BEST PROXIMITY POINT UNDER THE FRAME OF QUASI-PARTIAL METRIC SPACES

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The best proximity point plays an important role in applied sciences. In this paper, we introduce two types of contractions based on the notion of $P-$property in the notion of quasi-partial metric spaces. We use our new contractions to build and prove some new theorems of proximity type. Our results modify many existing known results. We close our paper by introducing an example to support our results.

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1. Introduction

The notion of quasi-partial metric spaces was introduced by Karapiner et al [1] as a generalization of partial metric spaces [2] in the sense that the commutative property between x and y coordinates is not allowed. Moreover, the study and construct fixed and common fixed points in frame of metric spaces, quasi metric spaces and quasi-partial metric spaces are very useful for the Scientists in many different branches such as Physics, Chemistry, and Engineering. For some works on metric and partial metric spaces, see [3]-[15]. Also, for some works on quasi-metric space and quasi-partial metric spaces, see [16]-[27].

The definition of partial metric spaces is given as follows:

Definition 1.1. [1] *The function $\rho : M \times M \to [0, \infty)$ is called a quasi-partial metric if ρ satisfies the following hypotheses:*

(1) *If $\rho(c, c) = \rho(c, d) = \rho(d, d)$, then $c = d$,*
(2) $\rho(c, c) \leq \rho(c, d)$,
(2) $\rho(c, c) \leq \rho(d, c)$, *and*
(2) $\rho(c, e) + \rho(d, d) \leq \rho(c, d) + \rho(d, e)$

for all $c, d, e \in X$. The pair (M, ρ) is said to be a quasi partial-metric space.

Definition 1.2. [1] *A sequence (a_n) in a quasi-partial metric space (M, ρ) converges to a point $a^* \in M$ if*

$$\lim_{n\to+\infty} \rho(a_n, a^*) = \lim_{n\to+\infty} \rho(a^*, a_n) = \rho(a^*, a^*).$$

Definition 1.3. [1] *On the quasi-partial metric space (M, ρ), a sequence is called Cauchy if $\lim_{n,m\to+\infty} \rho(a_n, a_m)$ and $\lim_{n,m\to+\infty} \rho(a_m, a_n)$ exist as a finite number.*

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Definition 1.4. [1] *The quasi-partial metric space (M, ρ) is called complete if every Cauchy sequence (a_n) in M converges to a point $a^* \in M$ such that*

$$\lim_{n,m \to +\infty} \rho(a_n, a_m) = \lim_{n \to +\infty} \rho(a_m, a_n) = \rho(a^*, a^*).$$

Moreover, Karapiner in [1] proved that the following assertions hold in quasi-partial metric space:

[1] If $\rho(c, d) = 0$, then $c = d$.

[2] If $c \neq d$, then $\rho(c, d) > 0$ and $\rho(d, c) > 0$.

The notion of best proximity point and the $P-$property in the metric space (M, d) were introduced by Samet et al [28].

Definition 1.5. [28] *Let M_1 and M_2 be subsets of a metric space (M, d). An element $m^* \in M_1$ is said to be a best proximity point of the mapping $h : M_1 \to M_2$ if*

$$d(m^*, hm^*) = d(M_1, M_2),$$

where

$$d(M_1, M_2) = \inf \Big\{ d(s, r) : s \in M_1 \text{ and } r \in M_2 \Big\}.$$

Definition 1.6. [28] *Let (M_1, M_2) be subsets of a metric space (M, d). Also, let*

$$M_1^0 := \Big\{ m_1 \in M_1 \ : \ \text{there exists } m_2 \in M_2 \text{ such that } d(m_1, m_2) = d(M_1, M_2) \Big\}, \text{ and}$$

$$M_2^0 := \Big\{ m_2 \in M_2 \ : \ \text{there exists } m_1 \in M_1 \text{ such that } d(m_1, m_2) = d(M_1, M_2) \Big\}.$$

Then, we say that (M_1, M_2) have the P-property if

$$\begin{pmatrix} d(m_1, m_2) = d(M_1, M_2) \\ d(k_1, k_2) = d(M_1, M_2) \end{pmatrix} \Rightarrow \quad d(m_1, k_1) = d(m_1, k_2).$$

For some works in fixed point theory, we ask the reader to see [29]-[36]. Also, for related works on P-property, see [36], [37].

2. Main Results

Let M be a nonempty set endowed with a quasi-partial metric space ρ. Also, let M_1 and M_2 be two nonempty subsets of M. From now on, we let

$$\begin{aligned} M_1^0 \quad := \quad & \Big\{ m_1 \in M_1 \ : \ \text{there exists } m_2 \in M_2 \\ & \text{such that } \rho(m_1, m_2) = \rho(M_1, M_2), \rho(m_2, m_1) = \rho(M_2, M_1) \Big\} \end{aligned}$$

and

$$\begin{aligned} M_2^0 \quad := \quad & \Big\{ m_2 \in M_2 \ : \ \text{there exists } m_1 \in M_1 \\ & \text{such that } \rho(m_1, m_2) = \rho(M_1, M_2), \rho(m_2, m_1) = \rho(M_2, M_1) \Big\}, \end{aligned}$$

where

$$\rho(M_1, M_2) := \inf\{\rho(m_1, m_2) : m_1 \in M_1, m_2 \in M_2\}.$$

Definition 2.1. *Let M_1 and M_2 be non-empty subsets of a quasi-partial metric space (M,ρ). We say that the pair (M_1,M_2) has the P-property if*

$$\left(\begin{array}{c}\rho(m_1,m_2)=\rho(M_1,M_2)\\ \rho(k_1,k_2)=\rho(M_1,M_2)\end{array}\right)\Rightarrow\quad \rho(m_1,k_1)=\rho(m_2,k_2).$$

By $\Sigma-$function, we mean a function $\sigma:[0,+\infty)^4\to[0,+\infty)$ posses the following conditions:

(1) σ is continuous in all of its variables.
(2) $\sigma(0,j,k,l)=0$ for all $j,k,l\in[0,+\infty)$,
(3) $\sigma(i,0,k,l)=0$ for all $i,k,l\in[0,+\infty)$,
(4) $\sigma(i,j,0,l)=0$ for all $i,j,l\in[0,+\infty)$, and
(5) $\sigma(i,j,k,0)=0$ for all $i,j,k\in[0,+\infty)$.

Example 2.1. *Define*

$$\sigma_1,\sigma_2,\sigma_3\colon[0,+\infty)^4\to[0,+\infty)$$

by

$$\sigma_1(i,j,k,l)=s\inf\{i,j,k,l\},s>0$$
$$\sigma_2(i,j,k,l)=e^{(sijkl}-1,s>0,$$

and

$$\sigma_3(i,j,k,l)=sijkl,\ s>0.$$

Then σ_1,σ_2, σ_3 are $\Sigma-$functions.

By a $c-$comparison function τ on $[0,+\infty)$, we mean a function posses the following assertions:

(1) τ is continuous and nondecreasing.
(2) For $s>0$, $\tau^n(s)\to+\infty$ as $n\to+\infty$.

In the rest of the present paper τ stands to a $c-$comparison function and σ stands to a $\Sigma-$function.

Definition 2.2. *Let ρ be a quasi-partial metric on a set M. Assume M_1 and M_2 be subsets of M. We call $h:M_1\to M_2$ is $(\rho,\tau,\sigma,M_1,M_2)-$contraction of type I if for all $r,s\in M_1$, we have*

$$\begin{aligned}\rho(hr,hs)\quad\leq\quad&\tau(\rho(r,s))+\sigma\Bigg(\rho(s,hr)-\rho(M_1,M_2),\\ &\rho(r,hs)-\rho(M_1,M_2),\rho(r,hr)-\rho(M_1,M_2),\rho(s,hs)-\rho(M_1,M_2)\Bigg).\end{aligned}$$

Our first result is:

Theorem 2.1. *Let (M,ρ) be a complete quasi-partial metric space. Suppose M_1 and M_2 be closed subsets of M with respect to ρ. Assume $h\colon M_1\to M_2$ be a mapping satisfies the following conditions:*

1) h is $(\rho,\tau,\sigma,M_1,M_2)-$contraction,
2) M_1^0 is nonempty.
3) $hM_1^0\subseteq M_2^0$, and
4) (M_1,M_2) posses the P-property,
Then h has a unique best proximity $m^*\in M_1$.

Proof. Choosing $u_0\in M_1^0$. Condition (3) implies that there is $u_1\in M_1^0$ such that $\rho(u_1,hu_0)=\rho(M_1,M_2)$ and $\rho(hu_0,u_1)=\rho(M_1,M_2)$. So we can construct a sequence $(u_n)\subseteq M_1^0$ such that

$$\rho(u_{n+1},hu_n)=\rho(M_1,M_2),\quad\forall n\in\mathbb{N}\cup\{0\}$$

and

$$\rho(hu_n, u_{n+1}) = \rho(M_2, M_1), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

For any $m, n \in \mathbb{N} \cup \{0\}$, the $P-$property of (M_1, M_2) ensures that $\rho(u_n, u_m) = \rho(hu_{n-1}, hu_{m-1})$ and $\rho(u_m, u_n) = \rho(hu_{m-1}, hu_{n-1})$.

If there is $i \in \mathbb{N} \cup \{0\}$, for which $\rho(u_i, u_{i+1}) = 0$, it follows $\rho(u_{i+1}, u_i) = 0$ and hence

$$\rho(M_1, M_2) \leq \rho(u_i, hu_i) \leq \rho(u_i, u_{i+1}) + \rho(u_{i+1}, hu_i) = \rho(u_{i+1}, hu_i) = \rho(M_1, M_2)$$

and

$$\rho(M_2, M_1) \leq \rho(hu_i, u_i) \leq \rho(hu_i, u_{i+1}) + \rho(u_{i+1}, u_i) = \rho(hu_i, u_{i+1}) = \rho(M_2, M_1),$$

hence $\rho(M_1, M_2) = \rho(u_i, hu_i)$ and $\rho(M_2, M_1) = \rho(hu_i, u_i)$. Therefore u_i is a best proximity point of h.

By assuming that $\rho(u_n, u_{n+1}) > 0$, $\forall\, n \geq 0$, we may deduce that $\rho(u_{n+1}, u_n) > 0$, $\forall\, n \geq 0$.

Using the fact that h is $(\rho, \tau, \sigma, M_1, M_2)-$contraction, we have

$$\begin{aligned}
&\rho(u_n, u_{n+1}) = \rho(hu_{n-1}, hu_n)\\
\leq\ & \tau(\rho(u_{n-1}, u_n)) + \sigma\Big(\rho(u_n, hu_{n-1}) - \rho(M_1, M_2), \rho(u_{n-1}, hu_n) - \rho(M_1, M_2),\\
&\qquad \rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_n, hu_n) - \rho(M_1, M_2)\Big)\\
=\ & \tau(\rho(u_{n-1}, u_n)) + \sigma\Big(0, \rho(u_{n-1}, hu_n) - \rho(M_1, M_2),\\
&\qquad \rho(u_{n-1}, hu_{n-1}) - \rho(U, V), \rho(u_n, hu_n) - \rho(U, V)\Big)\\
=\ & \tau(\rho(u_{n-1}, u_n)), \quad n \in \mathbb{N} \cup \{0\}
\end{aligned}$$

and

$$\begin{aligned}
&\rho(u_{n+1}, u_n) = \rho(hu_n, hu_{n-1})\\
\leq\ & \tau(\rho(u_n, u_{n-1})) + \sigma\Big(\rho(u_{n-1}, hu_n) - \rho(M_1, M_2), \rho(u_n, hu_{n-1}) - \rho(M_1, M_2),\\
&\qquad \rho(u_n, hu_n) - \rho(M_1, M_2), \rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2)\Big)\\
=\ & \tau(\rho(u_n, u_{n-1})) + \sigma\Big(\rho(u_n, hu_{n-}) - \rho(M_1, M_2), 0,\\
&\qquad \rho(u_n, hu_n) - \rho(M_1, M_2), \rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2)\Big)\\
=\ & \tau(\rho(u_n, u_{n-1})), \quad n \in \mathbb{N} \cup \{0\}.
\end{aligned}$$

By refining process, we obtain

$$\rho(u_n, u_{n+1}) \leq \tau^n(\rho(u_0, u_1)), \quad n \in \mathbb{N} \cup \{0\}$$

and

$$\rho(u_{n+1}, u_n) \leq \tau^n(\rho(u_1, u_0)), \quad n \in \mathbb{N} \cup \{0\}.$$

On the other hand, triangular inequality and condition (4) give us

$$\begin{aligned}
\rho(M_1, M_2) &\leq \rho(u_n, hu_n)\\
&\leq \rho(u_n, hu_{n-1}) + \rho(hu_{n-1}, hu_n) - \rho(hu_{n-1}, hu_{n-1})\\
&= \rho(M_1, M_2) + \rho(u_n, u_{n+1})
\end{aligned}$$

and

$$\begin{aligned}\rho(M_1, M_2) &\leq \rho(hu_n, u_n)\\ &\leq \rho(hu_n, hu_{n-1}) + \rho(hu_{n-1}, u_n) - \rho(hu_{n-1}, hu_{n-1})\\ &= \rho(u_{n+1}, u_n) + \rho(M_1, M_2).\end{aligned}$$

Allowing n tends to infinity, we deduce that

$$\lim_{n\to+\infty} \rho(u_n, hu_n) = \rho(M_1, M_2) \qquad (\mathbf{2}.1)$$

and

$$\lim_{n\to+\infty} \rho(hu_n, u_n) = \rho(M_1, M_2) \qquad (\mathbf{2}.2)$$

Now, given $\varepsilon > 0$, we adopt the induction on t to show that

$$\rho(u_n, u_t) < \varepsilon, \qquad \forall\ t, n > m_0 \qquad \text{for some } m_0 \in \mathbb{N} \cup \{0\}. \qquad (\mathbf{2}.3)$$

Without lose of generality, we shall prove that

$$\rho(u_n, u_t) < \varepsilon, \qquad \forall\ t > n > m_0 \qquad \text{for some } m_0 \in \mathbb{N} \cup \{0\}. \qquad (\mathbf{2}.4)$$

If $t = n + 1$, since $\frac{1}{2}(\varepsilon - \sigma(\varepsilon)) > 0$ and $\rho(u_n, u_{n+1}) \to 0$, we choose m_0 such that

$$\rho(u_n, u_{n+1}) < \frac{1}{2}(\varepsilon - \tau(\varepsilon)) \qquad \forall\ n \geq m_0. \qquad (\mathbf{2}.5)$$

Since $\frac{1}{2}(\varepsilon - \tau(\varepsilon)) < \varepsilon$, we deduce that $(\mathbf{2}.3)$ is true for $t = n + 1$.

Suppose $(\mathbf{2}.3)$ holds for $t = k$.
Now, we shall prove $(\mathbf{2}.3)$ holds for $t = k + 1$.
In view of definition of ρ and the $P-$property of (M_1, M_2), we see that

$$\begin{aligned}&\rho(u_n, u_{k+1}) \leq \rho(u_n, u_{n+1}) + \rho(u_{n+1}, u_{k+1})\\ &= \rho(u_n, u_{n+1}) + \rho(hu_n, hu_k)\\ \leq\ & \rho(u_n, u_{n+1}) + \tau(\rho(u_n, u_k)) + \sigma\Big(\rho(u_k, hu_n) - \rho(M_1, M_2), \rho(u_n, hu_k) - \rho(M_1, M_2),\\ &\qquad \rho(u_n, hu_n) - \rho(M_1, M_2), \rho(u_k, hu_k) - \rho(M_1, M_2))\end{aligned} \qquad (\mathbf{2}.6)$$

Using $(\mathbf{2}.1)$ and the continuity of θ, we get

$$\begin{aligned}&\liminf_{n\to+\infty} \sigma(\rho(u_k, hu_n) - \rho(U, V), \rho(u_n, hu_k) - \rho(M_1, M_2),\\ &\qquad \rho(u_n, hu_n) - \rho(U, V), \rho(u_k, hu_k) - \rho(M_1, M_2)) = 0.\end{aligned}$$

So, we can choose n_0 to be large enough such that for each $n > n_0$,

$$\begin{aligned}&\sigma(\rho(u_k, hu_n) - \rho(M_1, M_2), \rho(u_n, hu_k) - \rho(M_1, M_2),\\ &\rho(u_n, hu_n) - \rho(M_1, M_2), \rho(u_k, hu_k) - \rho(M_1, M_2)) < \frac{1}{2}(\varepsilon - \sigma(\varepsilon))\end{aligned} \qquad (\mathbf{2}.7)$$

By employing inequalities $(\mathbf{2}.5)$, and $(\mathbf{2}.7)$ in $(\mathbf{2}.6)$, we deduce

$$\rho(u_n, u_{k+1}) \leq \frac{1}{2}(\varepsilon - \tau(\varepsilon)) + \tau(\varepsilon) + \frac{1}{2}(\varepsilon - \tau(\varepsilon)).$$

Thus $\rho(u_n, u_{k+1}) < \varepsilon$, which implies that $\rho(u_n, u_t) < \varepsilon$, for all $t > n > n_0$. Imitate the above arguments, we deduce that $\rho(u_t, u_n) < \varepsilon$, for all $t > n > n_0$. Thus, we have

$$\lim_{n,m\to+\infty} \rho(u_m, u_n) = 0.$$

Therefore, we conclude that (u_n) is Cauchy in M_1. The closeness property of M_1 in the complete space (M,ρ) implies that $u^* \in U$ such that $u_n \to u^*$ as $n \to +\infty$ in (M,ρ); that is,

$$\begin{aligned}\lim_{n,m\to+\infty} \rho(u_n,u_m) &= \lim_{n,m\to+\infty} \rho(u_m,u_n)\\ &= \rho(u^*,u^*) = \lim_{n\to+\infty} \rho(u_n,u^*)\\ &= \lim_{n\to+\infty} \rho(u^*,u_n).\end{aligned}$$

Now, allowing $n \to +\infty$ in:

$$\begin{aligned}\rho(hu^*,hu_n) \quad &\leq \quad \tau(\rho(u^*,u_n)) + \sigma(\rho(u_n,hu^*) - \rho(M_1,M_2), \rho(u^*,hu_n) - \rho(M_1,M_2),\\ &\qquad \rho(u_n,hu_n) - \rho(M_1,M_2), \rho(u^*,hu^*) - \rho(M_1,M_2))\end{aligned}$$

and

$$\begin{aligned}\rho(hu_n,hu^*) \quad &\leq \quad \tau(\rho(u_n,u^*)) + \sigma(\rho(u^*,hu_n) - \rho(M_1,M_2), \rho(u_n,hu^*) - \rho(M_1,M_2),\\ &\qquad \rho(u^*,hu^*) - \rho(M_1,M_2), \rho(u_n,hu_n) - d(M_1,M_2)),\end{aligned}$$

we reach to

$$\lim_{n\to+\infty} \rho(hu^*,hu_n) = \lim_{n\to+\infty} \rho(hu_n,hu^*) = 0.$$

Applying triangle inequality of the definition of ρ, to get

$$\rho(u^*,hu^*) \leq \rho(u^*,u_n) + \rho(u_n,hu_n) + \rho(hu_n,hu^*) - \rho(u_n,u_n) - \rho(hu_n,hu_n)$$

and

$$\rho(hu^*,u^*) \leq \rho(hu^*,hu_n) + \rho(hu_n,u_n) + \rho(u_n,u^*) - \rho(hu_n,hu_n) - \rho(u_n,u_n).$$

On letting $n \to +\infty$ in above inequalities, we get $\rho(u^*,hu^*) \leq \rho(M_1,M_2)$ and $\rho(hu^*,u^*) \leq \rho(M_2,M_1)$. Thus $\rho(u^*,hu^*) = \rho(M_1,M_2)$ and $\rho(hu^*,u^*) = \rho(M_2,M_1)$. So u^* is best proximity point of h.

Now, suppose there is $a^* \in M_1$ such that $\rho(a^*,Ta^*) = \rho(M_1,M_2)$ and $\rho(Ta^*,a^*) = \rho(M_2,M_1)$. Then by $P-$property of (M_1,M_2), we get $\rho(a^*,u^*) = \rho(ha^*,hu^*)$. Thus,

$$\begin{aligned}\rho(a^*,u^*) \quad &= \quad \rho(ha^*,hu^*)\\ &\leq \quad \tau(\rho(a^*,u^*)) + \sigma(\rho(u^*,ha^*) - \rho(M_1,M_2), \rho(a^*,hu^*) - \rho(M_1,M_2),\\ &\qquad \rho(a^*,ha^*) - \rho(M_1,M_2), \rho(u^*,hu^*) - \rho(M_1,M_2))\\ &= \quad \tau(\rho(a^*,u^*)) + \sigma(\rho(u^*,ha^*) - \rho(M_1,M_2), \rho(a^*,hu^*) - \rho(M_1,M_2), 0, 0)\\ &\leq \quad \tau(\rho(a^*,y^*)),\end{aligned}$$

The last inequality holds only if $a^* = u^*$. So we conclude that the best proximity point of h is unique. □

Definition 2.3. *Let ρ be a quasi-partial metric on a set M. Assume M_1 and M_2 be subsets of M. We call $h : M_1 \to M_2$ is $(\rho,\tau,\sigma,M_1,M_2)-$contraction of type II if for all $r,s \in M_1$, we have*

$$\begin{aligned}\rho(hr,hs) \quad &\leq \quad \tau\Bigg(\max\Big\{\rho(r,hr) - \rho(M_1,M_2), \rho(s,hs) - \rho(M_1,M_2)\Big\}\Bigg)\\ &+ \quad \sigma\Bigg(\rho(s,hr) - \rho(M_1,M_2), \rho(r,hs) - \rho(M_1,M_2), \rho(r,hr) - \rho(M_1,M_2),\\ &\qquad \rho(s,hs) - \rho(M_1,M_2)\Bigg).\end{aligned}$$

Our second result is:

Theorem 2.2. *Let (M, ρ) be a complete quasi-partial metric space. Suppose M_1 and M_2 be closed subsets of M with respect to ρ. Assume $h\colon M_1 \to M_2$ be a mapping satisfies the following conditions:*

1) h is $(\rho, \tau, \sigma, M_1, M_2)-$contraction of type II,

2) M_1^0 is nonempty,

3) $hM_1^0 \subseteq M_2^0$, and

4) (M_1, M_2) posses the P-property.

Then h has a unique best proximity $m^* \in M_1$.

Proof. As in the proof of Theorem **2**.1, we generate a sequence (u_n) in M_1^0 such that

$$\rho(u_{n+1}, hu_n) = \rho(M_1, M_2) \qquad \forall\, n \in \mathbb{N},$$
$$\rho(hu_n, u_{n+1}) = \rho(M_2, M_1) \qquad \forall\, n \in \mathbb{N},$$

and

$$\rho(u_n, u_m) = \rho(hu_{n-1}, hu_{m-1}) \qquad \forall\, n, m \in \mathbb{N}.$$

If there is $t \in \mathbb{N}$ such that $\rho(u_t, u_{t+1}) = 0$, then by triangular inequality we deduce that $\rho(u_t, hu_t) = \rho(M_1, M_2)$ and $\rho(hu_t, u_t) = \rho(M_2, M_1)$. Thus, we may assume that $\rho(u_i, u_{i+1}) \neq 0$ for all $i \in \mathbb{N}$. So $\rho(u_{i+1}, u_i) \neq 0$

For $n \in \mathbb{N}$, Condition (1) tell us

$$\begin{aligned}
&\rho(u_n, u_{n+1}) = \rho(hu_{n-1}, hu_n)\\
&\leq \tau\Bigg(\max\Big\{\rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_n, hu_n) - \rho(M_1, M_2)\Big\}\Bigg)\\
&+ \sigma\Bigg(\rho(u_n, hu_{n-1}) - \rho(M_1, M_2), \rho(u_{n-1}, hu_n) - \rho(M_1, M_2), \rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2),\\
&\qquad \rho(u_n, hu_n) - \rho(M_1, M_2)\Bigg)\\
&\leq \tau\Bigg(\max\Big\{\rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_n, hu_n) - \rho(M_1, M_2)\Big\}\Bigg).
\end{aligned} \tag{2.8}$$

Now, suppose

$$\max\Big\{\rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_n, hu_n) - \rho(M_1, M_2)\Big\} = \rho(u_n, hu_n) - \rho(M_1, M_2).$$

From triangular inequality, we have

$$\begin{aligned}
&\rho(u_n, hu_n) - \rho(M_1, M_2)\\
&\leq \rho(u_n, u_{n+1}) + \rho(u_{n+1}, hu_n) - \rho(u_{n+1}, u_{n+1}) - \rho(M_1, M_2)\\
&\leq \rho(u_n, u_{n+1}).
\end{aligned}$$

From (**2**.8), we have $\rho(u_n, u_{n+1}) \leq \rho(u_n, u_{n+1}) - \rho(M_1, M_2)$, which is a contradiction. Thus

$$\max\Big\{\rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_n, hu_n) - \rho(M_1, M_2)\Big\} = \rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2).$$

Again,by applying triangle inequality, we get

$$\rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2) \leq \rho(u_{n-1}, u_n).$$

Inequality (**2**.8) leads us to

$$\rho(u_n, u_{n+1}) \leq \tau(\rho(u_{n-1}, u_n)), \tag{2.9}$$

and hence

$$\Big\{\rho(u_n, u_{n+1}) : \; n \in \mathbb{N} \cup \{0\}\Big\} \tag{2.10}$$

is a decreasing sequence of the set of real numbers. Repeating (**2**.9) n times, we produce

$$\rho(u_n, u_{n+1}) \leq \tau^n(\rho(u_0, u_1)). \qquad (\mathbf{2}.11)$$

Using the same arguments as above, we have

$$\rho(u_{n+1}, u_n) \leq \tau^n(\rho(u_1, u_0)) \qquad (\mathbf{2}.12)$$

and

$$\Big\{\rho(u_{n+1}, u_n) : \ n \in \mathbb{N} \cup \{0\}\Big\} \qquad (\mathbf{2}.13)$$

is a decreasing sequence of the set of real numbers.

Inequalities (**2**.11) and (**2**.12) imply that

$$\lim_{n\to+\infty} \rho(u_n, u_{n+1}) = \lim_{n\to+\infty} \rho(u_{n+1}, u_n) = 0.$$

For $t, n \in \mathbb{N}$ with $t > m$, we have

$$\begin{aligned}
&\rho(u_n, u_t) = \rho(hu_{n-1}, hu_{t-1}) \\
&\leq \tau\Bigg(\max\Big\{\rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_{t-1}, hu_{t-1}) - \rho(M_1, M_2)\Big\}\Bigg) \\
+ \quad &\sigma\Bigg(\rho(u_{t-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_{n-1}, hu_{t-1}) - \rho(M_1, M_2), \rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2), \\
&\rho(u_{t-1}, hu_{t-1}) - \rho(M_1, M_2)\Bigg) \qquad (\mathbf{2}.14)
\end{aligned}$$

and

$$\begin{aligned}
&\rho(u_t, u_n) = \rho(hu_{t-1}, hu_{n-1}) \\
&\leq \tau\Bigg(\max\Big\{\rho(u_{t-1}, hu_{t-1}) - \rho(M_1, M_2), \rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2)\Big\}\Bigg) \\
+ \quad &\sigma\Bigg(\rho(u_{n-1}, hu_{t-1}) - \rho(M_1, M_2), \rho(u_{t-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_{t-1}, hu_{t-1}) - \rho(M_1, M_2), \\
&\rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2)\Bigg). \qquad (\mathbf{2}.15)
\end{aligned}$$

For $s \in \mathbb{N} \cup \{0\}$, triangular inequality implies that

$$\rho(u_s, hu_s) - \rho(M_1, M_2) \leq \rho(u_s, u_{s+1}). \qquad (\mathbf{2}.16)$$

Employing (**2**.11) and (**2**.16) in (**2**.14) and (**2**.15), we get

$$\begin{aligned}
&\rho(u_n, u_t) \leq \tau(\rho(u_{n-1}, u_n)) \\
+ \quad &\sigma\Bigg(\rho(u_{t-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_{n-1}, hu_{t-1}) - \rho(M_1, M_2), \rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2), \\
&\rho(u_{t-1}, hu_{t-1}) - \rho(M_1, M_2)\Bigg) \qquad (\mathbf{2}.17)
\end{aligned}$$

and

$$\begin{aligned}
&\rho(u_t, u_n) \leq \tau(\rho(u_{n-1}, u_n) \\
+ \quad &\sigma\Bigg(\rho(u_{n-1}, hu_{t-1}) - \rho(M_1, M_2), \rho(u_{t-1}, hu_{n-1}) - \rho(M_1, M_2), \rho(u_{t-1}, hu_{t-1}) - \rho(M_1, M_2), \\
&\rho(u_{n-1}, hu_{n-1}) - \rho(M_1, M_2)\Bigg). \qquad (\mathbf{2}.18)
\end{aligned}$$

On other hand

$$\begin{aligned}\rho(M_1, M_2) &\leq \rho(u_n, hu_n)\\ &\leq \rho(u_n, u_{n+1}) + \rho(u_{n+1}, hu_n) - \rho(u_{n+1}, u_{n+1})\\ &\leq \rho(u_n, u_{n+1}) + \rho(M_1, M_2).\end{aligned}$$

Letting $n \to +\infty$, we get

$$\lim_{n\to+\infty} \rho(u_n, hu_n) = \rho(M_1, M_2) \quad (\mathbf{2}.19)$$

By taking the limit on (**2**.17) and (**2**.18) and using the continuity of σ and (**2**.19), we obtain

$$\lim_{n,t\to+\infty} \rho(u_n, u_t) = 0.$$

Thus (u_n) is a Cauchy sequence in M_1. Since M is complete, $u_n \to a^*$ for some $a^* \in M$. Since M_1 is closed in M, then $a^* \in M_1$. Moreover, we have

$$\begin{aligned}\lim_{n,t\to+\infty} \rho(u_n, u_t) &= \lim_{n,t\to+\infty} \rho(u_t, u_n)\\ &= \lim_{n\to+\infty} \rho(u_n, a^*)\\ &= \lim_{n,t\to+\infty} \rho(a^*, u_n)\\ &= \lim_{n,t\to+\infty} \rho(a^*, a^*).\end{aligned}$$

Our attention right now is to show that

$$\rho(a^*, ha^*) = \rho(M_1, M_2) \text{ and } \rho(ha^*, a^*) = \rho(M_2, M_1).$$

Suppose on the contrary, that are, $\rho(a^*, ha^*) \neq \rho(M_1, M_2)$ and $\rho(ha^*, a^*) \neq \rho(M_2, M_1)$ Since h is $(\rho, \tau, \sigma, M_1, M_2)-$contraction of type II, we get

$$\begin{aligned}&\rho(hu_n, ha^*)\\ &\leq \tau\Big(\max\Big\{\rho(u_n, hu_n) - \rho(M_1, M_2), \rho(a^*, ha*) - \rho(M_1, M_2)\Big\}\Big)\\ &+ \sigma\Big(\rho(a^*, hu_n) - \rho(M_1, M_2), \rho(u_n, ha^*) - \rho(M_1, M_2), \rho(u_n, hu_n) - \rho(M_1, M_2),\\ &\qquad \rho(a^*, ha^*) - \rho(M_1, M_2)\Big) \end{aligned} \quad (\mathbf{2}.20)$$

and

$$\begin{aligned}&\rho(ha^*, hu_n)\\ &\leq \tau\Big(\max\Big\{\rho(a^*, ha^*) - \rho(M_1, M_2), \rho(u_n, hu_n) - \rho(M_1, M_2)\Big\}\Big)\\ &+ \sigma\Big(\rho(u_n, ha^*) - \rho(M_1, M_2), \rho(a^*, hu_n) - \rho(M_1, M_2), \rho(a^*, ha^*) - \rho(M_1, M_2),\\ &\qquad \rho(u_n, hu_n) - \rho(M_1, M_2)\Big). \end{aligned} \quad (\mathbf{2}.21)$$

By allowing n goes to infinity in (**2**.20) and (**2**.21), we conclude that

$$\lim_{n\to+\infty} \rho(ha^*, hu_n) \leq \tau(\rho(a^*, ha^*) - \rho(M_1, M_2)) < \rho(a^*, ha^*) - \rho(M_1, M_2) \quad (\mathbf{2}.22)$$

and

$$\lim_{n\to+\infty} \rho(hu_n, ha^*) \leq \tau(\rho(ha^*, a^*) - \rho(M_1, M_2)) < \rho(ha^*, a^*) - \rho(M_1, M_2). \quad (\mathbf{2}.23)$$

Applying the triangular inequality, we get

$$\begin{aligned}\rho(a^*, ha^*) &\leq \rho(a^*, u_n) + \rho(u_n, hu_n) + \rho(hu_n, ha^*) - \rho(u_n, u_n) - \rho(hu_n, hu_n)\\ &\leq \rho(a^*, u_n) + \rho(u_n, hu_n) + \rho(hu_n, ha^*).\end{aligned}$$

On letting n tends to infinity in above inequality, we arrive to

$$\rho(a^*, ha^*) < \rho(M_1, M_2) + \rho(a^*, ha^*) - \rho(M_1, M_2),$$

which is a contradiction. So $\rho(a^*, ha^*) = \rho(M_1, M_2)$.
Similarly, by the aiding of (**2**.22) and triangular inequality, we may show that $\rho(ha^*, a^*) = \rho(M_2, M_1)$. at this end we proved that a^* is a best proximity point of h. To prove the uniqueness of fixed proximity point of h, we assume that $\rho(b^*, hb^*) = \rho(M_1, M_2)$ and $\rho(hb^*, b^*) = \rho(M_2, M_1)$. So $\rho(a^*, b^*) = \rho(ha^*, hb^*)$. Using condition (1) of the theorem, we get

$$\begin{aligned}&\rho(ha^*, hb^*) = \rho(ha^*, hb^*)\\ &\leq \tau\Big(\max\Big\{\rho(a^*, ha^*) - \rho(M_1, M_2), \rho(b^*, hb^*) - \rho(M_1, M_2)\Big\}\Big) \qquad (\mathbf{2}.24)\\ &+ \sigma\Big(\rho(b^*, ha^*) - \rho(M_1, M_2), \rho(a^*, hb^*) - \rho(M_1, M_2), \rho(a^*, ha^*) - \rho(M_1, M_2),\\ &\qquad \rho(b^*, hb^*) - \rho(M_1, M_2)\Big) \qquad (\mathbf{2}.25)\\ &= 0. \qquad (\mathbf{2}.26)\end{aligned}$$

Thus $\rho(a^*, b^*) = 0$, and hence $a^* = b^*$. □

Now, we employ our main results to derive more results:

Corollary 2.1. *Let (M, ρ) be a complete quasi-partial metric space. Let $h\colon M \to M$ be a mapping such that for $r, s \in M$, we have*

$$\rho(hr, hs) \leq \tau(\rho(r, s)) + \sigma\Big(\rho(s, hr), \rho(r, hs), \rho(r, hr), \rho(s, hs)\Big).$$

Then h has a unique fixed point $m^* \in M$.

Corollary 2.2. *Let (M, ρ) be a complete quasi-partial metric space. Let $h\colon M \to M$ be a mapping such that for $r, s \in M$, we have*

$$\rho(hr, hs) \leq \tau\Big(\Big\{\rho(r, hr)), \rho(s, hs)\Big\}\Big) + \sigma\Big(\rho(s, hr), \rho(r, hs), \rho(r, hr), \rho(s, hs)\Big).$$

Then h has a unique fixed point $m^* \in M$.

Corollary 2.3. *Let (M, ρ) be a complete quasi-partial metric space. Suppose M_1 and M_2 be closed subsets of M with respect to ρ. Let $h\colon M_1 \to M_2$ be a mapping such that for $r, s \in M_1$, we have*

$$\begin{aligned}\rho(hr, hs) \leq \tau(\rho(r, s)) + \inf\Big\{&\rho(s, hr) - \rho(M_1, M_2),\\ &\rho(r, hs) - \rho(M_1, M_2), \rho(r, hr) - \rho(M_1, M_2), \rho(s, hs) - \rho(M_1, M_2)\Big\}.\end{aligned}$$

Moreover assume the following conditions:

1) M_1^0 is nonempty,
2) $hM_1^0 \subseteq M_2^0$, and
3) (M_1, M_2) posses the P-property.

Then h has a unique best proximity $m^* \in M_1$.

Corollary 2.4. *Let (M,ρ) be a complete quasi-partial metric space. Suppose M_1 and M_2 be closed subsets of M with respect to ρ. Let $h\colon M_1 \to M_2$ be a mapping such that for $r,s \in M_1$, we have*

$$\rho(hr,hs) \le \tau\Big(\Big\{\rho(r,hr)-\rho(M_1,M_2),\rho(s,hs)-\rho(M_1,M_2)\Big\}\Big)$$
$$+\inf\Big\{\rho(s,hr)-\rho(M_1,M_2),\rho(r,hs)-\rho(M_1,M_2),$$
$$\rho(r,hr)-\rho(M_1,M_2),\rho(s,hs)-\rho(M_1,M_2)\Big\}.$$

Moreover assume the following conditions:

1) M_1^0 is nonempty,
2) $hM_1^0 \subseteq M_2^0$, and
3) (M_1,M_2) posses the P-property.
Then h has a unique best proximity $m^* \in M_1$.

Corollary 2.5. *Let (M,ρ) be a complete quasi-partial metric space. Let $h\colon M \to M$ be a mapping such that for $r,s \in M$, we have*

$$\rho(hr,hs) \quad\le\quad \tau(\rho(r,s))+\inf\Big\{\rho(s,hr),\rho(r,hs),\rho(r,hr),\rho(s,hs)\Big\}.$$

Then h has a unique fixed point $m^* \in M$.

Corollary 2.6. *Let (M,ρ) be a complete quasi-partial metric space. Let $h\colon M \to M$ be a mapping such that for $r,s \in M$, we have*

$$\rho(hr,hs) \quad\le\quad \tau\Big(\max\Big\{\rho(r,hr),\rho(s,hs)\Big\}\Big)+\inf\Big\{\rho(s,hr),\rho(r,hs),\rho(r,hr),\rho(s,hs)\Big\}.$$

Then h has a unique fixed point $m^* \in M$.

Example 2.2. *Define the quasi-partial metric ρ on*

$$M=\Big\{0,2,\frac{1}{2},3,\frac{1}{3},4,\frac{1}{4},\cdots\Big\}$$

by $\rho(a,b)=|a-b|+a$. Take $M_1=\Big\{0,2,3,4,\cdots\Big\}$ and $M_2=\Big\{0,\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\Big\}$. Define $h: M_1 \to M_2$ by

$$h(r)=\begin{cases} 0, & \text{if } r=0;\\ \frac{1}{r}, & \text{if } r\neq 0.\end{cases}$$

Define the function τ on $[0,+\infty)$ via

$$\tau(u)=\begin{cases} \frac{1}{2}u^2, & \text{if } u<1;\\ \frac{1}{2}u, & \text{if } u\ge 1.\end{cases}$$

Also, define

$$\sigma\colon M\times M\times M\times M \to [0,+\infty)$$

by $\sigma(i,j,k,r)=\inf\{i,j,k,r\}$. Then

(1) *$hM_1^0 \subseteq M_2^0$.*
(2) *(M_1,M_2) satisfies the P-property.*
(3) *h is an $(\rho,\tau,\sigma,M_1,M_2)$-contraction.*

Proof. Note that $M_1^0=\{0\}$, $M_2^0=\{0\}$, $\rho(M_1,M_2)=0$ and $\rho(M_2,M_1)=0$.

To prove that h is an $(\rho,\tau,\sigma,M_1,M_2)$-contraction. Given $r,s\in M_1$. we discuss the following cases:

Case 1: $r = s = 0$. Here, we have

$$\rho(h0, h0) = 0 \quad \leq \quad \tau(\rho(r,s)) + \sigma\Big(\rho(s,hr) - \rho(M_1,M_2), \rho(r,hs) - \rho(M_1,M_2), \rho(r,hr) - \rho(M_1,M_2), \rho(s,hs) - \rho(M_1,M_2)\Big).$$

Case 2: $r = 0$ and $s \neq 0$. Here, we have

$$\rho(h0, hs) = \frac{1}{s} \leq \frac{1}{2}s = \tau(\rho(0,s)) + \sigma\Big(\rho(s,0), \rho(0,\frac{1}{s}), \rho(0,h0), \rho(s,hs)\Big).$$

Case 3: $r \neq 0$ and $s = 0$. Here, we have

$$\rho(hr, h0) = \frac{2}{r} \leq r = \tau(\rho(r,0)) + \sigma\Big(\rho(0,\frac{1}{s}), \rho(r,0), \rho(r,\frac{1}{r}), \rho(0,h0)\Big).$$

Case 4: $r \neq 0$ and $s \neq 0$. Here, we have

$$\rho(hr, hs) = \rho\Big(\frac{1}{r}, \frac{1}{s}\Big) = \Big|\frac{1}{r} - \frac{1}{s}\Big| + \frac{1}{r} \tag{2.27}$$

and

$$\begin{aligned} &\tau(\rho(r,s)) + \sigma\Big(\rho(s,hr) - \rho(M_1,M_2), \rho(r,hs) - \rho(M_1,M_2), \\ &\qquad \rho(r,hr) - \rho(M_1,M_2), \rho(s,hs) - \rho(M_1,M_2)\Big) \\ &= \frac{1}{2}|r-s| + \frac{1}{2}r + \sigma\Big(|s - \frac{1}{r}| + s, |r - \frac{1}{s}| + r, \\ &\qquad |r - \frac{1}{r}| + r, |s - \frac{1}{s}| + s\Big). \end{aligned} \tag{2.28}$$

By comparing (**2**.27) with (**2**.28), we conclude that

$$\rho(hr, hs) \quad \leq \quad \tau(\rho(r,s)) + \sigma\Big(\rho(s,hr) - \rho(M_1,M_2), \rho(r,hs) - \rho(M_1,M_2), \rho(r,hr) - \rho(M_1,M_2), \rho(s,hs) - \rho(M_1,M_2)\Big).$$

Hence we deduce that h is an $(\rho, \tau, \sigma, M_1, M_2)$-contraction.

Theorem **2**.1, ensures that h has a best proximity point in M_1.

□

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