U.P.B. Sci. Bull., Series A, Vol. 78, Iss. 4, 2016 ISSN 1223-7027

ON EDGE IRREGULARITY STRENGTH OF TOEPLITZ GRAPHS

Ali Ahmad[1], Martin Bača[2], Muhammad Faisal Nadeem[3]

An edge irregular k-labeling of a graph G is a labeling of the vertices of G with labels from the set $\{1, 2, \ldots, k\}$ in such a way that for any two different edges xy and $x'y'$ their weights $w(xy)$ and $w(x'y')$ are distinct. The weight $w(xy)$ of an edge xy in G is the sum of the labels of the end vertices x and y. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$.

In this paper, we study the edge irregular k-labeling for Toeplitz graphs and determine the exact value for several classes of Toeplitz graphs.

Keywords: irregular assignment, irregularity strength, edge irregularity strength, Toeplitz graphs

MSC2010: 05C78

1. Introduction

Let G be a connected, simple and undirected graph with vertex set V and edge set E. By a *labeling* we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. If the domain is $V(G) \cup E(G)$ then we call the labeling *total labeling*. Thus, for an edge k-labeling $\delta : E(G) \to \{1, 2, \ldots, k\}$ the associated weight of a vertex $x \in V(G)$ is

$$w_\delta(x) = \sum \delta(xy),$$

where the sum is over all vertices y adjacent to x.

Chartrand *et al.* in [10] introduced edge k-labeling δ of a graph G such that $w_\delta(x) \neq w_\delta(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. The irregularity strength $s(G)$ can be interpreted as the

[1]College of Computer Science & Information Systems, Jazan University, Jazan, KSA, e-mail: ahmadsms@gmail.com

[2]Department of Applied Mathematics and Informatics, Technical University, Košice, Slovak Republic, e-mail: martin.baca@tuke.sk

[3]Department of Mathematics, COMSATS, Institute of Information Technology, Lahore, Pakistan, e-mail: mfaisalnadeem@ymail.com

smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different. This parameter has attracted much attention [5, 6, 9, 16].

Motivated by these papers, Ahmad *et al.* in [1] started to investigate an *edge irregularity strength*. A vertex k-labeling $\phi : V(G) \to \{1, 2, \dots, k\}$ is called an *edge irregular k-labeling* of the graph G if for every two different edges xy and $x'y'$ there is $w_\phi(xy) \neq w_\phi(x'y')$, where the weight of an edge $xy \in E(G)$ is $w_\phi(xy) = \phi(x) + \phi(y)$. The minimum k for which the graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G, denoted by $es(G)$. The notion of the edge irregularity strength was defined in [1]. There is estimated the lower bound of the edge irregularity strength as follows

Theorem 1.1. [1] *Let G be a simple graph with maximum degree $\Delta = \Delta(G)$. Then $es(G) \geq \max\{\lceil(|E(G)| + 1)/2\rceil, \Delta(G)\}$.*

In [1] it is proved that for path P_n, $n \geq 2$, $es(P_n) = \lceil n/2 \rceil$, for star $K_{1,n}$, $n \geq 1$, $es(K_{1,n}) = n$, for double star $S_{m,n}$, $3 \leq m \leq n$, $es(S_{m,n}) = n$ and for Cartesian product of two paths P_n and P_m, $m, n \geq 2$, $es(P_n \Box P_m) = \lceil (2mn - m - n + 1)/2 \rceil$. Al-Mushayt [4] determined the edge irregularity strength of products of certain families of graphs with path P_2.

2. **Toeplitz graph**

A simple undirected graph T of order p is called *Toeplitz graph* if its adjacency matrix $A(T)$ is Toeplitz. A *Toeplitz matrix* $A(T) = (a_{i,j})$, is a $(p \times p)$ symmetric matrix which has constant values along all diagonals parallel to the main diagonal, i.e. $a_{i,j} = a_{i+1,j+1}$ for each $i, j = 1, 2, \dots, p-1$. The p distinct diagonals of a $(p \times p)$ symmetric Toeplitz adjacency matrix will be labeled $0, 1, 2, \dots, p-1$. Diagonal 0 is the main diagonal and it contains only zeros, i.e. $a_{ii} = 0$ for all $i = 1, 2, \dots, p$ so that there are no loops in the Toeplitz graph. A Toeplitz graph T is uniquely defined by the first row of $A(T)$, a $(0-1)$-sequence. Let $t_1, t_2, \dots, t_s$ be the diagonals containing ones, $0 < t_1 < t_2 < \cdots < t_s < p$. Then, the corresponding Toeplitz graph will be denoted by $T_p\langle t_1, \dots, t_s \rangle$. That is, $T_p\langle t_1, \dots, t_s \rangle$ is the graph with the vertex set $V(T) = \{v_i : i = 1, 2, \dots, p\}$ in which two vertices u, v of T being connected by an edge if and only if $|u - v| \in \{t_1, t_2, \dots, t_s\}$. If t_j, $j = 1, 2, \dots, s$, is the diagonal containing ones then the diagonal elements a_{i,t_j+i}, $i = 1, 2, \dots, p - t_j$, determine edges $v_i v_{t_j+i}$ in the Toeplitz graph. Thus the edge set is $E(T) = \bigcup_{j=1}^{s} \{v_i v_{t_j+i} : i = 1, 2, \dots, p - t_j\}$, $|V(T)| = p$ and $|E(T)| = ps - \sum\limits_{j=1}^{s} t_j$.

Toeplitz graphs have been introduced by Sierksma and first been investigated by van Dal et al. [11] with respect to their hamiltonicity. Later

Heuberger [18] has extended this study in 2002. The properties of Toeplitz graphs; such as bipartitness, planarity and colourability, have been studied in [12, 13, 14, 15]. For more recent works on Toeplitz graphs see [8, 21, 22, 25]. A Toeplitz graph is not necessarily connected, see Figure 1.

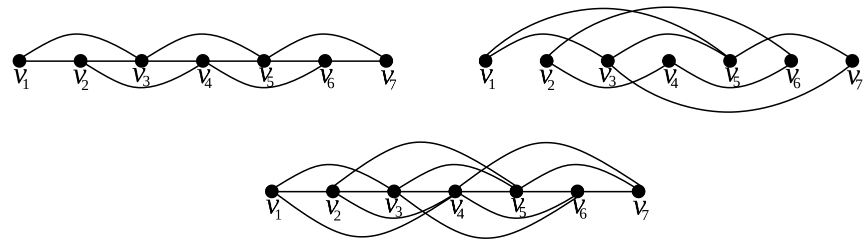

FIGURE 1. Toeplitz graphs $T_7\langle 1,2\rangle$, $T_7\langle 2,4\rangle$ and $T_7\langle 1,2,3\rangle$

The following result proved by van Dal et al. [11], provides a lower bound on the number of components of a Toeplitz graph.

Theorem 2.1. [11] *$T_p\langle t_1,\dots,t_s\rangle$ has at least $gcd(t_1,\dots,t_s)$ components.*

In the paper we investigate the existence of the edge irregularity strength for Toeplitz graphs.

3. Results

Next theorem gives the exact value of the edge irregularity strength of Toeplitz graph $T_n\langle 1,2\rangle$ which is bigger than the lower bound in Theorem 1.1.

Theorem 3.1. *Let $T_n\langle 1,2\rangle$ be a Toeplitz graph on $n \geq 3$ vertices. Then $es(T_n\langle 1,2\rangle) = n$.*

Proof. Let $T_n\langle 1,2\rangle$ be a Toeplitz graph with the vertex set $V(T_n\langle 1,2\rangle) = \{v_i : 1 \leq i \leq n\}$ and the edge set $E(T_n\langle 1,2\rangle) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+2} : 1 \leq i \leq n-2\}$. According to Theorem 1.1 we have that $es(T_n\langle 1,2\rangle) \geq n-1$. Since every two adjacent vertices in $T_n\langle 1,2\rangle$ are a part of complete graph K_3, therefore under every edge irregular labeling the smallest edge weight has to be at least 3 and the largest edge weight has to be at least $2n+2-t_1-t_2 = 2n-1$. Since the edge weight $2n-1$ is the sum of two labels, so at least one label is at least $\lceil (2n-1)/2 \rceil = n$. Therefore $es(T_n\langle 1,2\rangle) \geq n$. To prove the equality, it suffices to prove the existence of an optimal edge irregular n-labeling.

Let $\phi_1 : V(T_n\langle 1,2\rangle) \to \{1,2,\dots,n\}$ be the vertex labeling such that

$$\phi_1(v_i) = i, \quad \text{for} \quad 1 \leq i \leq n.$$

Since $w_{\phi_1}(v_iv_{i+1}) = \phi_1(v_i) + \phi_1(v_{i+1}) = 2i+1$, for $1 \leq i \leq n-1$ and $w_{\phi_1}(v_iv_{i+2}) = \phi_1(v_i) + \phi_1(v_{i+2}) = 2i+2$, for $1 \leq i \leq n-2$, so the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling ϕ_1 is an optimal edge irregular n-labeling. This completes the proof. □

Next theorem proves that the lower bound in Theorem 1.1 is tight.

Theorem 3.2. *Let $T_n\langle 1,3\rangle$ be a Toeplitz graph on $n \geq 4$ vertices. Then $es(T_n\langle 1,3\rangle) = n-1$.*

Proof. Let $T_n\langle 1,3\rangle$ be a Toeplitz graph with the vertex set $V(T_n\langle 1,3\rangle) = \{v_i : 1 \leq i \leq n\}$ and the edge set $E(T_n\langle 1,3\rangle) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+3} : 1 \leq i \leq n-3\}$. According to Theorem 1.1 we have that $es(T_n\langle 1,3\rangle) \geq \lceil (2n+1-4)/2 \rceil = n-1$. For the converse, we define a suitable edge irregular labeling $\phi_2 : V(T_n\langle 1,3\rangle) \to \{1,2,\ldots,n-1\}$ in the following way

For $n \equiv 1 \pmod 4$

$$\phi_2(v_i) = \begin{cases} i, & \text{if } i \equiv 1 \pmod 4 \text{ and } 1 \leq i < n-1 \\ i-1, & \text{if } i \equiv 2,3 \pmod 4 \\ i-1, & \text{if } i \equiv 0 \pmod 4 \text{ and } 1 \leq i \leq n-2 \\ i, & \text{if } i = n-1 \\ i-2, & \text{if } i = n \end{cases}$$

For $n \equiv 0,2,3 \pmod 4$

$$\phi_2(v_i) = \begin{cases} i, & \text{if } i \equiv 1 \pmod 4 \\ i-1, & \text{if } i \equiv 0,2,3 \pmod 4 \end{cases}$$

The edge weights are as follows

If $n \equiv 1 \pmod 4$

$$w_{\phi_2}(v_iv_{i+1}) = \begin{cases} 2i, & \text{if } i \equiv 0,1 \pmod 4 \text{ and } 1 \leq i < n-2 \\ 2i-1, & \text{if } i \equiv 2,3 \pmod 4 \text{ and } 1 \leq i < n-2 \\ 2i, & \text{if } i = n-2 \\ 2i-1, & \text{if } i = n-1 \end{cases}$$

If $n \equiv 0,2,3 \pmod 4$

$$w_{\phi_2}(v_iv_{i+1}) = \begin{cases} 2i, & \text{if } i \equiv 0,1 \pmod 4 \\ 2i-1, & \text{if } i \equiv 2,3 \pmod 4 \end{cases}$$

If $n \equiv 1 \pmod 4$

$$w_{\phi_2}(v_iv_{i+3}) = \begin{cases} 2i+2, & \text{if } i \equiv 1,2 \pmod 4 \text{ and } i < n-4 \\ 2i+1, & \text{if } i \equiv 0,3 \pmod 4 \\ 2i+3, & \text{if } i = n-4 \\ 2i, & \text{if } i = n-3 \end{cases}$$

If $n \equiv 0,2,3 \pmod 4$

$$w_{\phi_2}(v_iv_{i+3}) = \begin{cases} 2i+2, & \text{if } i \equiv 1,2 \pmod 4 \\ 2i+1, & \text{if } i \equiv 0,3 \pmod 4. \end{cases}$$

Since, the edge weights are distinct for all pairs of distinct edges, the vertex labeling ϕ_2 is a suitable edge irregular $(n-1)$-labeling. Hence, we have $es(T_n\langle 1,3\rangle) = n-1$. □

Next theorem gives the exact value of the edge irregularity strength for $T_n\langle 2,4\rangle$ and show that this value is bigger than the lower bound in Theorem 1.1.

Theorem 3.3. *Let $T_n\langle 2,4\rangle$, $n \geq 5$, be a Toeplitz graph. Then $es(T_n\langle 2,4\rangle) = n-1$.*

Proof. Let $T_n\langle 2,4\rangle$ be a Toeplitz graph with the vertex set $V(T_n\langle 2,4\rangle) = \{v_i : 1 \leq i \leq n\}$ and the edge set $E(T_n\langle 2,4\rangle) = \{v_iv_{i+2} : 1 \leq i \leq n-2\} \cup \{v_iv_{i+4} : 1 \leq i \leq n-4\}$. Let $\phi_3 : V(T_n\langle 2,4\rangle) \to \{1,2,\ldots,n-1\}$ be the vertex labeling such that

$$\phi_3(v_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd} \\ \left\lceil \frac{n+i-2}{2} \right\rceil, & \text{if } i \text{ is even.} \end{cases}$$

The edge weights are as follows:

$$w_{\phi_3}(v_iv_{i+2}) = \begin{cases} i+2, & \text{if } i \text{ is odd} \\ n+i-1, & \text{if } i \text{ and } n \text{ are even} \\ n+i, & \text{if } i \text{ is even and } n \text{ is odd} \end{cases}$$

$$w_{\phi_3}(v_iv_{i+4}) = \begin{cases} i+3, & \text{if } i \text{ is odd} \\ n+i, & \text{if } i \text{ and } n \text{ are even} \\ n+i+1, & \text{if } i \text{ is even and } n \text{ is odd.} \end{cases}$$

We can see that all vertex labels are at most $n-1$. The edge weights under the labeling ϕ_4 successively attain values $3,4,\ldots,n-1,n,n+2,n+3,\ldots,2n-3$ for n odd and $3,4,\ldots,n-2,n-1,n+1,n+2,\ldots,2n-3$ for n even. Thus the edge weights are distinct for all pairs of distinct edges and the labeling ϕ_3 provides the upper bound on $es(T_n\langle 2,4\rangle)$, i.e $es(T_n\langle 2,4\rangle) \leq n-1$.

Since every edge of $T_n\langle 2,4\rangle$ belongs to K_3, then under every edge irregular labeling the smallest possible edge weight is obtain as sum of the vertex labels 1 and 2. Then the largest edge weight has to be at least $|E(T_n\langle 2,4\rangle)|+2 = 2n-4$ and obtained as the sum of different vertex labels. Thus the largest edge weight is at least $2n-3$ and $es(T_n\langle 2,4\rangle) \geq \lceil (2n-3)/2 \rceil = n-1$. This provides the lower bound on $es(T_n\langle 2,4\rangle)$. Combining with previous upper bound, we get that $es(T_n\langle 2,4\rangle) = n-1$. □

The following theorem gives the exact value of the edge irregularity strength for Toeplitz graph $T_n\langle 1,2,3\rangle$ for $n \not\equiv 1 \pmod 4$.

Theorem 3.4. *Let $T_n\langle 1,2,3\rangle$, $n \geq 4$, be a Toeplitz graph. Then*

$$es(T_n\langle 1,2,3\rangle) = \begin{cases} \frac{3n}{2} - 1, & \text{if } n \text{ is even} \\ \frac{3n-3}{2}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Let $V(T_n\langle 1,2,3\rangle) = \{v_i : 1 \leq i \leq n\}$ be the vertex set and $E(T_n\langle 1,2,3\rangle) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+2} : 1 \leq i \leq n-2\} \cup \{v_iv_{i+3} : 1 \leq i \leq n-3\}$ be the edge set of $T_n\langle 1,2,3\rangle$ with $|E(T_n\langle 1,2,3\rangle)| = 3n-6$. According to Theorem 1.1 we have $es(T_n\langle 1,2,3\rangle) \geq \max\{\lceil (3n-4)/2 \rceil, 6\} = \lceil (3n-4)/2 \rceil$. Since every four consecutive vertices in $T_n\langle 1,2,3\rangle$ form a complete graph K_4, therefore under every edge irregular labeling no couple of adjacent vertices can be assigned by the same label. This implies that the smallest edge weight 2 is not possible. So if the smallest edge weight is 3 then the largest edge weight is at least $3n-4$. Since each edge weight is a sum of two labels, at least one label is at least $\lceil (3n-4)/2 \rceil$. Thus for $n = 4t+3$, $t \geq 1$, we have

$$es(T_n\langle 1,2,3\rangle) \geq \left\lceil \frac{3n-4}{2} \right\rceil = \left\lceil 6k+2+\frac{1}{2} \right\rceil = \frac{3n-3}{2}. \tag{1}$$

For n even the edge weight $3n-4$ is the sum of two the same labels $3n/2-2$ assigned to the adjacent vertices. Since it is not possible, then one label from the sum $3n-4$ has to be at least $\lceil (3n-3)/2 \rceil$. Hence we have

$$es(T_n\langle 1,2,3\rangle) \geq \left\lceil \frac{3n-3}{2} \right\rceil = \left\lceil \frac{3n}{2} - 1 - \frac{1}{2} \right\rceil = \frac{3n}{2} - 1. \tag{2}$$

For the converse, we define the vertex labeling ϕ_4 as follows:

$$\phi_4(v_i) = \begin{cases} \frac{3i-2}{2}, & \text{if } i \text{ is even} \\ \frac{3i-1}{2}, & \text{if } i \equiv 1 \pmod 4 \\ \frac{3i-3}{2}, & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

Observe that under the vertex labeling ϕ_4 all vertex labels are at most $3n/2-1$ for n even and $(3n-3)/2$ for $n \equiv 3 \pmod 4$. For $n \equiv 0,3 \pmod 4$ the edge weights successively attain values $3, 4, \ldots, 3n-4$ and for $n \equiv 2 \pmod 4$ the edges receive the weights $3, 4, \ldots, 3n-6, 3n-5, 3n-3$. It means that the edge weights are distinct for all pairs of distinct edges and the labeling ϕ_4 is a suitable edge irregular $(3n/2-1)$-labeling, respectively $((3n-3)/2)$-labeling. Thus the labeling ϕ_4 provides the upper bound on $es(T_n\langle 1,2,3\rangle)$. Combining with the lower bounds given by (1) and (2), produces the desired result. $\square$

The following theorem gives the upper bound of the edge irregularity strength for Toeplitz graph $T_n\langle 1,2,3\rangle$ for $n \equiv 1 \pmod 4$.

Theorem 3.5. *Let $T_n\langle 1,2,3\rangle$, be a Toeplitz graph for $n \equiv 1 \pmod 4$, $n \geq 5$. Then*

$$es(T_n\langle 1,2,3\rangle) \leq \frac{3n-1}{2}.$$

Proof. In view that $(3n-1)/2$ is an upper bound on the edge irregularity strength of graph $T_n\langle 1,2,3\rangle$ it suffices to prove the existence of a vertex labeling $\phi_5 : V(T_n) \to \{1,2\ldots,(3n-1)/2\}$ with edge irregular properties. Define the vertex labels as follows:

$$\phi_5(v_i) = \phi_4(v_i) \quad \text{for} \quad v_i \in V(T_n\langle 1,2,3\rangle).$$

It is a routine matter to verify that all vertex labels are at most $(3n-1)/2$ and the edge weights form the set of different integers, namely $\{3,4,\ldots,3n-6,3n-5,3n-3\}$. Thus the labeling ϕ_5 is desired edge irregular $((3n-1)/2)$-labeling. $\square$

4. **Conclusion**

In this paper we delt the existence of the edge irregularity strength for Toeplitz graphs. We determined the exact values of the edge irregularity strength of Toeplitz graphs $T_n\langle 1,2\rangle$, $T_n\langle 1,3\rangle$ and $T_n\langle 2,4\rangle$, namely we proved that $es(T_n\langle 1,2\rangle) = n$, $es(T_n\langle 1,3\rangle) = es(T_n\langle 2,4\rangle) = n-1$. Moreover we proved that $es(T_n\langle 1,2,3\rangle) = 3n/2-1$ for n even and $es(T_n\langle 1,2,3\rangle) = (3n-3)/2$ for $n \equiv 3 \pmod 4$. For $n \equiv 1 \pmod 4$ we showed that $es(T_n\langle 1,2,3\rangle) \leq (3n-1)/2$. We believe that this upper bound is the exact value therefore we propose the following conjecture.

Conjecture 1. *Let $n \equiv 1 \pmod 4$, $n \geq 5$. Then*

$$es(T_n\langle 1,2,3\rangle) = (3n-1)/2.$$

Acknowledgement. The research for this article was supported by Jazan University and by APVV-15-0116.

REFERENCES

[1] *A. Ahmad, O. Al-Mushayt and M. Bača*, On edge irregularity strength of graphs, Appl. Math. Comput. **243**(2014) 607-610.

[2] *A. Ahmad, M. Bača, Y. Bashir and M.K. Siddiqui*, Total edge irregularity strength of strong product of two paths, Ars Combin. **106**(2012) 449-459.

[3] *O. Al-Mushayt, A. Ahmad and M.K. Siddiqui*, On the total edge irregularity strength of hexagonal grid graphs, Australas. J. Combin. **53**(2012) 263-271.

[4] *O. Al-Mushayt*, On edge irregularity strength of products of certain families of graphs with path P_2, Ars Combin. to appear.

[5] *M. Aigner and E. Triesch*, Irregular assignments of trees and forests, SIAM J. Discrete Math. **3**(1990) 439-449.

[6] *D. Amar and O. Togni*, Irregularity strength of trees, Discrete Math. **190**(1998) 15-38.

[7] *M. Bača, S. Jendroľ, M. Miller and J. Ryan*, On irregular total labellings, Discrete Math. **307**(2007) 1378-1388.

[8] *M. Bača, Y. Bashir, M. F. Nadeem, A. Shabbir*, On super edge-antimagic total labeling of Toeplitz graphs, Mathematics in the 21st Century Springer Proceedings in Mathematics & Statistics, **98**(2015) 1-10.

[9] *T. Bohman and D. Kravitz*, On the irregularity strength of trees, J. Graph Theory **45**(2004) 241-254.

[10] *G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz and F. Saba*, Irregular networks, Congr. Numer. **64**(1988) 187-192.

[11] *R. van Dal, G. Tijssen, Z. Tuza, J. van der Veen, C.H. Zamfirescu, and T. Zamfirescu*, Hamiltonian properties of Toeplitz graphs, Discrete Math. **159**(1996) 69-81.

[12] *R. Euler, H. LeVerge and T. Zamfirescu*, A characterization of infinite, bipartite Toeplitz graphs, In: Ku Tung-Hsin (Ed.), Combinatorics and Graph Theory 95, vol. 1, Academia Sinica, World Scientific, Singapore, 1995, 119-130.

[13] *R. Euler*, Characterizing bipartite Toeplitz graphs, Theoretical Computer Science **263**(2001) 47-58.

[14] *R. Euler*, Coloring planar Toeplitz graphs and the stable set polytope, Discrete Math. **276**(2004) 183-200.

[15] *R. Euler and T. Zamfirescu*, On planar Toeplitz graphs, Graphs and Combinatorics **29**(2013) No. 5 1311-1327 (`DOI: 10.1007/s00373-012-1185-8`).

[16] *A. Frieze, R.J. Gould, M. Karonski, and F. Pfender*, On graph irregularity strength, J. Graph Theory **41**(2002) 120-137.

[17] *J.A. Gallian*, A Dynamic Survey Graph labeling, Electron. J. Combin. **18**(2011), # DS6.

[18] *C. Heuberger*, On hamiltonian Toeplitz graphs, Discrete Math. **245**(2002) 107-125.

[19] *J. Ivančo and S. Jendroľ*, Total edge irregularity strength of trees, Discussiones Math. Graph Theory **26**(2006) 449-456.

[20] *S. Jendroľ, J. Miškuf and R. Soták*, Total edge irregularity strength of complete graphs and complete bipartite graphs, Discrete Math. **310**(2010) 400-407.

[21] *S. Malik and A.M. Qureshi*, Hamiltonian cycles in directed Toeplitz graphs, Ars Combin. **109**(2013) 511-526.

[22] *S. Malik and T. Zamfirescu*, Hamiltonian connectedness in directed Toeplitz graphs, Bull. Math. Soc. Sci. Math. Roum. **53**(2010) No.2 145-156.

[23] *C.C. Marzuki, A.N.M. Salman and M. Miller*, On the total irregularity strength on cycles and paths, Far East J. Math. Sci. **82**(2013) No.1 121.

[24] K.M. Mominul Haque, Irregular total labellings of generalized Petersen graphs, Theory Comput. Syst. **50**(2012) 537-544.

[25] *S. Nicoloso and U. Pietropaoli*, On the chromatic number of Toeplitz graphs, Discrete Appl. Math. (`DOI:10.1016/j.dam.2011.07.012 (2012)`).

[26] *Nurdin, E.T. Baskoro, A.N.M. Salman and N.N. Gaos*, On the total vertex irregularity strength of trees, Discrete Math. **310**(2010) 3043-3048.

[27] *J. Przybylo*, Linear bound on the irregularity strength and the total vertex irregularity strength of graphs, SIAM J. Discrete Math. **23**(2009), 511-516.