then we have

$$\sum_{m=1}^{n_i^{(1)}} \mu_{im}^2 = \sum_{m=1}^{n_i^{(1)}} \left[q(x_i) - p(x_i)\left(\frac{m\pi}{x_i - x_{i-1}}\right)^2 \right]^2 = \sum_{m=1}^{n_i^{(1)}} \left[q(x_i) - p(x_i)\left(\frac{m\pi}{\delta}\right)^2 \right]^2. \tag{2.7}$$

From the relation $\left(\frac{m\pi}{\delta}\right)^2 \leq \left(\frac{t\pi}{\delta}\right)^2 \quad (m \leq t \leq m+1)$, we find that,
$\left[q(x_i) - p(x_i)\left(\frac{m\pi}{\delta}\right)^2 \right]^2 \geq \left[q(x_i) - p(x_i)\left(\frac{t\pi}{\delta}\right)^2 \right]^2, \quad (1 \leq m \leq n_i^{(1)} - 1).$

Hence we obtain that, $\int\limits_m^{m+1} \left[q(x_i) - p(x_i)\left(\frac{m\pi}{\delta}\right)^2 \right]^2 dt > \int\limits_m^{m+1} \left[q(x_i) - p(x_i)\left(\frac{t\pi}{\delta}\right)^2 \right]^2 dt$

or $\left[q(x_i) - p(x_i)\left(\frac{m\pi}{\delta}\right)^2 \right]^2 > \int\limits_m^{m+1} \left[q(x_i) - p(x_i)\left(\frac{t\pi}{\delta}\right)^2 \right]^2 dt \quad (1 \leq m \leq n_i^{(1)} - 1).$

By using these inequalities and the relation (2.7) we find that,

$$\sum_{m=1}^{n_i^{(1)}} \mu_{im}^2 = \sum_{m=1}^{n_i^{(1)}} \left[q(x_i) - p(x_i) \left(\frac{m\pi}{\delta} \right)^2 \right]^2 \geq \sum_{m=1}^{n_i^{(1)}-1} \left[q(x_i) - p(x_i) \left(\frac{m\pi}{\delta} \right)^2 \right]^2$$
$$> \sum_{m=1}^{n_i^{(1)}-1} \int_m^{m+1} \left[q(x_i) - p(x_i) \left(\frac{t\pi}{\delta} \right)^2 \right]^2 dt = \int_1^{n_i^{(1)}} \left[q(x_i) - p(x_i) \left(\frac{t\pi}{\delta} \right)^2 \right]^2 dt \qquad (2.8)$$

Moreover, from the inequality $p(x_i)\left(\frac{m\pi}{\delta}\right)^2 - q(x_i) < -\epsilon$, we obtain that,

$$\frac{\delta}{\pi} \sqrt{\frac{q(x_i) - \epsilon}{p(x_i)}} - 1 \leq n_i^{(1)} < \frac{\delta}{\pi} \sqrt{\frac{q(x_i) - \epsilon}{p(x_i)}}. \qquad (2.9)$$

From (2.8) and (2.9), we find that,

$$\sum_{m=1}^{n_i^{(1)}} \mu_{im}^2 > \int_0^{a-1} \left[q(x_i) - p(x_i) \left(\frac{t\pi}{\delta} \right)^2 \right]^2 dt - q^2(x_i) > \int_0^{a} \left[q(x_i) - p(x_i) \left(\frac{t\pi}{\delta} \right)^2 \right]^2 dt - 2q^2(x_i) \qquad (2.10)$$

where $a = \frac{\delta}{\pi} \sqrt{\frac{q(x_i)-\epsilon}{p(x_i)}}$.
when we calculate the integral on the right side of the relation (2.10), the result is

$$\int_0^{a} \left[q(x_i) - p(x_i) \left(\frac{t\pi}{\delta} \right)^2 \right]^2 dt = \frac{\delta}{15\pi} \sqrt{\frac{q(x_i) - \epsilon}{p(x_i)}} \left[8q^2(x_i) + 4q(x_i)\epsilon + 3\epsilon^2 \right]. \qquad (2.11)$$

From (2.10) and (2.11) we get $\sum_{m=1}^{n_i^{(1)}} \mu_{im}^2 > \frac{\delta}{15\pi} f(x_i, \epsilon) - 2q^2(x_i)$,

where $f(x_i, \epsilon) = \sqrt{\frac{q(x_i)-\epsilon}{p(x_i)}} \left[8q^2(x_i) + 4q(x_i)\epsilon + 3\epsilon^2 \right]$.□

Let $\gamma_{i1}'' \leq \gamma_{i2}'' \leq \cdots$ be eigenvalues of the operator L_i'' and $\gamma_{i1}^{(2)} \leq \gamma_{i2}^{(2)} \leq \cdots$ be eigenvalues of the operator $L_i^{(2)}$.
Let us define the numbers $n_i''(\lambda)$, $n_i^{(2)}(\lambda)$, $n_i''(\epsilon)$ and $n_i^{(2)}(\epsilon)$ as follows:

$$n_i''(\lambda) = \sum_{\gamma_{im}'' < -\lambda} 1 , \quad n_i^{(2)}(\lambda) = \sum_{\gamma_{im}^{(2)} < -\lambda} 1, \quad n_i''(\epsilon) = n_i'' , \quad n_i^{(2)}(\epsilon) = n_i^{(2)}.$$

Theorem 2.2. *For the eigenvalues smaller than $-\epsilon$ of the operator $L_i^{(2)}$, the inequality*

$$\sum_{m=1}^{n_i^{(2)}} (\gamma_{im}^{(2)})^2 < \frac{\delta}{15\pi} f(x_{i-1}, \epsilon) + q^2(x_{i-1})$$

is satisfied.

Proof: The eigenvalues of the operator $L_i^{(2)}$ are of the form

$$\gamma_{im}^{(2)} = p(x_{i-1}) \left(\frac{(m-1)\pi}{x_i - x_{i-1}} \right)^2 - q(x_{i-1}) \qquad (m = 1, 2, \cdots). \qquad (2.12)$$

By the relation $\left(\frac{(m-1)\pi}{x_i-x_{i-1}}\right)^2 \geq \left(\frac{t\pi}{\delta}\right)^2 \quad (m-2 \leq t \leq m-1;\ m=2,3,\cdots)$, we have the inequalities

$$p(x_{i-1})\left(\frac{(m-1)\pi}{\delta}\right)^2 \geq p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2$$

$$\left[q(x_{i-1}) - p(x_{i-1})\left(\frac{(m-1)\pi}{\delta}\right)^2\right]^2 \leq \left[q(x_{i-1}) - p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2\right]^2$$

$(m-2 \leq t \leq m-1;\ 2 \leq m \leq n_i^{(2)})$.
Therefore we get the inequalities

$$\int\limits_{m-2}^{m-1}\left[q(x_{i-1}) - \left(\frac{(m-1)\pi}{\delta}\right)^2\right]^2 dt < \int\limits_{m-2}^{m-1}\left[q(x_{i-1}) - p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2\right]^2 dt$$

or

$$\left[q(x_{i-1}) - p(x_{i-1})\left(\frac{(m-1)\pi}{\delta}\right)^2\right]^2 < \int\limits_{m-2}^{m-1}\left\{q(x_{i-1}) - p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2\right\}^2 dt \tag{2.13}$$

$(2 \leq m \leq n_i^{(2)})$.
By using (2.12) and (2.13) we obtain that,

$$\begin{aligned}
\sum_{m=1}^{n_i^{(2)}} (\gamma_{im}^{(2)})^2 &= \sum_{m=1}^{n_i^{(2)}}\left[q(x_{i-1}) - p(x_{i-1})\left(\frac{(m-1)\pi}{\delta}\right)^2\right]^2 \\
&= q^2(x_{i-1}) + \sum_{m=2}^{n_i^2}\left[q(x_{i-1}) - p(x_{i-1})\left(\frac{(m-1)\pi}{\delta}\right)^2\right]^2 \\
&< q^2(x_{i-1}) + \sum_{m=2}^{n_i^{(2)}}\int\limits_{m-2}^{m-1}\left[q(x_{i-1}) - p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2\right]^2 dt \\
&= q^2(x_{i-1}) + \int\limits_{0}^{n_i^{(2)}-1}\left[q(x_{i-1}) - p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2\right]^2 dt.
\end{aligned}$$

By the inequality $p(x_{i-1})\left(\frac{(m-1)\pi}{\delta}\right)^2 - q(x_{i-1}) < -\epsilon$, we get $n_i^{(2)} < \frac{\delta}{\pi}\sqrt{\frac{q(x_{i-1})-\epsilon}{p(x_{i-1})}} + 1$. By the last inequality we have

$$\sum_{m=1}^{n_i^{(2)}} (\gamma_{im}^{(2)})^2 < \int\limits_{0}^{b}\left[q(x_{i-1}) - p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2\right] dt + q^2(x_{i-1}), \tag{2.14}$$

here, $b = \frac{\delta}{\pi}\sqrt{\frac{q(x_{i-1})-\epsilon}{p(x_{i-1})}}$. From (2.11) and (2.14) we obtain that,

$$\sum_{m=1}^{n_i^{(2)}} (\gamma_{im}^{(2)})^2 < \frac{\delta}{15\pi}\sqrt{\frac{q(x_i)-\epsilon}{p(x_i)}}[8q^2(x_{i-1}) + 4q(x_{i-1})\epsilon + 3\epsilon^2] + q^2(x_{i-1})$$

or

$$\sum_{m=1}^{n_i^{(2)}} (\gamma_{im}^{(2)})^2 < \frac{\delta}{15\pi} f(x_{i-1}, \epsilon) + q^2(x_{i-1}).\square \tag{2.15}$$

Since $q(x_i) \leq q(x) \leq q(x_{i-1})$ in the interval $[x_{i-1},\ x_i]$, then $L_i' \leq\ L_i^{(1)}$ and $L_i'' \leq\ L_i^{(2)}$. In this case by [6], it is known that,

$$n_i'(\lambda) \geq n_i^{(1)}(\lambda) \quad and \quad n_i''(\lambda) \leq n_i^{(2)}(\lambda). \tag{2.16}$$

On the other hand, from the variation principles of R. Courant ([7]) we have

$$N'(\lambda) \geq \sum_{i=1}^{M} n_i'(\lambda), \quad N''(\lambda) \leq \sum_{i=1}^{M} n_i''(\lambda). \tag{2.17}$$

From (2.2), (2.3) and (2.17) we find that,

$$\sum_{i=1}^{M} n_i^{(1)}(\lambda) \leq N(\lambda) \leq \sum_{i=2}^{M} n_i^{(2)}(\lambda) + n_i''(\lambda).$$

By using the last relation, the inequalities

$$\sum_{i=1}^{M} \sum_{m=1}^{n_i^{(1)}} \mu_{im}^2 \leq \sum_{j=1}^{N(\epsilon)} \lambda_j^2 \leq \sum_{i=2}^{M} \sum_{m=1}^{n_i^{(2)}} (\gamma_{im}^{(2)})^2 + \sum_{m=1}^{n_1''} (\gamma_{im}'')^2 \tag{2.18}$$

can be proved.

Theorem 2.3. *If the functions $p(x)$ and $q(x)$ satisfy the conditions 1., 2., 3. and 4., then for small positive values of ϵ we have*

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_{\delta}^{g(\epsilon)} f(x,\ \epsilon)dx - cg^{\alpha}(\epsilon) \tag{2.19}$$

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \sum_{m=1}^{n_1''} (\gamma_{1m}'')^2 + \frac{1}{15\pi} \int_{0}^{g(\epsilon)} f(x,\ \epsilon)dx + cg^{\alpha}(\epsilon) \tag{2.20}$$

Here, $f(x,\epsilon) = \sqrt{\frac{q(x)-\epsilon}{p(x)}}[8q^2(x) + 4q(x)\epsilon + 3\epsilon^2]$ and c is a positive constant.

Proof : Since the function $q(x)$ is monotonous decreasing, then the function $f(x,\epsilon)$ is monotonous decreasing with respect to x for every ϵ satisfying the conditions $\epsilon \in (0,\ q(0))$ and $g^{\alpha}(\epsilon) \geq 2$.
Therefore we have

$$\delta f(x_i,\ \epsilon) = \int_{x_i}^{x_{i+1}} f(x_i,\ \epsilon)dx > \int_{x_i}^{x_{i+1}} f(x,\ \epsilon)dx \quad (1 \leq i \leq M-1) \tag{2.21}$$

By using theorem 2.1 and (2.21), we get

$$\sum_{i=1}^{n_i^{(1)}} \mu_{im}^2 > \frac{1}{15\pi} \int_{x_i}^{x_{i+1}} f(x,\ \epsilon)dx - 2q^2(0) \tag{2.22}$$

From (2.18) and (2.22), we obtain that,

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \sum_{i=1}^{M-1} \left[\frac{1}{15\pi} \int_{x_i}^{x_{i+1}} f(x,\, \epsilon)dx - 2q^2(0) \right] > \frac{1}{15\pi} \int_{x_1}^{x_M} f(x,\, \epsilon)dx - 2q^2(0)M \tag{2.23}$$

By (2.1) for small positive values of ϵ, we find that,

$$M = \frac{g(\epsilon)}{\delta} = [|^{\alpha}(\epsilon)|] + 1 < 2g^{\alpha}(\epsilon). \tag{2.24}$$

If we consider that $x_1 = \delta$ and $x_M = g(\epsilon)$ then from (2.23) and (2.24) we get

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_{\delta}^{p(\epsilon)} f(x,\, \epsilon)dx - cp^k(\epsilon).$$

So the inequality (2.19) is proved. Now, let us prove the inequality (2.20). Again, if we consider that for every ϵ satisfying the conditions $\epsilon \in (0,\, q(0))$ and $g^{\alpha}(\epsilon) \geq 2$, the function $f(x, \epsilon)$ is monotonous decreasing with respect to x then we obtain that,

$$\delta f(x_{i-1},\, \epsilon) = \int_{x_{i-2}}^{x_{i-1}} f(x_{i-1},\, \epsilon)dx < \int_{x_{i-2}}^{x_{i-1}} f(x,\, \epsilon)dx \quad (2 \leq i \leq M). \tag{2.25}$$

From theorem 2.2 and the relation (2.25), we find that,

$$\sum_{m=1}^{n_i^{(2)}} (\gamma_{im}^{(2)})^2 < \frac{1}{15\pi} \int_{x_{i-2}}^{x_{i-1}} f(x,\, \epsilon)dx + q^2(0). \tag{2.26}$$

From (2.18) and (2.26), we get

$$\begin{aligned} \sum_{j=1}^{N(\epsilon)} \lambda_j^2 &< \sum_{m=1}^{n_1''} (\gamma_{1m}'')^2 + \sum_{i=2}^{M} \left[\frac{1}{15\pi} \int_{x_{i-2}}^{x_{i-1}} f(x,\, \epsilon)dx + q^2(0) \right] \\ &= \sum_{m=1}^{n_1''} (\gamma_{1m}'')^2 + \frac{1}{15\pi} \int_{x_0}^{x_{M-1}} f(x,\, \epsilon)dx + (M-1)q^2(0) \\ &< \sum_{m=1}^{n_1''} (\gamma_{1m}'')^2 + \frac{1}{15\pi} \int_{0}^{x_M} f(x, \epsilon)dx + Mq^2(0). \end{aligned} \tag{2.27}$$

By (2.24) and (2.27) we obtain that,

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \sum_{m=1}^{n_1''} (\gamma_{1m}'')^2 + \frac{1}{15\pi} \int_{0}^{g(\epsilon)} f(x,\, \epsilon)dx + cg^{\alpha}(\epsilon).\Box$$

We can proof the following inequality for the sum $\sum_{m=1}^{n_1''} (\gamma_{1m}'')^2$ on the last inequality

$$\sum_{m=1}^{n_1''} (\gamma_{1m}'')^2 < c_3 \int_{0}^{\delta} f(x,\, \epsilon)dx + c_3 g^{\alpha}(\epsilon). \tag{2.28}$$

From (2.20) and (2.28), we find that,

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \frac{1}{15\pi} \int_0^{g(\epsilon)} f(x,\, \epsilon)dx + c_4 \int_0^{\delta} f(x,\, \epsilon)dx + c_4 g^{\alpha}(\epsilon) \tag{2.29}$$

Here, $c_4 > 0$ is a constant.

3. The Asymptotic Formulas For The Sum Of Squares Of The Negative Eigenvalues

In this section we will find some formulas for the sum $\sum\limits_{\lambda_j<-\epsilon} \lambda_j^2$ as $\epsilon \to +0$. First of all we suppose that the function $q(x)$ satisfies the following condition:

5. $\lim\limits_{x\to\infty} q(x)x^{k-\eta} = \lim\limits_{x\to\infty} (q(x)x^{k+\eta})^{-1} = 0$ for every $\eta > 0$.

where k is a constant which belongs to the interval $(0,\, \frac{2}{5})$.

Theorem 3.1. *If the functions $p(x)$ and $q(x)$ satisfy the conditions 1.,2.,3., and 5. , then the asymptotic formula*

$$\sum_{\lambda_j<-\epsilon} \lambda_j^2 = \frac{1}{15\pi}\big(1+O(\epsilon^{t_0})\big) \int_{q(x)\ge\epsilon} \sqrt{\frac{q(x)-\epsilon}{p(x)}}\Big[8q^2(x)+4q(x)\epsilon+3\epsilon^2\Big]dx$$

is satisfied as $\epsilon \to +0$. Here t_0 is a positive constant.

Proof: By Theorem 2.3, for small positive values of ϵ we have

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_0^{g(\epsilon)} f(x,\, \epsilon)dx - \frac{1}{15\pi} \int_0^{\delta} f(x,\, \epsilon)dx - cg^{\alpha}(\epsilon) \tag{3.1}$$

For the proof of theorem , we will limit the each term on the right side of the inequality (3.1). Since the function $q(x)$ is monotonous decreasing, then we have

$$q(x) \ge q[g(2\epsilon)] = 2\epsilon \qquad x \in [0, g(2\epsilon)].$$

Therefore we find that,

$$\int_0^{g(\epsilon)} f(x,\, \epsilon)dx > \int_0^{g(2\epsilon)} f(x,\, \epsilon)dx = \int_0^{g(2\epsilon)} \sqrt{\frac{q(x)-\epsilon}{p(x)}}\Big[8q^2(x)+4q(x)\epsilon+3\epsilon^2\Big]dx$$

$$> \int_0^{g(2\epsilon)} \sqrt{\frac{q(x)-\epsilon}{p(x)}}(8\epsilon^2+4\epsilon^2+3\epsilon^2)dx = 15\epsilon^{5/2} \int_0^{g(2\epsilon)} \frac{1}{\sqrt{p(x)}}dx. \tag{3.2}$$

Since $p(x) < c_2$, then we get

$$\int_0^{g(\epsilon)} f(x,\, \epsilon)dx > 15\epsilon^{5/2}\sqrt{c_2^{-1}}g(2\epsilon) > \epsilon^{5/2}\sqrt{c_2^{-1}}g(2\epsilon). \tag{3.3}$$

If the function $q(x)$ satisfies the condition 5. and $\lim\limits_{\epsilon\to\infty} g(\epsilon) = \infty$, then we obtain that $\lim\limits_{\epsilon\to 0}[q(g(2\epsilon))(g(2\epsilon))^{k+\eta}] = \infty$. Hence we find

$$g(2\epsilon) > \epsilon^{-\frac{1}{k+\eta}}, \tag{3.4}$$

for the small values with respect to η of ϵ. By using (3.3) and (3.4) we get

$$\int_0^{g(\epsilon)} f(x,\,\epsilon)dx > \sqrt{c_2^{-1}}\,\epsilon^{\frac{5}{2}-\frac{1}{k+\eta}} \tag{3.5}$$

Now let us limit the integral $\int_0^{\delta} f(x,\,\epsilon)dx$ on the inequality (3.1): Since the function $q(x)$ satisfies the condition 5., there is a function $f_1(\eta)$ such that

$$q(x) \le f_1(\eta)x^{\eta-k} \qquad (0<\eta<k), \tag{3.6}$$

where $f_1(\eta)$ has positive values. On the other hand since $p(x) \ge c_1$ $(c_1>0)$, then we have

$$\begin{aligned}
\int_0^{\delta} f(x,\,\epsilon)dx &= \int_0^{\delta}\sqrt{\frac{q(x)-\epsilon}{p(x)}}\Big[8q^2(x)+4\epsilon q(x)+3\epsilon^2\Big]dx\\
&< \int_0^{\delta}\sqrt{\frac{q(x)}{p(x)}}\Big[8q^2(x)+4q(x)\epsilon+3\epsilon^2\Big]dx\\
&< \int_0^{\delta}\frac{1}{\sqrt{p(x)}}\Big[8q^{5/2}(x)+4q^{3/2}(x)\epsilon+3q^{1/2}(x)\epsilon^2\Big]dx\\
&< \int_0^{\delta}\frac{1}{\sqrt{p(x)}}\Big[8q^{5/2}(x)+4q^{3/2}(x)q(x)+3q^{1/2}(x)q^2(x)\Big]dx\\
&< \frac{15}{\sqrt{c_1}}\int_0^{\delta} q^{5/2}(x)dx = \frac{15}{\sqrt{c_1}}\int_0^{1} q^{5/2}(x)dx + \frac{15}{\sqrt{c_1}}\int_1^{\delta} q^{5/2}(x)dx\\
&< c_6 + \frac{15}{\sqrt{c_1}}\int_1^{\delta} q^{5/2}(x)dx < c_6 + \frac{15}{\sqrt{c_1}}\,f_1^{5/2}(\eta)\int_1^{\delta} x^{\frac{5}{2}(\eta-k)}dx\\
&< c_6 + f_2(\eta)\,\delta^{\frac{5}{2}(\eta-k)+1}\\
&< f_3(\eta)\,\delta^{\frac{5}{2}(\eta-k)+1}
\end{aligned} \tag{3.7}$$

where $f_3(\eta)$ is a positive valued function with respect to η $(0<\eta<k)$. By the equality (2.1), for the small values of ϵ we find that,

$$\delta < g^{1-\alpha}(\epsilon) \qquad (\alpha \in (0,\,1)) \tag{3.8}$$

If we write $g(\epsilon)$ instead of x in (3.6), then we get

$$q(g(\epsilon)) \le f_1(\eta)\Big(g(\epsilon)\Big)^{\eta-k}$$

or

$$g(\epsilon) \le \Big[f_1(\eta)\Big]^{\frac{1}{k-\eta}}\,\epsilon^{-\frac{1}{k-\eta}} \tag{3.9}$$

By (3.7), (3.8) and (3.9), we obtain that,

$$\int_0^{\delta} f(x,\,\epsilon)dx < f_4(\eta)\,\epsilon^{-\frac{(1-\alpha)[5(\eta-k)+2]}{2(k-\eta)}}, \tag{3.10}$$

where $f_4(\eta)$ is a positive valued function with respect to η .
By (3.9) we get

$$g^{\alpha}(\epsilon) \le f_1(\eta)^{\frac{\alpha}{k-\eta}}\, \epsilon^{-\frac{\alpha}{k-\eta}}. \tag{3.11}$$

From (3.5), (3.10) and (3.11), we find the following inequalities:

$$\frac{\int\limits_0^{\delta} f(x,\, \epsilon)dx}{\int\limits_0^{g(\epsilon)} f(x,\, \epsilon)dx} < f_5(\eta)\, \epsilon^{\frac{-(1-\alpha)[5(\eta-k)+2]}{2(k-\eta)} - \frac{5}{2} + \frac{1}{k+\eta}} \tag{3.12}$$

$$\frac{g^{\alpha}(\epsilon)}{\int\limits_0^{g(\epsilon)} f(x,\, \epsilon)dx} < f_5(\eta)\, \epsilon^{-\frac{\alpha}{k-\eta} - \frac{5}{2} + \frac{1}{k+\eta}}. \tag{3.13}$$

Since $k \neq 0$, the functions $\frac{-[5(\eta-k)+2](1-\alpha)}{2(k-\eta)} - \frac{5}{2} + \frac{1}{k+\eta}$ and $-\frac{\alpha}{k-\eta} - \frac{5}{2} + \frac{1}{k+\eta}$ are continuous with respect to η at the point $\eta = 0$.
Consequently, for every $t > 0$, as $0 < \eta < \omega$ there is a number $\omega = \omega(t) > 0$ such that

$$-\frac{(1-\alpha)[5(\eta-k)+2]}{2(k-\eta)} - \frac{5}{2} + \frac{1}{k+\eta} > -\frac{\alpha(2-5k)}{2k} - t \tag{3.14}$$

and

$$-\frac{\alpha}{k-\eta} - \frac{5}{2} + \frac{1}{k+\eta} > \frac{2-5k-2\alpha}{2k} - t. \tag{3.15}$$

Here if we take $\alpha = \frac{2-5k}{4}$, $t = t_0 = \min\left\{\frac{(2-5k)^2}{16k},\ \frac{2-5k}{8k}\right\}$, then from (3.14) and (3.15) we obtain

$$\frac{-[5(\eta-k)+2](1-\alpha)}{2(k-\eta)} - \frac{5}{2} + \frac{1}{k+\eta} > \frac{(2-5k)^2}{8k} - t_0$$
$$\ge \frac{(2-5k)^2}{8k} - \frac{(2-5k)^2}{16k} = \frac{(2-5k)^2}{16k} \ge t_0 \tag{3.16}$$

and

$$-\frac{\alpha}{k-\eta} - \frac{5}{2} + \frac{1}{k+\eta} > \frac{2-5k}{4k} - t_0 \ge \frac{2-5k}{4k} - \frac{2-5k}{8k} = \frac{2-5k}{8k} \ge t_0. \tag{3.17}$$

From (3.12), (3.13), (3.16) and (3.17) we find that,

$$\frac{\int\limits_0^{\delta} f(x,\, \epsilon)dx}{\int\limits_0^{g(\epsilon)} f(x,\, \epsilon)dx} < c_7\, \epsilon^{t_0} \tag{3.18}$$

$$\frac{g^{\alpha}(\epsilon)}{\int\limits_0^{g(\epsilon)} f(x,\, \epsilon)dx} < c_7\, \epsilon^{t_0}, \tag{3.19}$$

where $c_7 = f_5(\eta(t_0)) \in (0, \infty)$ is a constant. From (3.1), (3.18) and (3.19) we get

$$\frac{\sum\limits_{j=1}^{N(\epsilon)} \lambda_j^2}{\frac{1}{15\pi} \int\limits_0^{g(\epsilon)} f(x,\, \epsilon)dx} > 1 - c_8\, \epsilon^{t_0} \tag{3.20}$$

From (2.29), (3.18) and (3.19) we find that,

$$\frac{\sum\limits_{j=1}^{N(\epsilon)} \lambda_j^2}{\frac{1}{15\pi} \int\limits_0^{g(\epsilon)} f(x,\, \epsilon) dx} < 1 + c_9\, \epsilon^{t_0} \tag{3.21}$$

From (3.20) and (3.21) we obtain the asymptotic formula

$$\frac{\sum\limits_{\lambda_j < -\epsilon} \lambda_j^2}{\frac{1}{15\pi} \int\limits_0^{g(\epsilon)} f(x,\, \epsilon) dx} - 1 = O(\epsilon^{t_0})$$

or $\sum\limits_{\lambda_j<-\epsilon} \lambda_j^2 = \frac{1}{15\pi}\Big[1 + O(\epsilon^{t_0})\Big] \int\limits_{q(x)\geq\epsilon} \sqrt{\frac{q(x)-\epsilon}{p(x)}} \Big[8q^2(x) + 4q(x)\,\epsilon + 3\epsilon^2\Big] dx$ as $\epsilon \to +0$. □

Let us denote the functions of the form $\ln_0 x = x$, $\ln_j x = \ln(\ln_{j-1} x)$ by $\ln_j x$ $(j = 0,\, 1,\, 2 \cdots)$.

Let us suppose that the function $q(x)$ satisfies the following condition:

6. There are a positive number $\xi > 0$ and a natural number $n \geq 1$ such that the function $q(x) - (\ln_n x)^{-\xi}$ is neither negative valued and nor monotonous increasing in an interval $[a,\, \infty)$ $(a > 0)$.

For large values of x, the inequality
$\ln_n\left(\frac{x}{\ln x}\right) < \ln_n x - \ln^{1-n} x, (n \geq 0)$ is satisfied. Let us prove this inequality:

If n=0, then we get $\frac{x}{\ln x} < x - \ln x$. Since $\lim\limits_{x\to\infty} \frac{\ln x}{x - \frac{x}{\ln x}} = 0$, $\ln x < x - \frac{x}{\ln x}$ is satisfied.

If n=1, then we get $\ln x - \ln(\ln x) < \ln x - 1$. Since $\lim\limits_{x\to\infty} \ln_2 x = \infty$, the inequality is satisfied.

For $n \geq 2$, let us use the induction method: If n=2, then we can show the equality $\ln_2(\frac{x}{\ln x}) = \ln_2 x + \ln(1 - \frac{\ln_2 x}{\ln x})$. Since $\lim\limits_{x\to\infty} \frac{\ln_2 x}{\ln x} = 0$, $\ln(1 - \frac{\ln_2 x}{\ln x}) \sim -\frac{\ln_2 x}{\ln x}$; as $x \to \infty$. From here we obtain that $\ln(1 - \frac{\ln_2 x}{\ln x}) < -\frac{1}{2}\frac{\ln_2 x}{\ln x}$; for large values of x. So we get $\ln_2 x + \ln(1 - \frac{\ln_2 x}{\ln x}) < \ln_2 x - \frac{1}{2}\frac{\ln_2 x}{\ln x} < \ln_2 x - \ln^{-1} x$. For n=m $(m \geq 2)$, let us suppose that the inequality $\ln_m(\frac{x}{\ln x}) < \ln_m x - \ln^{1-m} x$ is satisfied. For n=m+1, $\ln_{m+1}(\frac{x}{\ln x}) = \ln(\ln_m \frac{x}{\ln x}) < \ln(\ln_m x - \ln^{1-m} x) = \ln\Big[\ln_m x(1 - \ln_m^{-1} x \ln^{1-m} x)\Big]$. For large values of x, we can show $\ln(1 - \ln_m^{-1} x \ln^{1-m} x) < -\frac{1}{2}\ln_m^{-1} x \ln^{1-m} x$ similar to the above. From here we obtain that $\ln_{m+1}(\frac{x}{\ln x}) < \ln_{m+1} x + \ln(1 - \ln_m^{-1} x \ln^{1-m} x) < \ln_{m+1} x - \frac{1}{2}\ln_m^{-1} x \ln^{1-m} x$. Here since $\frac{\ln x}{2 \ln_m x} > 1, (m \geq 2)$, we find that $\ln_{m+1}(\frac{x}{\ln x}) < \ln_{m+1} x - \frac{1}{2}\ln_m^{-1} x \ln x \ln^{-m} x < \ln_{m+1} x - \ln^{-m} x$. If the function $q(x)$ satisfies the conditions 3., 4., and 6. ,then by using the last inequality, for the small positive values of ϵ, the inequality

$$q\Big(\frac{g(\epsilon)}{\ln g(\epsilon)}\Big) - \epsilon > \big(\ln g(\epsilon)\big)^{-(\xi+1)(n+1)} \tag{3.22}$$

can be proved.

Theorem 3.2. *If the functions $p(x)$ and $q(x)$ satisfy the conditions 1.,2., 3., 4.and 6., then the asymptotic formula*

$$\sum_{\lambda_j<-\epsilon} \lambda_j^2 = \frac{1}{15\pi}\left[1 + O(e^{-\epsilon^{-\beta}})\right] \int\limits_{q(x)\geq\epsilon} \sqrt{\frac{q(x)-\epsilon}{p(x)}}\Big[8q^2(x) + 4q(x)\,\epsilon + 3\epsilon^2\Big] dx$$

is satisfied as $\epsilon \to 0$*. Here* β *is a positive constant.*

Proof: From the theorem 2.3, for small positive values ϵ , we obtain that

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_0^{g(\epsilon)} f(x,\, \epsilon)dx - c_{10}\,\delta - cg^{\alpha}(\epsilon) \tag{3.23}$$

From (2.1) and (3.23), as $\alpha = \frac{1}{2}$, we find that,

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 > \frac{1}{15\pi} \int_0^{g(\epsilon)} f(x,\, \epsilon)dx - c_{11} g^{1/2}(\epsilon), \tag{3.24}$$

where c_{11} is a positive constant. Let us restrict the integral $\int_0^{g(\epsilon)} f(x,\, \epsilon)dx$ on the right side of the inequality (3.24).

$$\int_0^{g(\epsilon)} f(x,\, \epsilon)dx = \int_0^{g(\epsilon)} \sqrt{\frac{q(x)-\epsilon}{p(x)}} \left[8q^2(x) + 4q(x)\,\epsilon + 3\epsilon^2\right] dx > 15\epsilon^2 \int_0^{g(\epsilon)} \sqrt{\frac{q(x)-\epsilon}{p(x)}} dx \tag{3.25}$$

From (3.25), for the small positive values of ϵ , we get

$$\int_0^{g(\epsilon)} f(x,\, \epsilon)dx > \epsilon^2 \int_{1/2 f(\epsilon)}^{f(\epsilon)} \sqrt{\frac{q(x)-\epsilon}{p(x)}} dx > \frac{\epsilon^2 f(\epsilon)}{2} \sqrt{\frac{q(f(\epsilon))-\epsilon}{c_2}} \tag{3.26}$$

where $f(\epsilon) = g(\epsilon) \ln^{-1} g(\epsilon)$. From (3.22) and (3.26) we obtain that,

$$\int_0^{g(\epsilon)} f(x,\, \epsilon)dx > \frac{\epsilon^2}{2\sqrt{c_2}} \frac{g(\epsilon)}{\ln g(\epsilon)} \left(\ln g(\epsilon)\right)^{-\frac{1}{2}(\xi+1)(n+1)} > \epsilon^2 g^{3/4}(\epsilon)$$

so

$$\frac{g^{1/2}(\epsilon)}{\int_0^{g(\epsilon)} f(x,\, \epsilon)dx} < \epsilon^{-2} g^{-1/4}(\epsilon) \tag{3.27}$$

Since the function $q(x)$ satisfies the condition 6., then the inequality $\epsilon = q(p(\epsilon)) \geq (\ln_n p(\epsilon))^{-\xi}$ is satisfied. From this inequality, we get

$$g(\epsilon) \geq e^{\epsilon^{-\frac{1}{\xi}}}. \tag{3.28}$$

From (3.27) and (3.28) we obtain that,

$$\frac{g^{1/2}(\epsilon)}{\int_0^{g(\epsilon)} f(x,\, \epsilon)dx} < \epsilon^{-2} e^{-\frac{1}{4}\epsilon^{-\frac{1}{\xi}}} < e^{-\epsilon^{-\beta}} \tag{3.29}$$

From (3.24) and (3.29) we find that

$$\frac{\sum_{\lambda_j < -\epsilon} \lambda_j^2}{\frac{1}{15\pi} \int_{q(x) \geq \epsilon} f(x,\, \epsilon)dx} > 1 - c_{12} e^{-\epsilon^{-\beta}} \tag{3.30}$$

From (2.29) we get

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \frac{1}{15\pi} \int_0^{g(\epsilon)} f(x,\, \epsilon)dx + c_{13}\delta + c_4 g^{\alpha}(\epsilon) \tag{3.31}$$

From (2.1) and (3.31) as $\alpha = \frac{1}{2}$, we have

$$\sum_{j=1}^{N(\epsilon)} \lambda_j^2 < \frac{1}{15\pi} \int_0^{g(\epsilon)} f(x,\, \epsilon)dx + c_{14} g^{1/2}(\epsilon) \tag{3.32}$$

From (3.29) and (3.32) we find that,

$$\frac{\sum\limits_{\lambda_j < -\epsilon} \lambda_j^2}{\frac{1}{15\pi} \int\limits_{q(x)\geq\epsilon} f(x,\, \epsilon)dx} < 1 + c_{15} e^{-\epsilon^{-\beta}} \tag{3.33}$$

From (3.30) and (3.33) we obtain the asymptotic formula as $\epsilon \to 0$.

$$\frac{\sum\limits_{\lambda_j < -\epsilon} \lambda_j^2}{\frac{1}{15\pi} \int\limits_{q(x)\geq\epsilon} f(x,\, \epsilon)dx} - 1 = O\left(e^{-\epsilon^{-\beta}}\right)$$

or

$$\sum_{\lambda_j < -\epsilon} \lambda_j^2 = \frac{1}{15\pi}\Big[1 + O(e^{-\epsilon^{-\beta}})\Big] \int\limits_{q(x)\geq\epsilon} \sqrt{\frac{q(x)-\epsilon}{p(x)}}\Big[8q^2(x) + 4q(x)\epsilon + 3\epsilon^2\Big]dx.$$

□

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