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ON SOME DIFFERENTIAL SANDWICH THEOREMS OF P – VALENT FUNCTIONS

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In this paper we obtain some subordination and superordination results for higher-order derivatives of p-valent functions involving a generalized differential operator $D^{n}_{\lambda,p,l}(f*g)^{j}$ *and also we obtain sandwich-type theorems. Connections of the results obtained in this paper with known results are considerate and an example is presented.*

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1. Introduction on Subordination and Superordination

Certain aspects of the subordinations and superordinations of functions were considered by D.J. Hallenbeck, S.T. Ruscheweyh, J.A. Antonino, S. Romaguera, S.S. Miller, P.T. Mocanu, G.St. Sălăgean, and others (see [5], [6], [8] and [9]).

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}\mid |z|<1\}$ and let $\mathcal{A}(p)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the

form $$f(z)=z^{p}+\sum_{k=p+1}^{\infty}a_k z^k,\ \ p\in\mathbb{N}=\{1,2,...\},\tag{1.1}$$

which are p-valent in U. We write $\mathcal{A}(1)=\mathcal{A}$.

Let $\mathcal{H}[a,p]=\{f\in\mathcal{H}(U):\ f(z)=a+a_p z^p+a_{p+1}z^{p+1}+...,\ a\in\mathbb{C},\ p\in\mathbb{N}\}$.

Definition 1.1 *If* $f,g\in\mathcal{H}(U)$, *we say that* f *is subordinated to* g *or* g *is superordinate to* f, *if there exists a Schwartz function* $\omega(z)$ *in* U *with* $\omega(0)=0$ *and* $|\omega(z)|<1$, *for all* $z\in U$, *such that* $f(z)=g(\omega(z))$, $z\in U$. *In such a case we write* $f\prec g$, *or* $f(z)\prec g(z)$, $z\in U$.

Furthermore, if the function g is univalent in U, then we have the following equivalence (cf., e.g. [6], [8] and [9]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0)\ \text{and}\ f(U)\subset g(U).$$

In paper [2] were obtained results about the first order differential subordination and supraordination respectively. In the next section we will extend these results to second order differential subordination and superordination respectively. Therefore we introduce the following elements.

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Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let h be an univalent function in U and $q \in H[a, p]$.

Definition 1.2 *If p is analytic in U and satisfies the second order differential subordination:* $\psi\left(p(z), zp'(z), z^2 p''(z); z\right) \prec h(z)$, for $z \in U$, (1.2)

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply dominant, if $p \prec q$ for all p satisfying (1.2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominant q of (1.1) is said to be the best dominant of (1.1).

Remark 1.1 The best dominant is unique up to a rotation of U.

Remark 1.2 Based on results obtained in [8] by Miller and Mocanu, Bulboaca in [5] considered certain classes of first order differential superordinations as well as superordination (in [6]), preserving the integral operators. Ali et al. [1], have used the results of Bulboaca [5] to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}$ to satisfy $q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$, where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [11] obtained a sufficient condition for starlikeness of $f \in \mathcal{A}$ in terms of the quantity $\frac{f''(z) f(z)}{(f'(z))^2}$.

Recently, Shanmugam et al. [10] obtained sufficient conditions for the normalized analytic function $f \in \mathcal{A}$ to satisfy $q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$, $q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z)$.

Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let h analytic in U and $q \in H[a, p]$.

Definition 1.3 *If p and $\psi\left(p(z), zp'(z), z^2 p''(z); z\right)$ are univalent and if p satisfies the second order differential superordination*

$$h(z) \prec \psi\left(p(z), zp'(z), z^2 p''(z); z\right), z \in U, \tag{1.3}$$

then p is a solution of the differential superordination (1.2). An analytic function q is called a subordinant if $q \prec p$, for all p satisfying (1.3). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$, for all subordinant q of (1.2) is said to be the best subordinant.

Miller and Mocanu in [8] determined conditions on ψ such that n

$$h(z) \prec \psi\left(p(z), zp'(z), z^2 p''(z); z\right)$$

implies $q(z) \prec p(z)$, for all p functions that satisfies the above superordination. Moreover, they obtained sufficient conditions so that the q function is the largest function with this property, called the best subordinant of this subordination. Using these results, Bulboacă [5] considered certain classes of first order differential superordinations as well as superordination preserving integral operators. For two functions $f \in \mathcal{A}(p)$ given by (1.1) and $g \in \mathcal{A}(p)$ defined by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{1.1'}$$

the Hadamard product (or convolution product) is defined by

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_k b_k z^k=(g*f)(z). \tag{1.4}$$

Upon differentiating both sides of (1.4) j -times with respect to z, we have:

$$(f*g)^{(j)}(z)=\delta(p,j)z^{p-j}+\sum_{k=p+1}^{\infty}\delta(k,j)a_k b_k z^{k-j}, \tag{1.5}$$

Where $\delta(p,j)=\dfrac{p!}{(p-j)!},\ p>j,\ p\in\mathbb{N},\ j\in\mathbb{N}_0=\mathbb{N}\cup\{0\}.$ (1.6)

For functions $f,g\in A(p)$, we define the linear operator

$$D^n_{\lambda,p,l}(f*g)^{(j)}:A(p)\to A(p)$$

By

$$D^0_{\lambda,p,l}(f*g)^{(j)}(z)=(f*g)^{(j)}(z),$$

$$D^1_{\lambda,p,l}(f*g)^{(j)}(z)=D_{\lambda,p,l}(f*g)^{(j)}(z)=$$

$$(1-\lambda)(f*g)^{(j)}(z)+\frac{\lambda}{(p-j+l)z^{l-1}}\left(z^l\cdot(f*g)^{(j)}(z)\right)'=$$

$$\delta(p,j)z^{p-j}+\sum_{k=p+1}^{\infty}\left(\frac{p-j+l+\lambda(k-p)}{p-j+l}\right)\delta(k,j)a_k b_k z^{k-j},$$

$$D^2_{\lambda,p,l}(f*g)^{(j)}(z)=(1-\lambda)D_{\lambda,p,l}(f*g)(z)+\frac{\lambda}{(p-j+l)z^{l-1}}\left(z^l\cdot D_{\lambda,p,l}(f*g)(z)\right)'=$$

$$\delta(p,j)z^{p-j}+\sum_{k=p+1}^{\infty}\left(\frac{p-j+l+\lambda(k-p)}{p-j+l}\right)^2\delta(k,j)a_k b_k z^{k-j}$$

and (in general)

$$\begin{aligned}&D^n_{\lambda,p,l}(f*g)^{(j)}(z)=\\&(1-\lambda)D^{n-1}_{\lambda,p,l}(f*g)^{(j)}(z)+\frac{\lambda}{(p-j+l)z^{l-1}}\left(z^l\cdot D^{n-1}_{\lambda,p,l}(f*g)(z)\right)'=\\&\delta(p,j)z^{p-j}+\sum_{k=p+1}^{\infty}\left(\frac{p-j+l+\lambda(k-p)}{p-j+l}\right)^n\delta(k,j)a_k b_k z^{k-j},\\&\text{for}\ \lambda>0;l\ge 0;p>j;p\in\mathbb{N},n,j\in\mathbb{N}_0;z\in U.\end{aligned} \tag{1.7}$$

From (1.7), we can easily deduce that

$$\begin{aligned}&\lambda z\left(D^n_{\lambda,p,l}(f*g)^{(j)}(z)\right)'=(p-j+l)D^{n+1}_{\lambda,p,l}(f*g)^{(j)}(z)-\\&[(p-j)(1-\lambda)+l]D^n_{\lambda,p,l}(f*g)^{(j)}(z),\\&\text{for}\ \lambda>0;l\ge 0;p>j;p\in\mathbb{N},n,j\in\mathbb{N}_0;z\in U.\end{aligned} \tag{1.8}$$

We remark that the linear operator $D^n_{\lambda,p,l}(f*g)^{(j)}(z)$ reduces to several many other linear operators: (i) for $j=0$, we obtain the operator studied by Aouf et al [3];

(ii) for $j=0$ and $g(z)=\dfrac{z^p}{1-z}$, we obtain the operator $I^m_p(\lambda,l)$ introduced and studied by Cătaş [7];

(iii) for $l = 1$, we have $D^n_{\lambda,p,1}(f * g)^{(j)}(z) = D^n_{\lambda,p}(f * g)^{(j)}(z)$, where the operator $D^n_{\lambda,p}$ was introduced and studied by Aouf and El-Ashwah [2];

(iv) for $l = 1$, $\lambda = 1$, $g(z) = \frac{z^p}{1-z}$, the differential operator $D^n_p f^{(j)}(z)$ was introduced and studied by Aouf and Seoudy [4]. In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 1.4 *(Miller and Mocanu [8]) Denote by* Q *the set of all functions* f *that are analytic and injective on* $\overline{U} \setminus E(f)$, *where* $E(f) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\right\}$ *and are such that* $f'(\zeta) \neq 0$ *for* $\zeta \in \partial U \setminus E(f)$.

Lemma 1.1 *(Miller and Mocanu [9]) Let the function* q *be univalent in the unit disk* U *and* θ *and* Φ *be analytic in a domain* D *containing* $q(U)$ *with* $\Phi(\omega) \neq 0$ *when* $\omega \in q(U)$.

Let $Q(z) = z\, q'(z) \cdot \Phi(q(z))$ *and* $h(z) = \theta(q(z)) + Q(z)$. *Suppose that*

1. Q *is starlike univalent in* U *and 2.* $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$, *for* $z \in U$. *If* p *is analytic with* $p(0) = q(0)$, $p(U) \subseteq \mathrm{D}$ *and* $\theta(p(z)) + zp'(z)\Phi(p(z)) \prec \theta(q(z)) + zq'(z)\Phi(q(z))$ *then* $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 1.2 *(Bulboacă [5]) Let the function* q *be convex univalent in* U *and let* ν *and* Φ *be analytic in a domain* D *containing* $q(U)$. *Suppose that*

1. $\Re\left(\frac{\nu'(q(z))}{\Phi(q(z))}\right) > 0$, *for* $z \in U$ *and 2.* $\psi(z) = zq'(z)\Phi(q(z))$ *is starlike univalent in* U.

If $p(z) \in \mathrm{H}[q(0),1] \cap Q$, *with* $p(U) \subseteq \mathrm{D}$, $\nu(p(z)) + zp'(z)\Phi(p(z))$ *is univalent in* U *and* $\nu(q(z)) + zq'(z)\Phi(q(z)) \prec \nu(p(z)) + zp'(z)\Phi(p(z))$, *then* $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2. Subordination and Superordination Results

In this section we obtain sufficient conditions on analytic functions $f, g \in \mathrm{A}(p)$ (based on them we defined the linear operator $D^n_{\lambda,p,l}(f * g)^{(j)}$), such that to be verified the following relation: $q_1(z) \prec \left(\frac{aD^{n+1}_{\lambda,p,l}(f * g)^{(j)}(z) + bD^n_{\lambda,p,l}(f * g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu} \prec q_2(z)$,

where q_1 and q_2 are given univalent functions in U.

Unless otherwise mentioned, we shall assume throughout this paper that $\lambda > 0$, $l \geq 0$, $p \in \mathrm{N}$, $p > j$, $n, j \in \mathrm{N}_0$, $\mu \in \mathrm{C}$, $\mu \neq 0$, $z \in U$, $f, g \in \mathrm{A}(p)$ given by (1.1) and (1.1'), respectively.

Theorem 2.1 *Let* $\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\in \mathrm{H}(U)$*, Assume that*

$$\Re\left(1+\frac{\alpha}{\beta}q(z)-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right)>0,\ \text{for } \alpha,\beta\in\mathrm{C},\ \beta\neq 0 \text{ and} \tag{2.1}$$

$$\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z):=\alpha\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}+$$

$$\beta\cdot\mu\cdot\frac{a(p-j+l)D_{\lambda,p,l}^{n+2}(f*g)^{(j)}(z)+\{b(p-j+l)-a[(p-j)(2-\lambda)+l]\}D_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)}{\lambda\left(aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)\right)}-$$

$$\beta\cdot\mu\cdot\frac{b[(p-j)(2-\lambda)+l]D_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\lambda\left(aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)\right)}. \tag{2.2}$$

If q *satisfies the following subordination*

$$\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z)\prec\alpha q(z)+\frac{\beta zq'(z)}{q(z)},\ \text{for } \alpha,\beta\in\mathrm{C},\ \beta\neq 0, \tag{2.3}$$

then

$$\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\prec q(z), \tag{2.4}$$

and q *is the best dominant.*

Corollary 2.1 *Let* $q(z)=\frac{1+Az}{1+Bz}$, $z\in U$, $-1\leq B<A\leq 1$ *and assume that (2.1) holds. If*

$$\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z)\prec\alpha\frac{1+Az}{1+Bz}+\beta\frac{(A-B)z}{(1+Az)(1+Bz)}, \tag{2.6}$$

for $\alpha,\beta\in\mathrm{C}$, $\beta\neq 0$*, where* ψ_{λ}^{n} *is defined in (2.2), then*

$$\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\prec\frac{1+Az}{1+Bz},\mu\in\mathrm{C}, \tag{2.7}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

For $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$, we have the following corollary.

Corollary 2.2 *Assume that (2.1) holds for* $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$*. If*

$$\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z)\prec\alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+2\beta\gamma\frac{z}{1-z^{2}},$$

for $\alpha,\beta\in\mathrm{C}$, $\beta\neq 0$, $0<\gamma\leq 1$, *where* ψ_{λ}^{n} *is defined in (2.2), then*

$$\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\prec\left(\frac{1+z}{1-z}\right)^{\gamma},$$
$$z\in U,\mu\in\mathbb{C},\mu\neq 0,a,b\in\mathbb{C},a+b\neq 0,0<\gamma\leq 1,$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Taking $g(z)=\frac{z^p}{1-z}$ in Theorem 2.1, we obtain the following result.

Corollary 2.3 *Let* q *be univalent in* U *with* $q(0)=1$ *and assume that (2.1) holds. If* $f\in \mathcal{A}(p)$ *satisfies the following subordination condition:*

$$\alpha\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}+$$
$$\beta\cdot\mu\cdot\frac{a(p-j+l)D_{\lambda,p,l}^{n+2}(f*g)^{(j)}(z)+\{b(p-j+l)-a[(p-j)(2-\lambda)+l]\}D_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)}{\lambda\left(aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)\right)}-$$
$$\beta\cdot\mu\cdot\frac{b[(p-j)(2-\lambda)+l]D_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\lambda\left(aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)\right)}\prec\alpha q(z)+\frac{\beta zq'(z)}{q(z)},\qquad(2.8)$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$ *then*

$$\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\prec q(z),\qquad(2.9)$$

and $q(z)$ *is the best dominant.* Based on Lemma 1.2 we have the following theorem.

Theorem 2.2 *Let* q *be convex and univalent in* U *such that* $q(0)=1$. *Assume that*

$$\Re\left(\frac{\alpha}{\beta}q(z)\right)>0,\qquad(2.10)$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$. *If* $\mu,a,b\in\mathbb{C}$, $\mu\neq 0$, $a+b\neq 0$,

$$\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap Q,$$

and $\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z)$ *is univalent in* U, *where* $\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z)$ *is defined in (2.2),*

$$\text{then }\ \alpha q(z)+\beta\frac{zq'(z)}{q(z)}\prec\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z),\qquad(2.11)$$

$$\text{Implies }\ q(z)\prec\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu},\qquad(2.12)$$

and q *is the best subordinant.* Taking $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.4 *Let* $q(z)=\dfrac{1+Az}{1+Bz}$, $-1\le B<A\le 1$ *and assume that (2.10) holds. If*

$$\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\in \mathrm{H}[q(0),1]\cap Q$$

$$\text{for}\,\mu\in\mathrm{C},\mu\neq 0,a,b\in\mathrm{C},a+b\neq 0,$$

and $\alpha\dfrac{1+Az}{1+Bz}+\beta\dfrac{(A-B)z}{(1+Az)(1+Bz)}\prec\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z)$, *for* $\alpha,\beta\in\mathrm{C}$, $\beta\neq 0$, $-1\le B<A\le 1$, *where* ψ_{λ}^{n} *is defined in (2.2), then*

$$\frac{1+Az}{1+Bz}\prec\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best subordinant.*

Corollary 2.5 *Assume that (2.10) holds for* $q(z)=\left(\dfrac{1+z}{1-z}\right)^{\gamma}$. *If*

$$\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\in \mathrm{H}[q(0),1]\cap Q$$

for $\mu\in\mathrm{C},\mu\neq 0,a,b\in\mathrm{C},a+b\neq 0$; $\alpha\left(\dfrac{1+z}{1-z}\right)^{\gamma}+2\beta\gamma\dfrac{z}{1-z^{2}}\prec\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z)$,

for $\alpha,\beta\in\mathrm{C}$, $\beta\neq 0$, $0<\gamma\le 1$, *where* ψ_{λ}^{n} *is defined in (2.2), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu},$$

$$z\in U,\mu\in\mathrm{C},\mu\neq 0,a,b\in\mathrm{C},a+b\neq 0,0<\gamma\le 1,$$

and $\left(\dfrac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.* Taking $g(z)=\dfrac{z^{p}}{1-z}$ in Theorem 2.2, we obtain the following result.

Corollary 2.6 *Let* q *be convex and univalent in* U *such that* $q(0)=1$. *Assume that*

$$\Re\left(\frac{\alpha}{\beta}q(z)\right)>0,\ \text{for}\ \alpha,\beta\in\mathrm{C},\ \beta\neq 0.$$

If $\mu,a,b\in\mathrm{C}$, $\mu\neq 0$, $a+b\neq 0$,

$$\left(\frac{aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\in \mathrm{H}[q(0),1]\cap Q,$$

and $\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z)$ *is univalent in* U, *where* $\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z)$ *is defined by*

$$\psi_{\lambda}^{n}(a,b,\alpha,\beta,\mu;z):=\alpha\left(\frac{aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}+$$

$$\beta\cdot\mu\cdot\frac{a(p-j+l)D^{n+2}_{\lambda,p,l}f^{(j)}(z)+\{b(p-j+l)-a[(p-j)(2-\lambda)+l]\}D^{n+1}_{\lambda,p,l}f^{(j)}(z)}{\lambda\left(aD^{n+1}_{\lambda,p,l}f^{(j)}(z)+bD^{n}_{\lambda,p,l}f^{(j)}(z)\right)}-$$

$$\beta\cdot\mu\cdot\frac{b[(p-j)(2-\lambda)+l]D^{n}_{\lambda,p,l}f^{(j)}(z)}{\lambda\left(aD^{n+1}_{\lambda,p,l}f^{(j)}(z)+bD^{n}_{\lambda,p,l}f^{(j)}(z)\right)}.$$

Then $\alpha q(z)+\beta\frac{zq'(z)}{q(z)}\prec\psi^n_\lambda(a,b,\alpha,\beta,\mu;z)$,

Implies $q(z)\prec\left(\frac{aD^{n+1}_{\lambda,p,l}f^{(j)}(z)+bD^{n}_{\lambda,p,l}f^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^\mu$, *and* q *is the best subordinant.*

3. Sandwich Results

Combining Theorem 2.1 and Theorem 2.2, we obtain the following sandwich theorem.

Theorem 3.1 *Let* q_1 *and* q_2 *be convex and univalent in* U *such that* $q_1(z)\neq 0$ *and* $q_2(z)\neq 0$, *for all* $z\in U$. *Suppose that* q_1 *satisfies (2.1) and* q_2 *satisfies (2.10). If* $\left(\frac{aD^{n+1}_{\lambda,p,l}(f*g)^{(j)}(z)+bD^{n}_{\lambda,p,l}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^\mu\in\mathcal{H}[q(0),1]\cap Q$ *for* $\mu\in\mathbb{C}$, $\mu\neq 0$, $a,b\in\mathbb{C}$, $a+b\neq 0$ *and* $\psi^n_\lambda(a,b,\alpha,\beta,\mu;z)$ *is univalent in* U *and is defined in (2.2), then* $\alpha q_1(z)+\frac{\beta zq_1'(z)}{q_1(z)}\prec\psi^n_\lambda(a,b,\alpha,\beta,\mu;z)\prec\alpha q_2(z)+\frac{\beta zq_2'(z)}{q_2(z)}$, *for* $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$ *implies*

$$q_1(z)\prec\left(\frac{aD^{n+1}_{\lambda,p,l}(f*g)^{(j)}(z)+bD^{n}_{\lambda,p,l}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^\mu\prec q_2(z),\ \mu\in\mathbb{C},\mu\neq 0,$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z)=\frac{1+A_1z}{1+B_1z}$, $q_2(z)=\frac{1+A_2z}{1+B_2z}$, where $-1\le B_2<B_1<A_1<A_2\le 1$ we have the following corollary.

Corollary 3.1 *Assume that (2.1) and (2.10) hold for* $q_1(z)=\frac{1+A_1z}{1+B_1z}$ *and* $q_2(z)=\frac{1+A_2z}{1+B_2z}$, *respectively. If* $\left(\frac{aD^{n+1}_{\lambda,p,l}(f*g)^{(j)}(z)+bD^{n}_{\lambda,p,l}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^\mu\in\mathcal{H}[q(0),1]\cap Q$,

and $\alpha\frac{1+A_1z}{1+B_1z}+\beta\frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)}\prec\psi^n_\lambda(a,b,\alpha,\beta,\mu;z)\prec\alpha\frac{1+A_2z}{1+B_2z}+\beta\frac{(A_2-B_2)z}{(1+A_2z)(1+B_2z)}$,

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\le B_2<B_1<A_1<A_2\le 1$, *where* ψ^n_λ *is defined in (2.2), then*

$$\frac{1+A_1z}{1+B_1z} \prec \left(\frac{aD_{\lambda,p,l}^{n+1}(f*g)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f*g)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}} \right)^{\mu} \prec \frac{1+A_2z}{1+B_2z},$$
$$z\in U,\ \mu\in\mathbb{C},\mu\neq 0, a,b\in\mathbb{C}, a+b\neq 0,$$

hence $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are the best subordinant and the best dominant, respectively.*

Taking $g(z)=\frac{z^p}{1-z}$ in Theorem 3.1, we obtain the following sandwich-type result.

Corollary 3.2 *Let* q_1 *and* q_2 *be convex and univalent in* U *such that* $q_1(z)\neq 0$ *and* $q_2(z)\neq 0$*, for all* $z\in U$*. Suppose that* q_1 *satisfies (2.1) and* q_2 *satisfies (2.10). If*

$$\left(\frac{aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}} \right)^{\mu} \in \mathcal{H}[q(0),1]\cap Q,$$

for $\mu\in\mathbb{C}$, $\mu\neq 0$, $a,b\in\mathbb{C}$, $a+b\neq 0$ *and* $\psi_\lambda^n(a,b,\alpha,\beta,\mu;z)$ *is univalent in* U *and is defined by* $\psi_\lambda^n(a,b,\alpha,\beta,\mu;z):=\alpha\left(\frac{aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}} \right)^{\mu}+$

$$\beta\cdot\mu\cdot\frac{a(p-j+l)D_{\lambda,p,l}^{n+2}f^{(j)}(z)+\{b(p-j+l)-a[(p-j)(2-\lambda)+l]\}D_{\lambda,p,l}^{n+1}f^{(j)}(z)}{\lambda\left(aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)\right)}-$$
$$\beta\cdot\mu\cdot\frac{b[(p-j)(2-\lambda)+l]D_{\lambda,p,l}^{n}f^{(j)}(z)}{\lambda\left(aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)\right)}.$$

Then $\alpha q_1(z)+\frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_\lambda^n(a,b,\alpha,\beta,\mu;z) \prec \alpha q_2(z)+\frac{\beta z q_2'(z)}{q_2(z)}$, *for* $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, *implies* $q_1(z)\prec\left(\frac{aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}} \right)^{\mu}\prec q_2(z)$, $\mu\in\mathbb{C},\mu\neq 0,$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

The following example indicates the possible applications of the above results.

Example 3.1 Let $q_1(z)=\frac{z+2}{2}$, $q_2(z)=\frac{1+z}{1-z}$, $g(z)=\frac{z^p}{1-z}$.

If (2.1) and (2.12) hold,

$\left(\frac{aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}} \right)^{\mu} \in \mathcal{H}[q(0),1]\cap Q, z\in U$, $\mu\in\mathbb{C}$, $\mu\neq 0$, $a\in\mathbb{C}$, $a\neq 0$,

$\psi_\lambda^n(a,b,\alpha,\beta,\mu;z)$ defined by

$$\psi_\lambda^n(a,b,\alpha,\beta,\mu;z) := \alpha\left(\frac{aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu} +$$
$$+\beta\cdot\mu\cdot\frac{a(p-j+l)D_{\lambda,p,l}^{n+2}f^{(j)}(z)+\{b(p-j+l)-a[(p-j)(2-\lambda)+l]\}D_{\lambda,p,l}^{n+1}f^{(j)}(z)}{\lambda\left(aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)\right)}$$
$$-\beta\cdot\mu\cdot\frac{b[(p-j)(2-\lambda)+l]D_{\lambda,p,l}^{n}f^{(j)}(z)}{\lambda\left(aD_{\lambda,p,l}^{n+1}f^{(j)}(z)+bD_{\lambda,p,l}^{n}f^{(j)}(z)\right)}.$$

is univalent in U and $\alpha\frac{z+2}{2}+\beta\frac{z}{2+z}\prec\psi_\lambda^n(a,b,\alpha,\beta,\mu;z)\prec\alpha\frac{1+z}{1-z}+\beta\frac{1}{1-z}$, for $\alpha,\beta\in\mathrm{C},\beta\neq 0$.

Theorem 3.1 affirms that $\frac{z+2}{2}\prec\left(\frac{aD_{\lambda,p,l}^{n+1}(f)^{(j)}(z)+bD_{\lambda,p,l}^{n}(f)^{(j)}(z)}{\delta(p,j)\cdot(a+b)z^{p-j}}\right)^{\mu}\prec\frac{1+z}{1-z}$,

$$z\in U,\mu\in\mathrm{C},\mu\neq 0,a,b\in\mathrm{C},a+b\neq 0,$$

and $\frac{z+2}{2}$ and $\frac{1+z}{1-z}$ are the best subordinant and the best dominant, respectively.

Remark 3.1 The above results are true for the relations of the strong differential subordination and strong differential superordination, too.

4. Conclusions

Complex-valued analytic functions have many properties that are not necessarily true for real-valued functions. One of the most important parts in the geometric function theory is the study of certain subclasses of holomorphic complex-valued functions which are defined by differential subordination, differential superordination, extremal functional conditions and differential operators.

Thus, were defined a generalized differential operator based on two analytical functions with complex-valued (using Hadamard convolution product). Some subordination and superordination results, in the form of sufficient conditions, for higher order derivatives of the p-valent functions involving the defined generalized differential operator have been determined. Sandwich type theorems have been obtained.

Relevant connections of some of the new results obtained in this paper with those in earlier works are also provided.

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